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On a class of thermal blow-up patterns

Doron Levy, Philip Rosenau

School of Mathematical Sciences, Tel Aviv University, Tel Aviv 69978, Israel

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Abstract

We study the patterns of a thermal explosion as described via $u_t = (\Delta + 1)u^m$, $m > 1$. These processes, characterized by an intrinsic length scale, always converge into very simple, universal, space-time separable, axisymmetric pattern(s) with a compact support – referred to as *dissipative compactons*. When the initial datum is specified on an axisymmetric annulus, though the evolving pattern seems to preserve this symmetry, at a later stage, it collapses very quickly to the center. In a perturbed annulus, local axisymmetric patches of blow-up form instead of a collapse. For a planar, homogeneous, Dirichlet problem, the space-time separability of the emerging pattern is preserved as well, but the competition between the intrinsic and extrinsic characteristic scales generates a wider variety of spatial patterns, with the self-localization taking place on large domains. As the width of the domain diminishes, then depending on the width–length ratio, the emerging pattern first partially, and then fully, attaches to the boundaries. With further decrease of the domain, the emerging separable pattern, instead of exploding, decays algebraically in time. © 1997 Elsevier Science B.V.

1. Introduction

The model problem

$$u_t = \Delta(u^m) + u^m, \quad m > 1, \quad x \in \mathbb{R}^n \tag{1}$$

describes a very simple model of a thermal explosion. This particular nonlinear balance between dissipation and volumetric heat production of the sources endows the problem with an intrinsic length scale and a special symmetry. This symmetry generates space-time separable solutions, *which will be shown to play a crucial role in the description of its asymptotic behavior.* The simplicity of the later-time behavior makes the problem accessible to analysis and enables its characterization.

We shall describe in some detail how the nonlinear system evolves from an arbitrary initial datum (but with a compact support) into *unique attractor(s)* fairly independent of the initial start-up. This attractor is stationary and has a compact support. However, the evolution is of a runaway type as the system explodes in a finite time. We shall provide accurate estimates of the time of the blow-up as a functional of the initial datum, and explore the basin of its attractors.

The fact that diffusive systems with sources may explode is well known and a significant amount of research was devoted to this subject. For semi-linear parabolic equations (i.e., when the diffusion is linear) the phenomenon of the blow-up is pretty well understood, cf. Refs. [1–3], with the later stages of evolution very often assuming self-similar patterns. For quasi-linear problems like the one considered here, the main activity was carried out in the former Soviet Union (for a recent summary see Ref. [3] and references therein). Most of these efforts were directed at one-dimensional (1D) problems. In contrast, the main

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thrust of our work is in the study of blow-up in higher dimensions, but as we shall show, much remains to be understood even in 1D. We stress that it is the presence of the intrinsic length scale in our problem (which cannot occur in a semi-linear problem unless it becomes completely linear), that endows it with its unique features. When no characteristic length is involved, the typical blow-up becomes self-similar. Instead, in our case we obtain stationary compact structures which, for reasons to be clarified shortly, will be referred to as *dissipative compactons.*

On the line, the evolution is quite simple and can be characterized by a complete localization of the exploding pattern. In higher dimensions, the study of which most of our efforts are devoted to, a larger topological variety of pattern distribution becomes feasible. Nevertheless, a great measure of simplicity underlies the dynamics in higher dimensions as well, with axisymmetric patterns in the plane and spherically symmetric patterns in 3D being the most stable states, always emerging from an arbitrary initial datum. Even when the initial datum is specified on an axisymmetric annulus, though the evolving pattern seems to preserve this symmetry for a long time, suddenly, when the pattern seems to fully converge upon its asymptotic shape, a sudden collapse of the annulus occurs and the ultimate pattern assumes the simplest axisymmetric form.

2. Dissipative compactons on the line

We start by considering the exploding patterns of

$$u_t = (u^2)_{xx} + u^2, \quad x \in (-\infty, \infty). \tag{2}$$

The solution that serves as an attractor is found via a separation of variables $u = \phi(t)Z(x)$. For $\phi(t)$ we have $\phi(t) = 1/(t_* - t)$ while the spatial part is found among the solutions of

$$(Z^2)_{xx} + Z^2 - Z = 0. \tag{3}$$

Among the solutions of Eq. (3), the one found to be relevant to our efforts is

$$u = \frac{4}{3(t_* - t)} \cos^2(x/4), \quad |x| < 2\pi, \tag{4}$$

and vanishes elsewhere. Thus, the intrinsic length scale in our problem is 4π (note that the coefficient, 1, of the source term has dimensions of $1/L^2$). Notably, our solution satisfies Eq. (2) in the classical sense. For a partial motivation to the very basic question of why this specific solution should be of any relevance to the problem at hand, we refer to a recent work [4], where a quasi-linear dispersive equation,

$$u_t + (u^2)_x + (u^2)_{xxx} = 0, \quad x \in (-\infty, \infty), \tag{5}$$

was found to support solitons with a compact support, the *compactons,*

$$u = \frac{4\lambda}{3} \cos^2[\tfrac{1}{4}(x - \lambda t)], \quad |x - \lambda t| \leqslant 2\pi, \tag{6}$$

vanishing elsewhere. Unlike a conventional soliton, due to the existence of an intrinsic characteristic in Eq. (5), the width of the compacton (6) is fixed and independent of the amplitude. The compactness is the result of the degeneracy of the third operator at $u = 0$. In a travelling frame, the ordinary differential equation that describes these compactons is identical to Eq. (4); it was this observation that hinted to the special role that solution (4) may play, and also motivated us to refer to these solutions as *dissipative compactons.*

The possibility to sustain compact pulse(s) for a finite time (determined by the time of explosion) is consistent with the following.

(i) The flux $\sim -(u^2)_x$ vanishes on the front where $u \sim x^2$.

(ii) For nonlinear diffusion on a line, if the initial datum is under a parabola, it takes a finite time before the front can move [5]. This effect, known as *the waiting time*, is caused by the degeneracy of the thermal conductivity at $u = 0$. A gradient build-up takes a finite time, and only upon the crossing of a certain threshold, the motion of the interface begins. In a cold background, this degeneracy also causes the thermal waves to propagate with a finite speed. Nonlinear diffusion also has an important impact on reaction-diffusion processes, cf. Refs. [6-8]. With the addition of a nonlinear source, the unbounded growth attracts the "mass" into the center of growth. This counteracts the natural diffusion, and thus further enhances the delay effect on the front.

We now turn to considering the attraction basin(s) of Eq. (2). To this end, it is useful to turn to Figs. 1-4, where several different evolution scenarios are considered. In Fig. 1 the support of the initial datum is fully

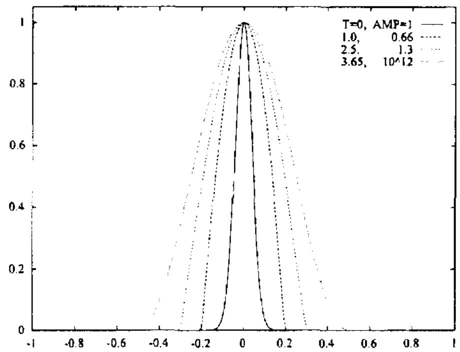

Fig. 1. Narrow initial datum. Observe how with the progress of time the evolving pulse approaches the dissipative compacton. Note that in this and all other figures the domain is scaled by a factor 4π. For comparison, here and elsewhere, the displayed amplitudes are normalized to one. Note the immense disparity of scales between the various pulses.

contained within the fundamental interval. The consequent evolution is normalized at the peak and clearly shows a convergence to the separable solution (4). The explosion time is then numerically computed to be $t_* = 3.65$. Since Eq. (2) is invariant under

$$u \to Au, \quad t \to t/A, \tag{7}$$

an initial increase in amplitude by a factor of A shortens the explosion time by the same factor. This provides the explosion times for all stretchings (or compressings) of a given initial distribution. The scaling property is also extensively used in all our numerical studies; it enables us to renormalize the amplitudes, so that the numerically challenging approach to the blow-up manifests itself through a compression in time.

In Fig. 2 the initial distribution is wider than the fundamental width (it is actually twice the fundamental width). The display of the resulting evolution, as in Fig. 1, is always normalized to one. One observes that the evolving pattern approaches and engulfs the dissipative compacton from the outside. The evolving pattern pulse does not expand much beyond its initial span. Its growth is localized within the fundamental interval, and stays so until the moment of explosion. In normalized units the tail becomes negligible, as the moment of explosion is approached. The pattern becomes indistinguishable from the dissipative compacton quite late in the evolution. By then it has acquired quite a large, and sometimes very large amplitude; see, for instance, Fig. 2. Depending on the process at hand, this may or may not be an observable phenomenon.

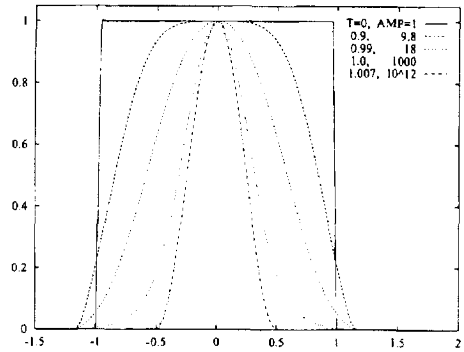

Fig. 2. Evolution of a wide initial square as it converges from the outside upon the dissipative compacton.

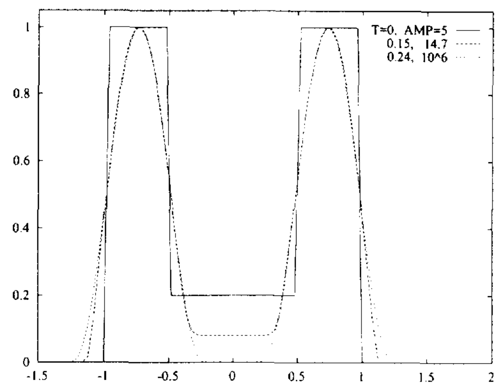

Fig. 3. Emergence of two dissipative compactons out of a symmetric initial datum.

We have seen that when the initial datum is given on a single interval, things are quite simple. If, however, the initial datum consists of a number of disjointed or inhomogeneous patches, things may become far more involved. Let us follow Fig. 3, where the same overall support (which is identical to the support in Fig. 2) is made of two disjointed patches. The separation between their centers is larger than the fundamental length, and hence two modes emerge.

For asymmetric initializations, one observes a sort of thermal cannibalization; bigger pulses prey on the smaller ones; cf. Fig. 4. In actuality, the small pulse grows at a slower rate, but in units normalized by the quick growth its growth seems suppressed. Even more puzzling is the evolution shown in Fig. 5; the inhomogeneous distribution partially shifts the support of the

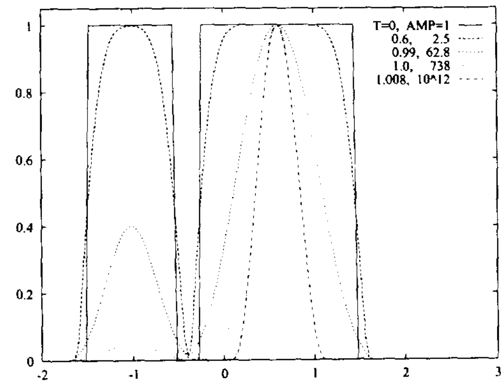

Fig. 4. Asymmetric initial data. Note the effect of cannibalization. The larger pulse appears to prey on the smaller one. In actuality the smaller pulse grows as well. However, in units normalized by the largest peak this growth appears to be suppressed.

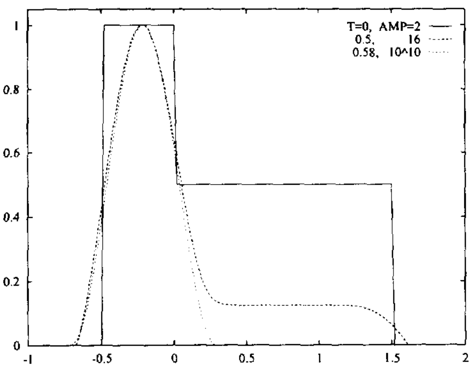

Fig. 5. A shift in the location of the center of explosion due to an uneven initial distribution. This figure exemplifies the difficulty to locate the center of explosion; though ultimately only one compacton emerges, its support is shifted with respect to the original support.

emerging compacton outside of the initial support.

We thus come to an open problem: *Given an initial datum, how are the center(s) of explosion selected?*

This problem is far more difficult than it may seem. One hopes that an analogy with the purely diffusive case will do. We thus consider the conservation laws for Eq. (2). Multiplying it by $f(x)$ and integrating by parts we obtain

$$\int_{\Omega} f(x)u(x,t)\,\mathrm{d}x = \text{const},$$

provided that

$$f''(x) + f(x) = 0 \qquad (8)$$

is satisfied. In the purely dissipative case Eq. (8) now implies $f'' = 0 \Rightarrow f = m_0 + m_1 x$, where m_0 represents conservation of mass. The conservation of its first moment m_1 is then used to locate the center of the self-similar solution in the center of the mass of the initial distribution. In our case $f_1(x) = \cos(x)$ and $f_2(x) = \sin(x)$. Since f_1 and f_2 are non-monotonical functions, the conservation of $\int \cos(x)u(x,t)\,\mathrm{d}x$ and $\int \sin(x)u(x,t)\,\mathrm{d}x$, and their periodic extension, do not seem to be of much help in determining the explosion center(s). The problem remains: What are the selection rules for these center(s)?

2.1. The time of explosion

We have seen that an increase in the amplitude shortens the explosion time by the same factor. We now consider the impact of widening the support. Let $u(x,0) = 1$ for $0 < x < L$ and zero elsewhere. The corresponding explosion time was evaluated numerically and is drawn in Fig. 6 in $L/4\pi$ units. Two effects are seen: For $L < 4\pi$ the shorter the initial pulse is, the longer it takes it to explode (the spread-out takes time). On the other hand, once $L/4\pi > 1$, the explosion time becomes practically independent of the initial width.

One can easily derive an estimate of the explosion time. If $M(t) = \max_x u(x,t)$, then

$$\dot{M} < M^2 \quad \text{and thus } t_* \geqslant 1/M(0). \qquad (9)$$

This bound is *independent* of the choice of the initial datum and is thus a lower bound on the explosion. How

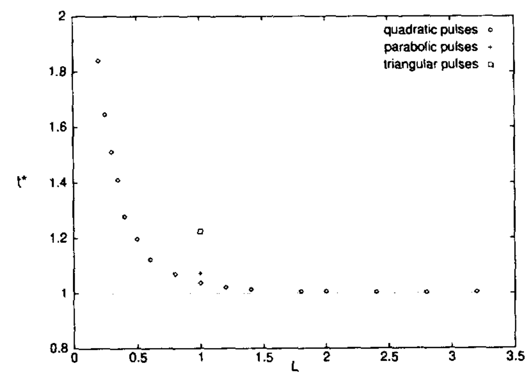

Fig. 6. Explosion time for a square pulse as a function of its width. Note that parabolic and triangular pulses need a larger time to explode.

sharp is this estimate? Comparing it with the explosion time of the square pulses we see that it fairly well represents square pulses wider than 4π. Of course, if initially $u = 1$ on the whole line, then $t_* = 1$ and spatially the solution remains homogeneous. If, however, the homogeneous distribution has a finite span it will, ultimately, converge towards the special distributed shape. Notably, if ab initio we assume $u(x,0) = \cos^2(x/4)$, then the explosion time is $t_* = 4/3$, which *is not only larger than for the square pulse, but is also the largest among the other initial convex shapes having the same initial maximal amplitude and span.* For instance, a triangular (or parabolic) pulse with the same initial amplitude and support explodes quicker than the asymptotic form (see Fig. 6). This means that throughout the reorganization towards the asymptotic stage, the growth of the pulse is quicker than in its asymptotic form. Note that since among all initial distributions, the homogeneous start-up appears to be first to explode, one can consider this initialization to be the most unstable one.

A sharper estimate on the explosion time of a particular initialization is obtained exploiting a maximum principle. Using *sub* and *super* solutions $\underline{u}$ and $\bar{u}$ (with respective explosion times $\underline{t}_*$ and $\bar{t}_*$) such that

$$\underline{u}(x,0) \leqslant u(x,0) \leqslant \bar{u}(x,0), \tag{10}$$

both the solution and the explosion time of a given initial pulse must satisfy

$$\underline{u}(x,t) \leqslant u(x,t) \leqslant \bar{u}(x,t),$$
$$\bar{t}_* \leqslant t_* \leqslant \underline{t}_*. \tag{11}$$

To prove that Eq. (10) implies Eq. (11), we assume that $u(x,t)$ and $v(x,t)$ are both solutions of Eq. (2) with $u_0(x) = u(x,0) \geqslant 0$, $v_0(x) = v(x,0) \geqslant 0$ and $u_0(x) - v_0(x) \geqslant 0$. Denoting by w the difference, $w(x,t) := u(x,t) - v(x,t)$, we have

$$w_t = w_{xx}(u+v) + 2w_x(u+v)_x$$
$$+ w[(u+v)_{xx} + (u+v)]. \tag{12}$$

Denoting $y(t) = \min_x w(x,t)$, and utilizing the fact that a solution of Eq. (2) remains non-negative for non-negative initial data, Eq. (12) yields

$$\dot{y} \geqslant a(t)y, \quad a(t) = [(u+v)_{xx} + (u+v)]|_{x(t),t)},$$
$$y(0) \geqslant 0.$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\exp\left(-\int^t a(s)\,\mathrm{d}s\right)y(t)\right] \geqslant 0,$$

which implies (by the smoothness of the solution), that $\forall t$, $y(t) \geqslant 0$, and the proof is complete.

The quality of these estimates depends to a large extent on how closely the sub and super solutions bound a given datum. For that purpose we can either use the numerically established data for the explosion times of square pulses (see Fig. 6) and/or the explosion time of $A\cos^2(x/4)$, which is $t = 4/3A$.

2.2. *Other values of m*

The same space-time separability emerges for other m's. The emerging shape is obtained via the solution of $(V^m)_{xx} + V^m = V$ with a vanishing flux at the edge, and takes the form

$$V = \alpha\cos^{2/(m-1)}(\beta x).$$

In particular,

$$m = 3, \quad V = \sqrt{3/2}\cos(x/3),$$
$$-3\pi/2 \leqslant x \leqslant 3\pi/2,$$
$$m = 4, \quad V = (8/5)^{1/3}\cos^{2/3}(3x/8),$$
$$-4\pi/3 \leqslant x \leqslant 4\pi/3,$$

and vanishes elsewhere. Note that as m increases, the solution narrows, and for $m > 3$, they become weak solutions.

3. Thermal explosion in higher dimensions

We consider mainly the planar case,

$$u_t = (u^2)_{xx} + (u^2)_{yy} + u^2,$$
$$u(x,y,t=0) = F(x,y), \tag{13}$$

and quote the results for the spherical case.

As in the one-dimensional case, the patterns that emerge asymptotically always have a space-time separable form, and blow up in a finite time (see Figs. 7–9).

The ultimate spatial shape of each of the patches *always appears to be the same axisymmetric pattern*

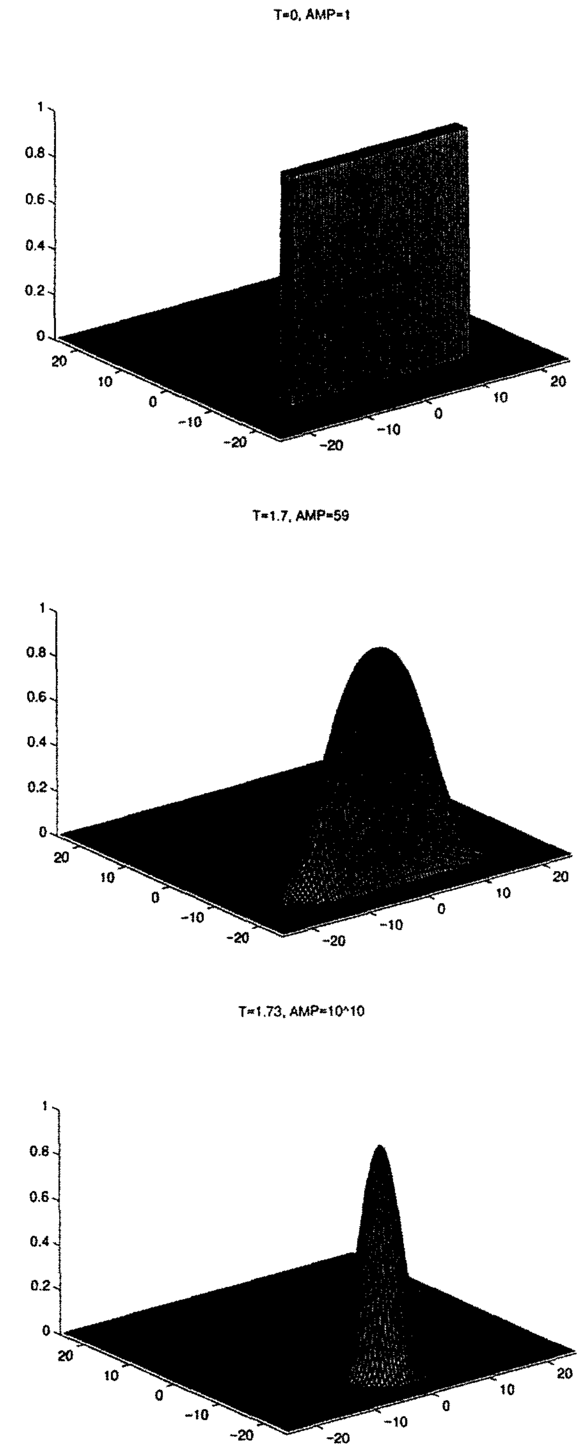

Fig. 7. Emergence of axisymmetric dissipative compactons out of an initial datum specified in a narrow strip.

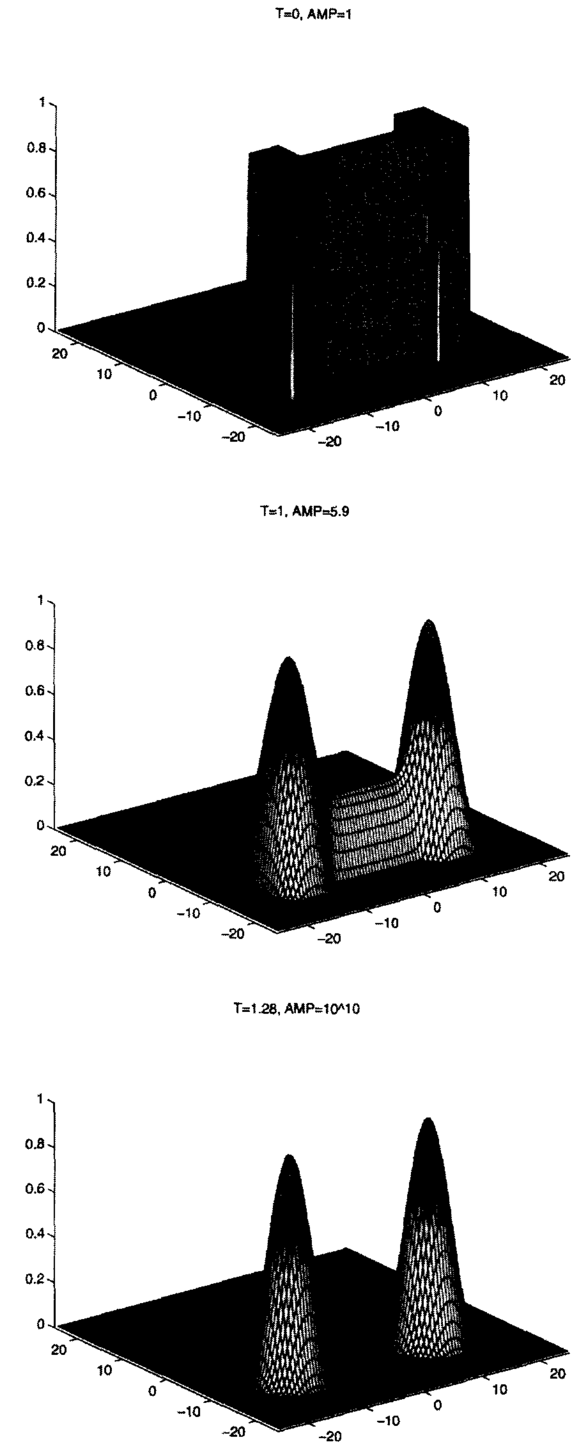

Fig. 8. A change of the initial distribution may affect the number of emerging dissipative compactons and their location.

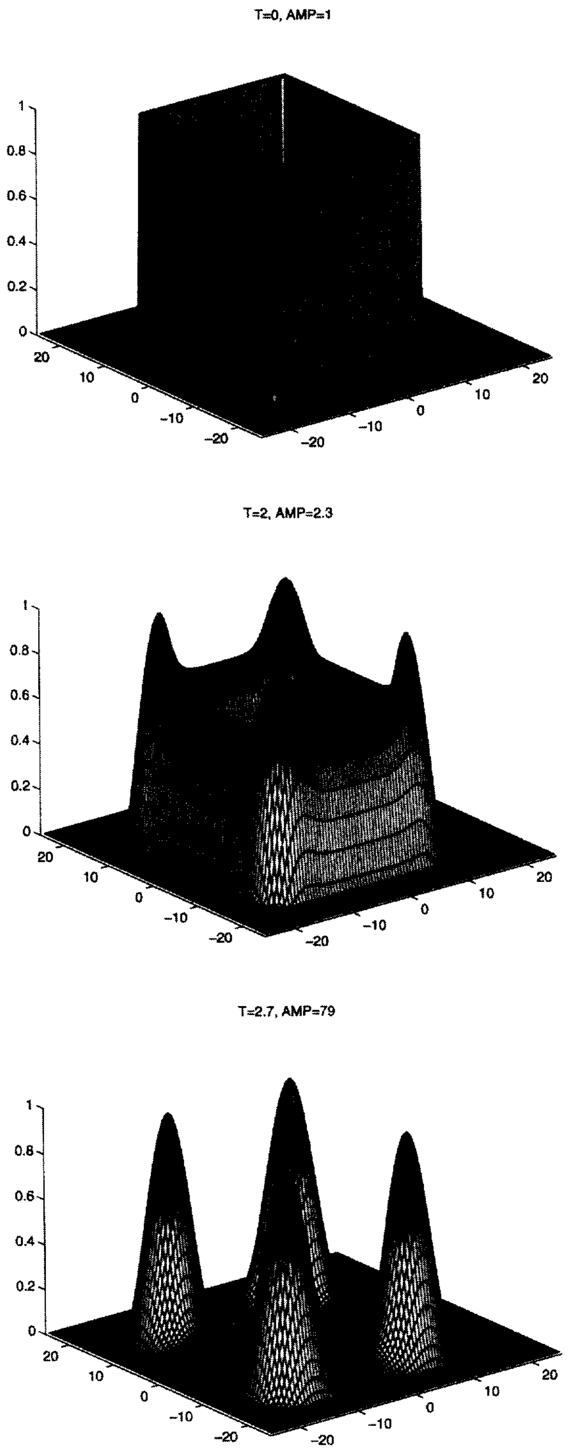

Fig. 9. Thermal evolution for an initial distribution of a square shell.

regardless of the initial distribution. As in 1D, to find this radial compacton, we turn to solve Eq. (13) via a separation of variables, $u = \phi(t)R(r)$. For $\phi(t)$ we have $\phi(t) = 1/(t_* - t)$ while the radial part is found among the solutions of

$$(R^2)_{rr} + \frac{1}{r}(R^2)_r + R^2 - R = 0. \qquad (14)$$

We were not able to solve this equation analytically and had to do it numerically. The diameter of this radial dissipative compacton was numerically estimated to be ≈ 14.1, which is larger than the span of the 1D compacton, 4π.

The location and the number of emerging patches depend on the inhomogeneity of the initial distribution, the characteristic radius of each initial patch, and its distance from its neighbors, see Figs. 7–9 for self-explanatory examples. Similarly to 1D, the conservation laws for Eq. (13) are found via

$$\Delta f(x,y) + f(x,y) = 0. \qquad (15)$$

As before, the resulting conservation laws do not seem to pin the location of the center(s) of the explosion.

Most of our one-dimensional computations can be straightforwardly repeated in the multi-dimensional setup. Eq. (13) is also invariant under Eq. (7), with similar consequences. Analogously to Eq. (9),

$$t_* \geqslant \frac{1}{\max_{x,y} u(x,y,t=0)},$$

is a lower bound on the explosion time, independent of the choice of the initial datum.

Following the 1D arguments (see Eq. (11)), a sharper estimate of the explosion time is obtained via a maximum principle. If initially

$$\underline{u}(x,y,0) \leqslant u(x,y,0) \leqslant \bar{u}(x,y,0), \qquad (16)$$

then both the solution and the explosion time satisfy

$$\underline{u}(x,y,t) \leqslant u(x,y,t) \leqslant \bar{u}(x,y,t)$$
$$\bar{t}_* \leqslant t_* \leqslant \underline{t}_*. \qquad (17)$$

The most remarkable effect, which cannot exist on the line, is the effect of *thermal collapse*. This unique phenomenon is demonstrated in Figs. 10 and 11 for two and three space dimensions. Initially the datum is specified on an axisymmetric annular strip. At the

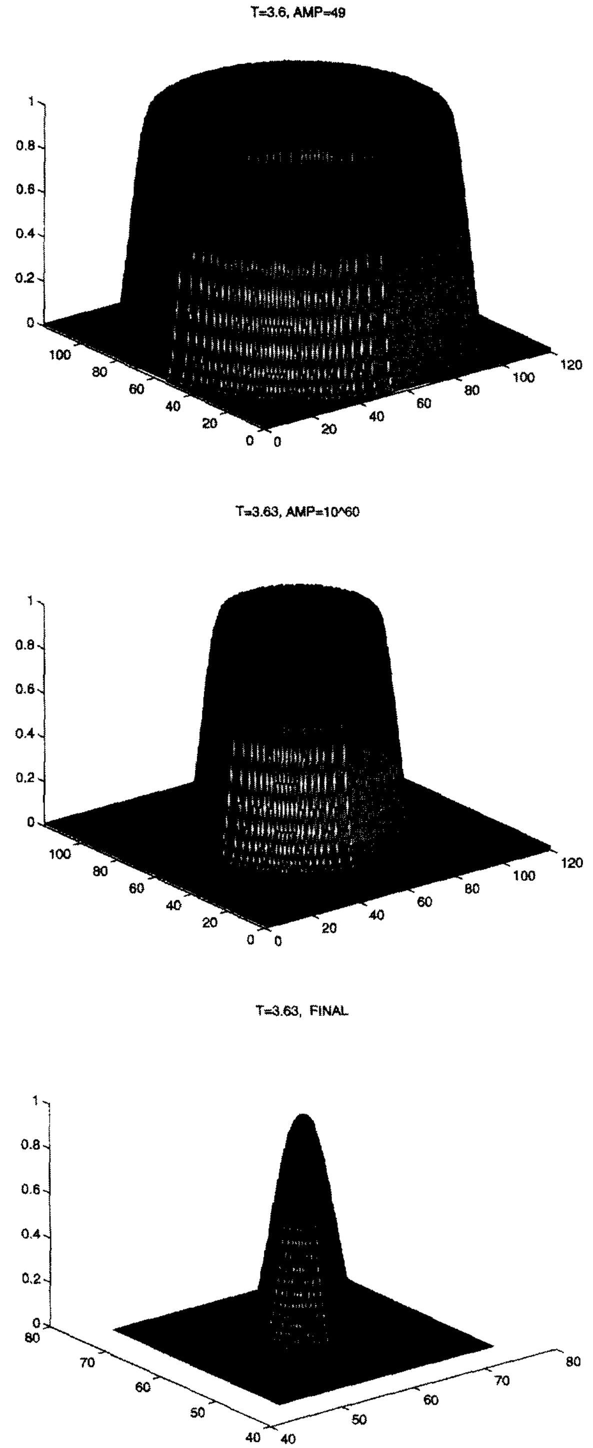

Fig. 10. Collapse of axisymmetric annulus in 2D. Note that in (c) we zoomed in on the domain.

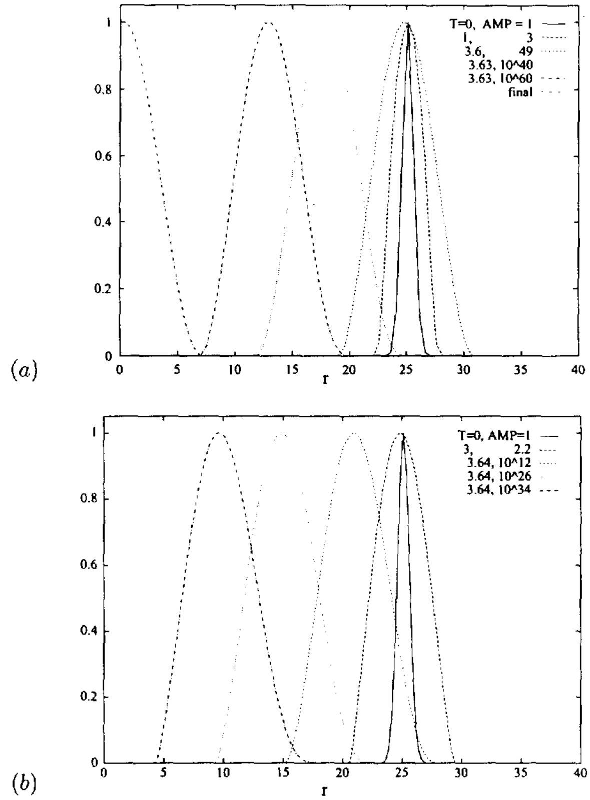

Fig. 11. Evolution and collapse of (a) an axisymmetric annulus (compare with Fig. 10), and (b) a spherical shell (3D). Note that the collapse is almost immediate.

first stage of evolution, the pulse seems to evolve towards the universal shape around the center of the annular strip, and while doing so, it acquires a sizable amplitude. When this process seems to be almost accomplished, it suddenly starts to collapse very quickly towards the center, where it ultimately explodes. At every stage of the evolution, even a slight perturbation of the front from its axisymmetric state (which may be caused by, say, truncation errors, or due to a Cartesian representation of the axisymmetric data) can destroy the front. Indeed, such a perturbation may bring the collapse process to a halt and instead of a collapse, local patch(es) of explosion develop. Consequently, thermal collapse may be considered to be an inherently unstable process.

4. The 2D Dirichlet problem

We have seen that the intrinsic length built into Eq. (1) is responsible for the emergence of dissipative compactons. Imposing boundaries with homogeneous boundary conditions introduces a second, extrinsic length scale into the problem. The ratio between these two scales to a very great extent determines the evolutionary outcome in the present problem.

In Ref. [3], the authors demonstrate that the first eigenvalue, λ_1, of the associated linear problem,

$$\Delta w_j + \lambda_j w_j = 0, \quad j = 1, 2, \ldots,$$

determines whether there is a blow-up ($\lambda_1 < 1$), or an algebraic decay ($\lambda_1 > 1$), without saying much more about the forming pattern. We shall demonstrate that the decay mode assumes a space-time separable form. In larger domains, characterized by $\lambda_1 < 1$, though there is always a blow-up, the plot thickens – for now the intrinsic length enters into the play. On the line, the characterization is simple: If the size of the domain, L, is $\pi < L < 4\pi$, then we observe the emergence of a space-time separable blow-up within the restraints of a homogeneous Dirichlet problem. The spatial distribution is found by solving

$$(V^m)_{xx} + V^m - V = 0, \quad V'(0) = V(L/2) = 0. \quad (18)$$

It is also easy to see that Eq. (18) implies

$$L = 2 \int_0^{V_0^2} \frac{dz}{\sqrt{\frac{4}{3}z^{3/2} - z^2 + V_0^4 - \frac{4}{3}V_0^3}},$$
$$V_0 = V(0). \quad (19)$$

A numerical computation of the integral (19) is shown in Fig. 12. Clearly, the only feasible values of L are those between $\pi < L < 4\pi$. Below π, the solution does not blow up; instead, a decay is observed (for decaying solutions the sign of the last term in Eq. (18) changes). When $V_0 = 4/3$ the integral (19) is elementary, and we obtain a solution identical with Eq. (4). At this limiting value of V_0, the flux on the boundaries vanishes, and hence, in this case, one can remove the boundaries. Thus, effectively, this solution is a dissipative compacton. When $L > 4\pi$, a domain filling Dirichlet solution is no longer possible. A self-localized solution confined to 4π emerges, pretty much ignoring the boundary conditions. This self-localized solution has a space-time separable structure and is thus recognized as a dissipative compacton. When $L \geqslant 8\pi$, under favorable initial conditions two or more compactons may form.

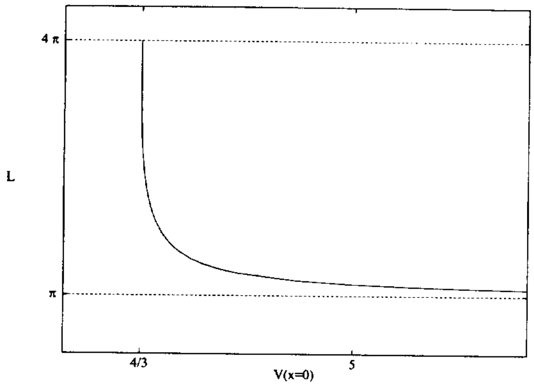

Fig. 12. The span, L, of the Dirichlet solution as a function of the peak value at the center. Note that L is confined to $\pi < L < 4\pi$.

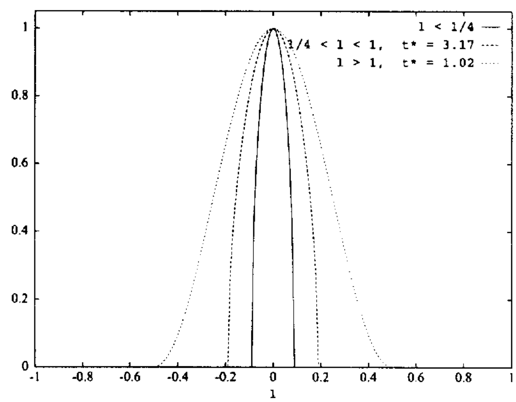

Fig. 13. The universal spatial profiles of the homogeneous Dirichlet problem. The widest pulse represents the self-localized pattern (which develops if $l = L/4\pi > 1$) that is detached from the boundaries. It is followed by a Dirichlet blow-up ($\pi < L < 4\pi$) and then by a decaying pattern ($L < \pi$). In the last two cases the profiles span the whole domain.

Fig. 13 displays three examples of solutions of the Dirichlet problem for a symmetric initial datum, centered around 0. The inner solution describes the spatial distribution of the decaying solution of the Dirichlet problem. The second profile corresponds to domain filling, an exploding solution of the Dirichlet problem. In both cases, a finite flux is deposited at the boundaries. The widest profile represents a self-localized pattern. If, however, like in Fig. 5, the initial datum

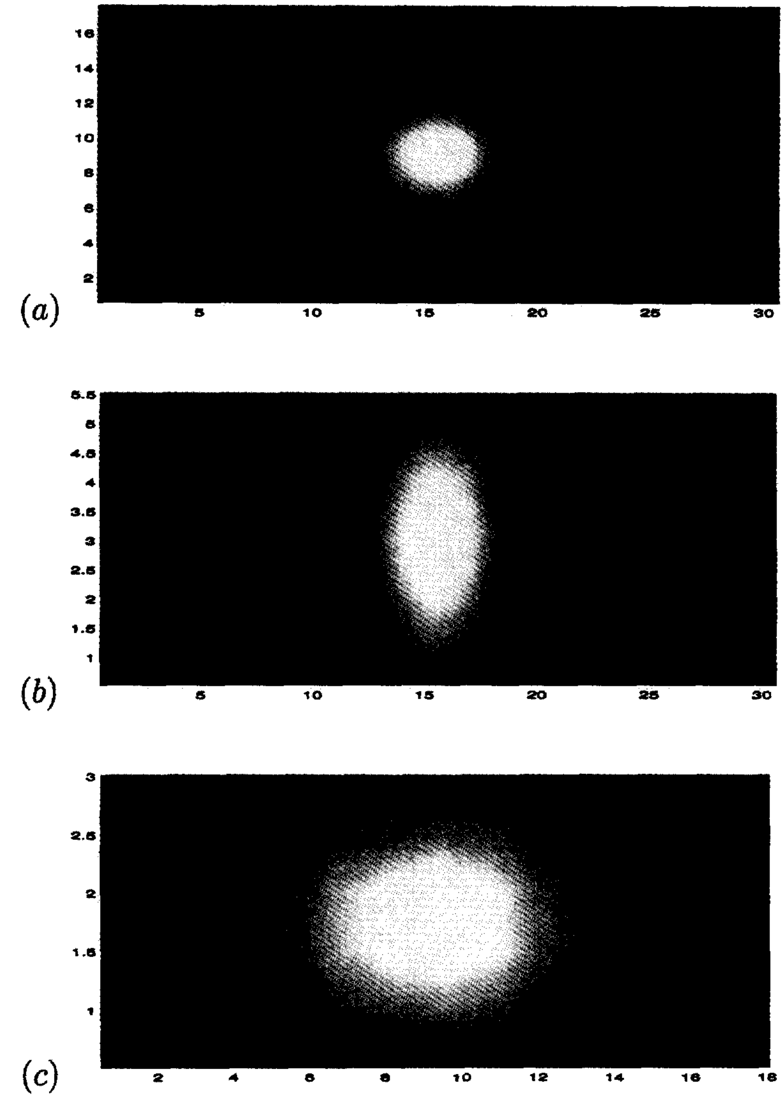

Fig. 14. Example of 2D blow-up with Dirichlet boundary conditions for different sizes of the domain. The initial datum is assumed to be 1 inside the domain and 0 on the boundary. (a) Fully localized pattern, (b) semi-localized pattern, (c) a genuine Dirichlet blow-up. Here the pattern fills the domain.

has an asymmetric distribution, the emerging pattern may no longer be symmetrically located, and its emergence occurs at exceedingly high amplitudes.

Before turning to higher dimensions, we note that the space-time separable solutions since some time ago are known to be attractors [9] for the purely diffusive, initial-boundary value problem with homogeneous Dirichlet boundary conditions.

In higher dimensions, the interaction between the two scales becomes even more interesting, for now new possibilities arise. For one, we may obtain a new mode of semi-localization. In this mode, in one direction we have a bona fide Dirichlet problem, while in the other, there is a self-localization. This is a hybrid mode which occurs in long and narrow strips. It is also noteworthy that on the line there is unique topological order with respect to the length of the domain (localization on long intervals, blow-up with heat loss on the boundaries for intermediate lengths, followed by a decay when $L < \pi$).

In following the emerging patterns in Fig. 14, it helps to note that for rectangular domains (L being the length and d the width), the critical dimensions ($\lambda = 1$) for the onset of decay are given via

$$\sqrt{\left(\frac{\pi}{L}\right)^2 + \left(\frac{\pi}{d}\right)^2} = 1, \tag{20}$$

which makes it clear that in thin domains with $d < \pi$, only a decaying mode can be observed. Elongated domains will have a critical width slightly above the minimal width of π. Similarly, since for a square the critical length is $(2\pi)^{1/2}$, any planar domain into which such a square can be inserted will undergo thermal blow-up. If also a circle of diameter 14.1 can be fitted into this domain, localization of the blow-up will be observed as well.

Fig. 14a shows that a fully localized, axisymmetric pattern is seen to emerge. If we now keep the length fixed but reduce the width, a semi-localized pattern emerges. As we keep reducing the width, the shape of the patterns becomes more and more elongated, but insofar as blow-up occurs, the pattern remains semi-localized (see Fig. 14b). At the critical width at which the pattern fills the domain, *the resulting Dirichlet problem describes a decay* (not shown here). In other words, in this geometrical setup, we bypass the Dirichlet blow-up.

If, on the other hand, the initial rectangle has the same width but a shorter length, say, 18, as we decrease the width we observe, as before, localized states followed by semi-localized ones. We can now pinpoint the difference between these two cases; for the same width for which the previous case provided a semi-localized pattern, in the present case there is a Dirichlet blow-up (see Fig. 14c). Note that due to numerical roundoff, the actual width in case (c) is closer to 3.5 than to 3, the value given in Fig. 14c. With a further decrease in width, Dirichlet decay (not shown) emerges. Thus in 2D, three modes of blow-up are possible, all having in common space-time separable attractors. One remark regarding the numerics is in order: Though for the first example in Fig. 14 we did not find a Dirichlet blow-up, it is still possible that close to the critical width there exists a "thin boundary layer" of widths where such a phenomenon may occur. This calls for a far more detailed study than intended within the scope of the present Letter.

As a final remark in this section we note that our conclusions are based on numerical experiments in simple planar domains. Thus, in higher dimensions a greater variety of partial localizations may be expected to emerge. For instance, while in the plane we did not observe decaying semi-localized states, they should not be precluded in higher dimensions. Similarly, in more complex geometries a larger variety of patterns may emerge.

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