MATH 464 HW 5

3.15 (a) $\int_{-\infty}^{\infty} \frac{\sin ax}{x} = \int_{-\infty}^{\infty} \pi \frac{\sin(\pi \cdot \frac{a}{\pi} x)}{\pi \cdot \frac{a}{\pi} x} d\left(\frac{ax}{\pi}\right)$

$\overset{y=\frac{a}{\pi}x}{=} \pi \int_{-\infty}^{\infty} \frac{\operatorname{sinc}(y)}{y} dy = \pi \Pi(0) = \pi$

(b) $\int_{-\infty}^{\infty} \left(\frac{\sin ax}{x}\right)^2 dx = a\pi \int_{-\infty}^{\infty} |\operatorname{sinc}(y)|^2 dy$

$= a\pi \int_{-\infty}^{\infty} |\Pi(s)|^2 ds = a\pi.$

(c) $\int_{-\infty}^{\infty} \left(\frac{\sin ax}{x}\right)^3 dx = \pi a^2 \int_{-\infty}^{\infty} \operatorname{sinc}^3(y)\, dy$

$= \pi a^2 \int_{-\infty}^{\infty} \Pi(s) \cdot \Lambda(s)\, ds = \frac{3}{4}\pi a^2$

(d) $\int_{-\infty}^{\infty} \left(\frac{\sin ax}{x}\right)^4 dx$

$= \pi a^3 \int_{-\infty}^{\infty} |\operatorname{sinc}^2(y)|^2 dy = \pi a^3 \int_{-\infty}^{\infty} \Lambda^2(s)\, ds$

$= \pi a^3 \cdot 2\int_0^1 (1-s)^2 ds = \frac{2\pi a^3}{3}$

3.22 (b) $\overline{f(x)}$

(d) $\frac{1}{2} f\left(\frac{1}{2}x\right)$

(f) $\int_{-\infty}^{\infty} \cos(2\pi s) F(s) \cdot e^{2\pi i s x} ds$

$= \frac{1}{2} \int_{-\infty}^{\infty} \left[e^{2\pi i s} + e^{-2\pi i s}\right] F(s)\, e^{2\pi i s x} ds$

$= \frac{1}{2} f(x+1) + \frac{1}{2} f(x-1)$

(h) $\frac{1}{8\pi i} f'(-\frac{x}{2})$ ②

Note that $\int_{-\infty}^{\infty} s F(2s) e^{-2\pi i s x} ds$

$$= \int_{-\infty}^{\infty} (-s) F(-2s) e^{2\pi i s x} ds$$

$$\mathcal{F}\{f'(-\tfrac{x}{2})\} = 2 \cdot 2\pi i (-2s) F(-2s) = -8\pi i s F(-2s)$$

i.e. $\mathcal{F}\left(\frac{1}{8\pi i} f'(-\frac{x}{2})\right) = -s F(-2s)$.

3.28 (a) Assume f is even, then

$$\int_{u=0}^{\infty} f(u) \cos(2\pi u x) du = \frac{1}{2} \int_{-\infty}^{\infty} f(u) e^{2\pi i u x} du$$

Taking Fourier transform of $\int_{-\infty}^{\infty} \frac{1}{2} f(u) e^{2\pi i u x} du$

$\Rightarrow \frac{1}{2} f(s)$

On the other hand, we know $\mathcal{F}(e^{-|x|}) = \frac{2}{1+4\pi^2 s^2}$

Hence $f(s) = \frac{4}{1+4\pi^2 s^2}$ for $x \geq 0$ and this is indeed an even function.

(b) Recall in Exercise 1.3, we derived the sine transform pair:

$$g(x) = 2\int_{s=0}^{\infty} G(s)\sin(2\pi s x)\,ds, \quad x>0$$

$$G(s) = 2\int_{x=0}^{\infty} g(x)\sin(2\pi s x)\,dx, \quad s>0$$

Take $g(x) = 2\Pi(x-\frac{1}{2})$, $G(s) = f(s)$

$$2\int_{u=0}^{\infty} f(u)\sin(2\pi u x)\,du = 2\Pi(x-\tfrac{1}{2}) \qquad x \geq 0$$

Hence

$$f(s) = G(s) = 2\int_{x=0}^{\infty} g(x)\sin(2\pi s x)\,dx$$

$$= 4\int_{x=0}^{\infty} \Pi(x-\tfrac{1}{2})\sin(2\pi s x)\,dx$$

$$= 4\int_{0}^{1} \sin(2\pi s x)\,dy$$

$$= \frac{4}{2\pi s}\left[1-\cos(2\pi s)\right] \qquad \text{for } s>0.$$

Remark: the same strategy can be applied to 3.28 (a).

(d) $f(x) * f(x) = e^{-\pi x^2} \Rightarrow F^2(s) = e^{-\pi s^2}$

$F(s) = e^{-\frac{\pi}{2}s^2} \Rightarrow f(x) = \sqrt{2}\,e^{-2\pi x^2}$

4.2

(b) $$F(k) = \int_{-\frac{p}{4}}^{\frac{p}{4}} \cos\left(\frac{2\pi\xi}{p}\right) e^{-2\pi i k \xi}\, d\xi$$

$$= \int_{-\infty}^{\infty} \frac{e^{i\frac{2\pi\xi}{p}} + e^{-i\frac{2\pi\xi}{p}}}{2} \cdot \Pi\left(\frac{2}{p}\xi\right) e^{-2\pi i k\xi}\, d\xi$$

$\frac{1}{2}$ $\frac{p}{4}$

$$\mathcal{F}\{e^{i\pi x}\Pi(x)\} = \text{sinc}\left(s - \frac{1}{2}\right)$$

$$\mathcal{F}\left\{e^{i\frac{2\pi}{p}\xi}\,\Pi\left(\frac{2}{p}\xi\right)\right\} = \frac{p}{2}\text{sinc}\left(\frac{p}{2}s - \frac{1}{2}\right)$$

$$\mathcal{F}\left\{e^{-i\frac{2\pi\xi}{p}}\,\Pi\left(\frac{2}{p}\xi\right)\right\} = \frac{p}{2}\text{sinc}\left(\frac{p}{2}s + \frac{1}{2}\right)$$

Hence $$F(k) = \frac{p}{2}\left[\text{sinc}\left(\frac{p}{2}k - \frac{1}{2}\right) + \text{sinc}\left(\frac{p}{2}k + \frac{1}{2}\right)\right]$$

FS: $$\sum_{k=-\infty}^{\infty} \frac{1}{p} F\left(\frac{k}{p}\right) e^{2\pi i k x/p}$$

$$= \sum_{-\infty}^{\infty} \frac{1}{2}\left[\text{sinc}\left(\frac{k}{2} - \frac{1}{2}\right) + \text{sinc}\left(\frac{k}{2} + \frac{1}{2}\right)\right] e^{2\pi i k x/p}$$

$$\text{sinc}\left(\frac{k}{2} - \frac{1}{2}\right) + \text{sinc}\left(\frac{k}{2} + \frac{1}{2}\right) = \begin{cases} -\dfrac{4}{\pi(k^2-1)} & k = 4n \\ +\dfrac{4}{\pi(k^2-1)} & k = 4n+2 \\ 0 & k = 4n+1,\ 4n+3 \end{cases}$$

Hence $G[k] = \begin{cases} \dfrac{-2}{\pi(k^2-1)} & k = 4n \\ \dfrac{2}{\pi(k^2-1)} & k = 4n+2 \\ 0 & k = 4n+1,\ 4n+3 \end{cases}$

FS: $\sum_{k=-\infty}^{\infty} G[k]\, e^{2\pi i k x/p}$

(d) $F(k) = \mathcal{F}\{e^{-a|x|}\} = \dfrac{1}{a} \dfrac{2}{1+4\pi^2(\frac{k}{a})^2}$

FS: $\sum_{k=-\infty}^{\infty} \dfrac{2}{ap} \dfrac{1}{1+4\pi^2(\frac{k}{ap})^2} e^{2\pi i k x/p}$

4.3 (a) f: 1-periodic

$F[k] = \int_0^1 x \cdot e^{-2\pi i s x} dx = \int_0^1 x\cos(2\pi s x)dx + i\int_0^1 x\sin(2\pi s x)dx$

F.S: $\sum_{k=-\infty}^{\infty} F[k]\, e^{2\pi i k x}$

(b) $F[k] = \int_{-1}^{0} (-x) e^{-2\pi i s x} dx + \int_0^1 x e^{-2\pi i s x} dy$

$= \int_0^1 x(e^{2\pi i s x} + e^{-2\pi i s x}) dx$

$= 2\int_0^1 x\cos(2\pi s x)\, dx$

F.S. $\sum_{k=-\infty}^{\infty} \frac{1}{2} F[\frac{k}{2}] e^{\pi i k y}$

(c) $F[k] = \int_{-1}^{1} x e^{-2\pi i s x} dy$

F.S: $\sum_{k=-\infty}^{\infty} \frac{1}{2} F[\frac{k}{2}] e^{\pi i k y}$

4.3 (a) f: 1-periodic $\qquad f(x) := \sum_{m=-\infty}^{\infty} g(x-mp)$

$$G[0] = \int_0^1 x\,dx = \frac{1}{2}$$

$k \neq 0 \quad G[k] = \int_0^1 x e^{-2\pi i k x}\,dx = \frac{e^{-2\pi i k}}{-2\pi i k} - \frac{1}{(2\pi i k)^2}\left[e^{-2\pi i k} - 1\right]$

$$= \left[\frac{1}{-2\pi i k} + \frac{1}{4\pi^2 k^2}\right] e^{-2\pi i k} - \frac{1}{4\pi^2 k^2} = -\frac{1}{2\pi i k}$$

$$\frac{1}{2} + \sum_{k\neq 0} G[k] e^{2\pi i k x} = \frac{1}{2} - \sum_{k\neq 0} \frac{e^{2\pi i k x}}{2\pi i k}$$

$$= \frac{1}{2} - \frac{1}{2\pi i} \sum_{k=1}^{\infty} \frac{2i \sin(2\pi k x)}{k}$$

$$= \frac{1}{2} - \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{\sin(2\pi k x)}{k}$$

(b) $G[k] = \int_{-\infty}^{\infty} \left[\Lambda(x-1)\right] e^{-2\pi i k x}\,dy$

$$= \begin{cases} e^{-2\pi i k} \operatorname{sinc}^2(k) & k \neq 0 \\ 1 & k = 0 \end{cases}$$

$$F[k] = \frac{1}{2} G\left[\frac{k}{2}\right]$$

F.S: $\frac{1}{2} + \frac{1}{2} \sum_{k\neq 0} \cancel{e^{-\pi i k}} \operatorname{sinc}^2\left(\frac{k}{2}\right) \cancel{e^{\pi i k}}$

$$= \frac{1}{2} + \frac{1}{2} \sum_{k\neq 0} \operatorname{sinc}^2\left(\frac{k}{2}\right) = \frac{1}{2} + \sum_{k=1}^{\infty} \operatorname{sinc}^2\left(\frac{k}{2}\right)$$

(c)

$$F[k] = \frac{1}{2}\int_{-1}^{1} x \cdot e^{-\pi i k x}\, dx$$

$$= -\frac{i}{2}\int_{-1}^{1} x\sin(\pi k x)\, dx$$

$$= -i\int_{0}^{1} x\sin(\pi k x)\, dx$$

$$= \frac{i}{\pi k}\int_{0}^{1} x\, d(\cos(\pi k x))$$

$$= \frac{i}{\pi k}\left[\cos(\pi k) - \int_{0}^{1}\cos(\pi k x)\, dx\right]$$

$$= \frac{i}{\pi k}(-1)^k$$

$$F[0] = 0$$

F.S: $$\sum_{k\neq 0} \frac{i}{\pi k}(-1)^k e^{\pi i k x}$$

$$= 2\sum_{k=1}^{\infty} \frac{i}{\pi k}(-1)^k\, i\sin(\pi k x)$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\pi k} 2\sin(\pi k x)$$