Math 464

Midterm #1 - Solutions.

① (a). $e^{-x} h(x) \longrightarrow \frac{1}{1+2\pi i s}$ $\quad h(x) = \text{heaviside function.}$

$$\Rightarrow \frac{1}{1+2\pi i x} \longrightarrow e^{s} h(-s)$$

$$f(x) = \frac{1}{x+i} = -i \frac{1}{1+2\pi i\left(\frac{x}{-2\pi}\right)}$$

$$\Rightarrow F(s) = (-2\pi i) e^{-2\pi s} h(2\pi s) = (-2\pi i) e^{-2\pi s} h(s)$$

(b) $f(x) = \int_{-\infty}^{\infty} F(s) e^{2\pi i s x} ds \Rightarrow F(s) = e^{-s^4}.$

② $f(x) = \frac{\sin 2x}{x} = 2 \operatorname{sinc}\left(\frac{2x}{\pi}\right)$

Parseval implies $\int_{-\infty}^{\infty} \left(\frac{\sin 2x}{x}\right)^2 \left(\frac{\sin 2x}{x}\right) dx =$

$$= \int_{-\infty}^{\infty} \left(2 \operatorname{sinc}\left(\frac{2x}{\pi}\right)\right)^2 \left(2 \sin\left(\frac{2x}{\pi}\right)\right) dx = 2\pi^2 \int_{-\infty}^{\infty} \Lambda\left(\frac{\pi s}{2}\right) \Pi\left(\frac{\pi s}{2}\right) ds$$

$$= 2\pi^2 \int_{-1/\pi}^{1/\pi} \Lambda\left(\frac{\pi s}{2}\right) ds = 4\pi^2 \int_{0}^{1/\pi} \left(1 - \frac{\pi s}{2}\right) ds = 3\pi.$$

(3) $(f*g)(x) = \int_{-\infty}^{\infty} f(u)g(x-u)\,du$

$$\Rightarrow (FT(f*g))(s) = \int_{x=-\infty}^{\infty} \left(\int_{u=-\infty}^{\infty} f(u)g(x-u)\,du\right) e^{-2\pi i s x}\,dx$$

$$= \int_{u=-\infty}^{\infty} du\, f(u) \int_{x=-\infty}^{\infty} g(x-u) e^{-2\pi i s x}\,dx = \cancel{\int du\, f(u) \int g(x-u) e^{-2\pi i s}}$$

$$= \int_{u=-\infty}^{\infty} du\, f(u)\, e^{-2\pi i s u} \int_{x=-\infty}^{\infty} g(x-u)\, e^{-2\pi i s(x-u)}\,d(x-u) = F(s)\cdot G(s).$$

(4). The Fourier Transform is:

$$F(s) = \int_0^{1/4} x\, e^{-2\pi i s x}\,dx = \left.\frac{x\, e^{-2\pi i s x}}{-2\pi i s}\right|_0^{1/4} - \left.\frac{e^{-2\pi i s x}}{(-2\pi i s)^2}\right|_0^{1/4}$$

$$= \frac{1/4}{-2\pi i s}\, e^{-\frac{\pi i s}{2}} + \frac{1}{4\pi^2 s^2}\left(e^{-\frac{\pi i s}{2}} - 1\right)$$

$$= e^{-\frac{\pi i s}{2}}\left(\frac{1}{4\pi^2 s^2} - \frac{1}{8\pi i s}\right) - \frac{1}{4\pi^2 s^2} \qquad s \neq 0.$$

If $s=0$: $F(0) = \int_0^{1/4} x\,dx = \frac{1}{2}x^2\Big|_0^{1/4} = \frac{1}{32}$.

Poisson $\Rightarrow$ $G[k] = \frac{1}{p} F\left(\frac{k}{p}\right) \overset{(p=1)}{=} F(k)$.

And the Fourier Series is $\sum_{k=-\infty}^{\infty} F[k]\, e^{\frac{2\pi i k x}{p}}$

i.e. since $p=1$: $\sum_{k=-\infty}^{\infty} F[k]\, e^{2\pi i k x}$