Midterm #1 - Solutions

1. $$g(x) = \frac{2}{x^2-4x+5} = \frac{2}{1+(x-2)^2}$$

We know that the transform of $\frac{2}{1+4\pi^2x^2}$ is $e^{-|s|}$.

Hence, the transform of $\frac{1}{1+x^2} = \frac{1}{1+4\pi^2\left(\frac{x}{2\pi}\right)^2}$ is $\frac{1}{2}2\pi e^{-2\pi|s|}$

and the transform of $g(x)$ is

$$G(s) = 2\pi e^{-4\pi i s} e^{-2\pi|s|}.$$

2. $$f(x) = \left(\frac{\sin ax}{x}\right)^2 = a^2\left(\frac{\sin\left(\pi\frac{ax}{\pi}\right)}{\pi\frac{ax}{\pi}}\right)^2 = a^2\operatorname{sinc}^2\left(a\frac{x}{\pi}\right).$$

$$F(s) = \frac{a^2}{a}\pi\Lambda\left(\frac{\pi s}{a}\right) = a\pi\Lambda\left(\frac{\pi s}{a}\right).$$

By Parseval:

$$I = \int_{-\infty}^{\infty}\left(\frac{\sin ax}{x}\right)^2\left(\frac{\sin ax}{x}\right)^2 dx = \int_{-\infty}^{\infty}(F(s))^2 ds =$$

$$= 2a^2\pi^2\int_0^{a/\pi}\left(\Lambda\left(\frac{\pi s}{a}\right)\right)^2 ds = 2a^2\pi^2\int_0^{a/\pi}\left(1-\frac{\pi s}{a}\right)^2 ds$$

$$= 2a^2\pi^2\left[s - 2\frac{\pi}{a}\frac{s^2}{2} + \frac{1}{3}\frac{\pi^2}{a^2}s^3\right]\Big|_0^{a/\pi}$$

$$= 2a^2\pi^2\frac{a}{\pi}\cdot\frac{1}{3} = \frac{2}{3}a^3\pi.$$

③. $$\int_{-\infty}^{\infty} f(u)\, f(x-u)\, du = e^{-\pi x^2}.$$

$$f * f(x) = e^{-\pi x^2}$$

$$(F(s))^2 = e^{-\pi s^2} \implies F(s) = e^{-\frac{\pi}{2} s^2} = e^{-\pi \left(\frac{s}{\sqrt{2}}\right)^2}$$

$$f(x) = \sqrt{2}\, e^{-2\pi x^2}.$$

4. ⓐ $$-(2\pi i s)^2 F(s) + F(s) = G(s)$$

$$g(x) = e^{-2|x|}$$

$$G(s) = \frac{1}{2} \cdot \frac{2}{1+4\pi^2 \left(\frac{s}{2}\right)^2} = \frac{1}{1+\pi^2 s^2}.$$

$$\implies F(s) = \frac{1}{1+4\pi^2 s^2} \cdot \frac{1}{1+\pi^2 s^2}.$$

ⓑ. $$F(s) = \frac{A}{1+4\pi^2 s^2} + \frac{B}{1+\pi^2 s^2} = \frac{A(1+\pi^2 s^2) + B(1+4\pi^2 s^2)}{(\quad)\cdot(\quad)} = \frac{1}{(\ )(\ )}$$

$$A + B = 1 \qquad -4B + B = 1 \implies B = -\frac{1}{3}$$

$$A\cancel{\pi^2} + B\cancel{\pi^2}4 = 0 \implies A = -4B \implies A = \frac{4}{3}.$$

$$F(s) = \frac{4/3}{1+4\pi^2 s^2} - \frac{1/3}{1+\pi^2 s^2}.$$

$$\implies f(x) = \frac{2}{3} e^{-|x|} - \frac{1}{3} e^{-2|x|}.$$

Notes on problem #1:

$$\mathcal{F}\left(\frac{2}{1+4\pi^2x^2}\right) = e^{-|s|}$$

$$\frac{1}{1+x^2} = \frac{1}{1+4\pi^2\left(\frac{x}{2\pi}\right)^2} \Rightarrow \mathcal{F}\left(\frac{1}{1+x^2}\right) = \frac{1}{\cancel{2}}\cancel{2}\pi e^{-2\pi|s|}$$

$$\Rightarrow \mathcal{F}\left(\frac{2}{1+(x-2)^2}\right) = 2\cdot e^{-4\pi i s}\pi e^{-2\pi|s|}.$$

It is wrong to first compute $x-2$ and then divide by 2π. If you first want to do the shift, write $\frac{x-2}{2\pi} = \frac{x}{2\pi} - \frac{1}{\pi}$. Shift by $\frac{1}{\pi}$ and then divide by 2π.

Otherwise, it is like saying that knowing $\mathcal{F}\left(\frac{2}{1+4\pi^2x^2}\right)$ provides us $\mathcal{F}\left(\frac{2}{1+4\pi^2(x-2)^2}\right)$ by shifting. But

then, from here, if we divide by 2π that should only be x we are dividing by:

$$\mathcal{F}\left(\frac{2}{1+4\pi^2\left(\frac{x}{2\pi}-2\right)^2}\right)$$

and that is not what we want.