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A stable semi-discrete central scheme for the two-dimensional incompressible Euler equations

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We derive a second-order, semi-discrete central-upwind scheme for the incompressible 2D Euler equations in the vorticity formulation. The reconstructed velocity field preserves an exact discrete incompressibility relation. We state a local maximum principle for a fully discrete version of the scheme and prove it using a convexity argument. We then show how similar convexity arguments can be used to prove that the scheme maps certain Orlicz spaces into themselves. The consequences of this result on the convergence of the scheme are discussed. Numerical simulations support the expected properties of the scheme.

Keywords: incompressible Euler equations; central schemes; maximum principle.

1. Introduction

We are interested in approximating solutions of the incompressible 2D Euler equations in the vorticity formulation

$$\omega_t + (u\omega)_x + (v\omega)_y = 0. \tag{1.1}$$

Here, ω denotes the vorticity and (u, v) denotes the velocity field. The incompressibility condition, $u_x + v_y = 0$, implies that (1.1) can be rewritten in a convective form

$$\omega_t + u\omega_x + v\omega_y = 0. \tag{1.2}$$

While the velocity and the vorticity are connected through a global relation (e.g. via the Biot–Savart law), the convective form (1.2) shows that the vorticity propagates with a finite speed as long as the velocity remains bounded.

The existence of a finite speed of propagation and the conservative form (1.1) motivate us to use schemes that were originally developed for hyperbolic conservation laws for approximating solutions of (1.1). In this work we focus on high-order Godunov-type central schemes of which the prototypes are the first-order Lax–Friedrichs (LxF) scheme (Friedrichs & Lax, 1971) and the second-order Nessyahu–Tadmor scheme (Nessyahu & Tadmor, 1990). The main advantage of central schemes is that they avoid Riemann problems (by averaging over the Riemann fans) which makes them particularly suitable for multi-dimensional problems and complicated geometries. In order to reduce the somewhat excessive numerical dissipation of the LxF scheme, various high-order methods have been developed in the past years (see Tadmor, 1998, and the references therein).

In a previous work (Levy & Tadmor, 1997) we derived a second-order fully discrete approximation of (1.1). Our method was based on the 2D scheme of Jiang & Tadmor (1998) with a particular choice of a velocity reconstruction that preserves an exact discrete incompressibility relation. This property then allowed us to state and prove a local maximum principle on our scheme. A related work (Kupferman &

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Tadmor, 1997) dealt with fully discrete central schemes for the Euler and the Navier–Stokes equations in the velocity formulation.

In this work we present a second-order semi-discrete scheme that is based on the semi-discrete scheme of Kurganov and Tadmor for hyperbolic conservation laws (Kurganov & Tadmor, 2000). Due to the structure of the scheme, the velocity reconstruction of Levy & Tadmor (1997) has to be replaced by a different velocity reconstruction that preserves an exact discrete incompressibility relation (one that suits the new scheme). We state and prove a local maximum principle for a fully discrete version of the scheme. This is done in two stages. First, a theorem is proved for a forward Euler discretization of the time derivative. This result is then extended to second- and third-order Runge–Kutta methods, at least those that can be written as a convex combination of first-order forward Euler methods (following Gottlieb *et al.*, 2001; Shu, 1988; Shu & Osher, 1988). A similar result is stated for certain multi-level Runge–Kutta schemes.

Following the work of Lopes Filho *et al.* (2000), we show that the convexity argument that is used to prove the maximum principle also implies that the evolution mapping for our scheme, $\omega(\cdot, 0) \mapsto \omega(\cdot, t^n)$, maps any Orlicz space into itself, i.e. the method is H^{-1}_{loc}-stable. In view of Lopes Filho *et al.* (2000), this result can serve as a step in proving the convergence of the scheme, at least for its first-order version.

We note in passing that certain variations in this scheme can be found in several places: a third-order, semi-discrete central scheme for the Euler and the Navier–Stokes equations was derived in Kurganov & Levy (2000). No theoretical results were provided there. Another related work is on the vorticity–streamfunction formulation of the incompressible 2D Navier–Stokes equations (Kupferman, 2001). There, the central scheme of Kurganov & Tadmor (2000) was used for the discretization of the convective term only, the velocity reconstruction was based on a misinterpretation of the results of Levy & Tadmor (1997) and the overall scheme was implicit. More details will be provided in Section 3.

This paper is organized as follows: In Section 2 we briefly review the derivation of semi-discrete central schemes for 2D hyperbolic conservation laws. Our scheme for the 2D incompressible Euler equations is then presented in Section 3. In Section 4 we state and prove a local maximum principle for our scheme. Convergence issues are discussed in Section 5. We conclude Section 6 with numerical simulations that demonstrate the desired properties of the scheme.

2. Semi-discrete central schemes for conservation laws

We consider the 2D conservation law

$$w_t + f(w)_x + g(w)_y = 0, \tag{2.1}$$

subject to the initial data, $w(x, y, 0) = w_0(x, y)$. We are interested in approximating solutions of (2.1) that are computed in terms of cell-averages on a computational grid with cells $I_{j,k}$ of a fixed dimension $\Delta x \times \Delta y$ that are centered around $(x_j, y_k) = j\Delta x, k\Delta y$.

Following Kurganov & Tadmor (2000), we start with known cell-averages on $I_{j,k}$ at time t^n, $\bar{w}^n_{j,k}$, which are then used to reconstruct a non-oscillatory piecewise-linear function of the form

$$\tilde{w}(x, y, t^n) = \sum_{j,k} \left[\bar{w}^n_{j,k} + (w^n_x)_{j,k}(x - x_j) + (w^n_y)_{j,k}(y - y_k) \right] \chi_{j,k}(x, y). \tag{2.2}$$

Here, $\chi_{j,k}(x, y)$ is the characteristic function of the cell $I_{j,k}$ and $(w^n_x)_{j,k}$, $(w^n_y)_{j,k}$ are approximations of the derivatives in the x- and y-directions, respectively. In order to avoid spurious oscillations, these derivatives should be estimated using non-linear limiters (Sweby, 1984). One possible option is the MinMod limiter: for $1 \leqslant \theta \leqslant 2$, set

$$
\begin{aligned}
(w_x^n)_{j,k} &= \mathcal{MM}\left(\theta \frac{\bar{w}^n_{j,k} - \bar{w}^n_{j-1,k}}{\Delta x}, \frac{\bar{w}^n_{j+1,k} - \bar{w}^n_{j-1,k}}{2\Delta x}, \theta \frac{\bar{w}^n_{j+1,k} - \bar{w}^n_{j,k}}{\Delta x}\right), \\
(w_y^n)_{j,k} &= \mathcal{MM}\left(\theta \frac{\bar{w}^n_{j,k} - \bar{w}^n_{j,k-1}}{\Delta y}, \frac{\bar{w}^n_{j,k+1} - \bar{w}^n_{j,k-1}}{2\Delta y}, \theta \frac{\bar{w}^n_{j,k+1} - \bar{w}^n_{j,k}}{\Delta y}\right),
\end{aligned}
\tag{2.3}
$$

where the MinMod limiter, $\mathcal{MM}$, is given by

$$
\mathcal{MM}(x_1, x_2, \ldots, x_n) = \begin{cases} \min(x_i), & x_i \geqslant 0, \ \forall\, i, \\ \max(x_i), & x_i \leqslant 0, \ \forall\, i, \\ 0, & \text{otherwise}. \end{cases} \tag{2.4}
$$

The reconstruction (2.2) can be used to obtain the point-values at the interfaces, i.e.

$$
\begin{aligned}
w^+_{j+\frac{1}{2},k} &= \bar{w}^n_{j+1,k} - \frac{\Delta x}{2}(w^n_x)_{j+1,k}, \quad w^-_{j+\frac{1}{2},k} = \bar{w}^n_{j,k} + \frac{\Delta x}{2}(w^n_x)_{j,k}, \\
w^+_{j,k+\frac{1}{2}} &= \bar{w}^n_{j,k+1} - \frac{\Delta y}{2}(w^n_y)_{j,k+1}, \quad w^-_{j,k+\frac{1}{2}} = \bar{w}^n_{j,k} + \frac{\Delta y}{2}(w^n_y)_{j,k}.
\end{aligned}
\tag{2.5}
$$

The values in (2.5) can now be used to estimate the maximum local speeds of propagation of information from the interfaces. These speeds can be estimated as

$$
a^x_{j+\frac{1}{2},k} = \max_{w \in \mathcal{C}\left(w^-_{j+\frac{1}{2},k}, w^+_{j+\frac{1}{2},k}\right)} \rho\left(\frac{\partial f}{\partial w}(w)\right), \qquad a^y_{j,k+\frac{1}{2}} = \max_{w \in \mathcal{C}\left(w^-_{j,k+\frac{1}{2}}, w^+_{j,k+\frac{1}{2}}\right)} \rho\left(\frac{\partial g}{\partial w}(w)\right). \tag{2.6}
$$

Here, $\rho(\cdot)$ denotes the spectral radius and $\mathcal{C}(a, b)$ is the curve in the phase space connecting the states a and b via the Riemann fan. We now use these local speeds of propagation to cover the x–y plane in a way that is portrayed in Fig. 1. We distinguish between three types of cells: interior, edge and corner cells. In reality, due to the possibly different speeds from different reconstructions around the corners, the corner cells may overlap the other cells (unlike Fig. 1). This will make no difference in the semi-discrete limit since the area of the corner cells is proportional to $(\Delta t)^2$ and hence their contribution will vanish in the limit $\Delta t \to 0$.

Similar to other Godunov-type schemes, the reconstruction (2.2) can now be evolved in time in each rectangle and projected on cell-averages in these rectangles. The result is then averaged back to the original mesh using another non-oscillatory reconstruction. In such a way we get the cell-averages at the next time-step, $\bar{w}^{n+1}_{j,k}$. A semi-discrete scheme is then obtained in the limit as $\Delta t \to 0$, i.e.

$$
\frac{\mathrm{d}}{\mathrm{d}t}\bar{w}_{j,k} = \lim_{\Delta t \to 0} \frac{\bar{w}^{n+1}_{j,k} - \bar{w}^n_{j,k}}{\Delta t} = \cdots
$$

We skip the details (these can be found in Kurganov & Tadmor, 2000). The resulting 2D, semi-discrete, central scheme is given by

$$
\frac{\mathrm{d}}{\mathrm{d}t}\bar{w}_{j,k} = -\frac{H^x_{j+\frac{1}{2},k} - H^x_{j-\frac{1}{2},k}}{\Delta x} - \frac{H^y_{j,k+\frac{1}{2}} - H^y_{j,k-\frac{1}{2}}}{\Delta y}. \tag{2.7}
$$

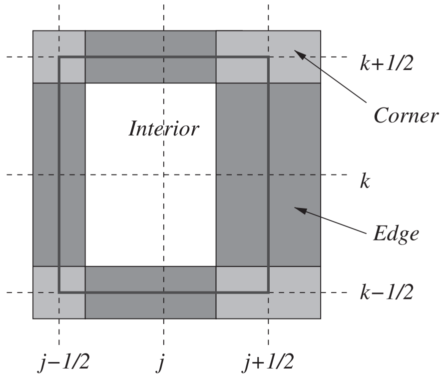

FIG. 1. Covering the x–y plane.

Here, we use the standard notations for the fixed mesh ratios, i.e. $\lambda = \Delta t/\Delta x$ and $\mu = \Delta t/\Delta y$. The numerical fluxes in the x- and y-directions are given as

$$\begin{aligned} H^x_{j+\frac{1}{2},k} &= \frac{f\left(w^+_{j+\frac{1}{2},k}\right) + f\left(w^-_{j+\frac{1}{2},k}\right)}{2} - \frac{a^x_{j+\frac{1}{2},k}}{2}\left[w^+_{j+\frac{1}{2},k} - w^-_{j+\frac{1}{2},k}\right], \\ H^y_{j,k+\frac{1}{2}} &= \frac{g\left(w^+_{j,k+\frac{1}{2}}\right) + g\left(w^-_{j,k+\frac{1}{2}}\right)}{2} - \frac{a^y_{j,k+\frac{1}{2}}}{2}\left[w^+_{j,k+\frac{1}{2}} - w^-_{j,k+\frac{1}{2}}\right]. \end{aligned} \tag{2.8}$$

The local speeds of propagation, $a^x_{j+\frac{1}{2},k}$, $a^y_{j,k+\frac{1}{2}}$, are given by (2.6), and the point-values, $w^\pm_{j+\frac{1}{2},k}$, $w^\pm_{j,k+\frac{1}{2}}$, are given by (2.5).

REMARKS

1. A similar scheme with a reduced dissipation was derived in Kurganov *et al.* (2001). This was done by collecting somewhat more accurate information regarding the local speeds of propagation, i.e. instead of assuming identical local speeds of propagation to the right and to the left (in the x- and y-coordinates), it is possible to provide different bounds on the maximum possible speed of propagation in each direction which is precisely what was done in Kurganov *et al.* (2001).
2. The scheme (2.7)–(2.8) together with the MinMod approximation of the derivatives (2.3) is second-order accurate in space. The accuracy in time is determined by the accuracy of the ODE solver. The order of accuracy of the method can be easily increased by using a more accurate reconstruction and a high-order ODE solver. Examples for such schemes can be found in Kurganov & Levy (2000) and Kurganov & Petrova (2001).
3. The main motivation for writing schemes in a semi-discrete form was in order to have a convenient treatment for dissipative terms. Since this will become handy when we comment later on the Navier–Stokes equations, we briefly comment on a possible discretization of such terms. Starting from a convection–diffusion equation of the form

$$w_t + f(w)_x + g(w)_y = Q^x(w, w_x, w_y)_x + Q^y(w, w_x, w_y)_y, \tag{2.9}$$

an approximation of (2.9) can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{\omega}_{j,k} = -\frac{H^x_{j+\frac{1}{2},k} - H^x_{j-\frac{1}{2},k}}{\Delta x} - \frac{H^y_{j,k+\frac{1}{2}} - H^y_{j,k-\frac{1}{2}}}{\Delta y} + \frac{P^x_{j+\frac{1}{2},k} - P^x_{j-\frac{1}{2},k}}{\Delta x} + \frac{P^y_{j,k+\frac{1}{2}} - P^y_{j,k-\frac{1}{2}}}{\Delta y}.$$

The convective fluxes, H^x, H^y, are given by (2.8), while the diffusion fluxes, P^x, P^y, can be obtained with a simple centered differencing:

$$P^x_{j+\frac{1}{2},k} = \frac{1}{2}\left[Q^x\!\left(\bar{\omega}_{j,k}, \frac{\bar{\omega}_{j+1,k} - \bar{\omega}_{j,k}}{\Delta x}, (\omega^n_y)_{j,k}\right) + Q^x\!\left(\bar{\omega}_{j+1,k}, \frac{\bar{\omega}_{j+1,k} - \bar{\omega}_{j,k}}{\Delta x}, (\omega^n_y)_{j+1,k}\right)\right],$$
$$P^y_{j,k+\frac{1}{2}} = \frac{1}{2}\left[Q^y\!\left(\bar{\omega}_{j,k}, (\omega^n_x)_{j,k}, \frac{\bar{\omega}_{j,k+1} - \bar{\omega}_{j,k}}{\Delta y}\right) + Q^y\!\left(\bar{\omega}_{j,k+1}, (\omega^n_x)_{j,k+1}, \frac{\bar{\omega}_{j,k+1} - \bar{\omega}_{j,k}}{\Delta y}\right)\right].$$

3. A scheme for the incompressible Euler equations

In this section we present a semi-discrete central scheme for the 2D incompressible Euler equations. This is done by making the necessary adjustments to the scheme (2.7)–(2.8). We start with the conservative formulation of the vorticity equation

$$\omega_t + (u\omega)_x + (v\omega)_y = 0. \tag{3.1}$$

The incompressibility condition, $u_x + v_y = 0$, allows us to rewrite (3.1) in the convective form

$$\omega_t + u\omega_x + v\omega_y = 0. \tag{3.2}$$

As long as the velocity field (u, v) remains bounded, (3.2) implies a finite speed of propagation. We approximate solutions of (3.1) as follows. Given cell-averages of the vorticity at time t^n, $\bar{\omega}^n_{j,k}$, we first reconstruct a non-oscillatory piecewise-linear interpolant (compare with (2.2))

$$\omega(x, y, t^n) = \sum_{j,k}\left[\bar{\omega}^n_{j,k} + (\omega^n_x)_{j,k}(x - x_j) + (\omega^n_y)_{j,k}(y - y_k)\right]\chi_{j,k}(x, y). \tag{3.3}$$

The discrete derivatives of the vorticity, $(\omega^n_x)_{j,k}$ and $(\omega^n_y)_{j,k}$, are computed with the MinMod limiter (2.3). The scheme (2.7)–(2.8) when written for the time evolution of the cell-averages of the vorticity reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{\omega}_{j,k} = -\frac{H^x_{j+\frac{1}{2},k} - H^x_{j-\frac{1}{2},k}}{\Delta x} - \frac{H^y_{j,k+\frac{1}{2}} - H^y_{j,k-\frac{1}{2}}}{\Delta y}, \tag{3.4}$$

with numerical fluxes that are given as

$$\begin{aligned} H^x_{j+\frac{1}{2},k} &= \frac{u_{j+\frac{1}{2},k}\left(\omega^+_{j+\frac{1}{2},k} + \omega^-_{j+\frac{1}{2},k}\right)}{2} - \frac{a^x_{j+\frac{1}{2},k}}{2}\left[\omega^+_{j+\frac{1}{2},k} - \omega^-_{j+\frac{1}{2},k}\right], \\ H^y_{j,k+\frac{1}{2}} &= \frac{v_{j,k+\frac{1}{2}}\left(\omega^+_{j,k+\frac{1}{2}} + \omega^-_{j,k+\frac{1}{2}}\right)}{2} - \frac{a^y_{j,k+\frac{1}{2}}}{2}\left[\omega^+_{j,k+\frac{1}{2}} - \omega^-_{j,k+\frac{1}{2}}\right]. \end{aligned} \tag{3.5}$$

The point-values of the vorticity, $\omega^\pm_{j+\frac{1}{2},k}$, $\omega^\pm_{j,k+\frac{1}{2}}$, are computed from the reconstruction (3.3). As long as it is valid to switch to the convective formulation of the vorticity equation (3.2), we can set the local

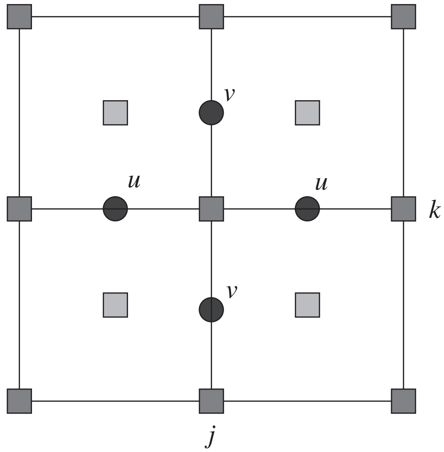

FIG. 2. The reconstruction of the velocity field.

speeds of propagation as $a^x_{j+\frac{1}{2},k} = |u_{j+\frac{1}{2},k}|$, $a^y_{j,k+\frac{1}{2}} = |v_{j,k+\frac{1}{2}}|$. All that remains is to obtain the values of the velocity field on the interfaces, i.e. $u_{j+\frac{1}{2},k}$, $v_{j,k+\frac{1}{2}}$ (consult Fig. 2).

One possible way of computing these velocities is the following: first, obtain the values of the streamfunction at the integer grid-points by solving the 5-point Laplacian

$$\Delta \psi_{jk} = -\bar{\omega}^n_{jk}. \tag{3.6}$$

The point-values of the streamfunction can be then used to define the velocity field as

$$\begin{aligned} u_{j+\frac{1}{2},k} &= \frac{1}{2\Delta y}\left(\frac{\psi_{j,k+1}+\psi_{j+1,k+1}}{2} - \frac{\psi_{j,k-1}+\psi_{j+1,k-1}}{2}\right), \\ v_{j,k+\frac{1}{2}} &= \frac{1}{2\Delta x}\left(-\frac{\psi_{j+1,k}+\psi_{j+1,k+1}}{2} + \frac{\psi_{j-1,k}+\psi_{j-1,k+1}}{2}\right). \end{aligned} \tag{3.7}$$

Note that the velocities defined in (3.7) satisfy the discrete incompressibility relation

$$\frac{u_{j+\frac{1}{2},k} - u_{j-\frac{1}{2},k}}{\Delta x} + \frac{v_{j,k+\frac{1}{2}} - v_{j,k-\frac{1}{2}}}{\Delta y} = 0. \tag{3.8}$$

REMARKS

1. To demonstrate the importance of the discrete incompressibility condition (3.8), we assume a constant vorticity, $\bar{\omega}^n = \text{Const}$. Then a necessary consistency condition for $\bar{\omega}^{n+1}$ to retain the same constant value is precisely (3.8).
2. Following the scheme for convection–diffusion equations of the form (2.9), it is straightforward to extend this scheme to the Navier–Stokes equations. This was done in the third-order case in Kurganov & Levy (2000). Another second-order scheme for the Navier–Stokes equations in a vorticity–streamfunction formulation was proposed in Kupferman (2001). There, however, the

reconstruction of the velocities was based on a misinterpretation of the results of Levy & Tadmor (1997). The procedure that was proposed in Kupferman (2001, p. 9) for the reconstruction of the velocity suggested defining the velocities as an "average of the velocities" on the integer grid-point, i.e.

$$u_{j-\frac{1}{2},k} = \frac{1}{2}(u_{j,k} + u_{j-1,k}). \tag{3.9}$$

Contrary to what was written in Kupferman (2001), the velocity reconstruction of Levy & Tadmor (1997) made use of (3.9) to define the velocities on the integer grid-points which are exactly where they were needed for the fully discrete scheme, and not the other way around.

4. A maximum principle

In this section we state a local maximum principle for a fully discrete version of the scheme from Section 3. First, we state and prove our result for a forward Euler discretization of the time derivative. An identical result for certain second- and third-order Runge–Kutta methods immediately follows.

THEOREM 4.1 Consider the scheme

$$\bar{\omega}^{n+1}_{j,k} = \bar{\omega}^{n}_{j,k} - \lambda\left(H^x_{j+\frac{1}{2},k} - H^x_{j-\frac{1}{2},k}\right) - \mu\left(H^y_{j,k+\frac{1}{2}} - H^y_{j,k-\frac{1}{2}}\right), \tag{4.1}$$

where $\lambda = \Delta t/\Delta x$ and $\mu = \Delta t/\Delta y$ are the fixed mesh ratios, and the numerical fluxes are given by

$$\begin{aligned} H^x_{j+\frac{1}{2},k} &= \frac{1}{2}\left[\left(\omega^+_{j+\frac{1}{2},k} + \omega^-_{j+\frac{1}{2},k}\right)u_{j+\frac{1}{2},k} - \left|u_{j+\frac{1}{2},k}\right|\left(\omega^+_{j+\frac{1}{2},k} - \omega^-_{j+\frac{1}{2},k}\right)\right],\\ H^y_{j,k+\frac{1}{2}} &= \frac{1}{2}\left[\left(\omega^+_{j,k+\frac{1}{2}} + \omega^-_{j,k+\frac{1}{2}}\right)v_{j,k+\frac{1}{2}} - \left|v_{j,k+\frac{1}{2}}\right|\left(\omega^+_{j,k+\frac{1}{2}} - \omega^-_{j,k+\frac{1}{2}}\right)\right]. \end{aligned} \tag{4.2}$$

We assume that the point-values of the vorticity on the interfaces, $\omega^{\pm}_{j+\frac{1}{2},k}, \omega^{\pm}_{j,k+\frac{1}{2}}$, are given by (3.3), with a MinMod limiter approximation of the derivative (2.3). We also assume that the discrete velocities $\{u_{j+\frac{1}{2},k}\}, \{v_{j,k+\frac{1}{2}}\}$, satisfy (3.8). Finally, assume the Courant-Friedrichs-Lewy (CFL) condition

$$\max\left(\lambda \max_{j,k}\left|u_{j+\frac{1}{2},k}\right|, \mu \max_{j,k}\left|v_{j,k+\frac{1}{2}}\right|\right) \leqslant \frac{1}{4}. \tag{4.3}$$

Then

$$\min_{\substack{-1\leqslant l\leqslant 1, -1\leqslant m\leqslant 1\\ |l|+|m|\leqslant 1}} \bar{\omega}^n_{j+l,k+m} \leqslant \bar{\omega}^{n+1}_{j,k} \leqslant \max_{\substack{-1\leqslant l\leqslant 1, -1\leqslant m\leqslant 1\\ |l|+|m|\leqslant 1}} \bar{\omega}^n_{j+l,k+m}. \tag{4.4}$$

Proof. The idea is to rewrite the scheme as a convex combination of the participating point-values on the interfaces which will then provide the desired result in view of the non-linear limiting that is part of the reconstruction stage. Clearly, the piecewise-linear reconstruction implies that

$$\bar{\omega}^n_{j,k} = \frac{1}{4}\left(\omega^-_{j+\frac{1}{2},k} + \omega^+_{j-\frac{1}{2},k} + \omega^-_{j,k+\frac{1}{2}} + \omega^+_{j,k-\frac{1}{2}}\right).$$

Hence, the scheme (4.1)–(4.2) can be rewritten as

$$
\begin{aligned}
\bar{\omega}_{j,k}^{n+1} = & \left[\frac{1}{4} - \frac{\lambda}{2}\left(u_{j+\frac{1}{2},k} + \left|u_{j+\frac{1}{2},k}\right|\right)\right]\omega^-_{j+\frac{1}{2},k} + \frac{\lambda}{2}\left(\left|u_{j+\frac{1}{2},k}\right| - u_{j+\frac{1}{2},k}\right)\omega^+_{j+\frac{1}{2},k} \\
& + \left[\frac{1}{4} - \frac{\lambda}{2}\left(\left|u_{j-\frac{1}{2},k}\right| - u_{j-\frac{1}{2},k}\right)\right]\omega^+_{j-\frac{1}{2},k} + \frac{\lambda}{2}\left(\left|u_{j-\frac{1}{2},k}\right| + u_{j-\frac{1}{2},k}\right)\omega^-_{j-\frac{1}{2},k} \\
& + \left[\frac{1}{4} - \frac{\mu}{2}\left(v_{j,k+\frac{1}{2}} + \left|v_{j,k+\frac{1}{2}}\right|\right)\right]\omega^-_{j,k+\frac{1}{2}} + \frac{\mu}{2}\left(\left|v_{j,k+\frac{1}{2}}\right| - v_{j,k+\frac{1}{2}}\right)\omega^+_{j,k+\frac{1}{2}} \\
& + \left[\frac{1}{4} - \frac{\mu}{2}\left(\left|v_{j,k-\frac{1}{2}}\right| - v_{j,k-\frac{1}{2}}\right)\right]\omega^+_{j,k-\frac{1}{2}} + \frac{\mu}{2}\left(\left|v_{j,k-\frac{1}{2}}\right| + v_{j,k-\frac{1}{2}}\right)\omega^-_{j,k-\frac{1}{2}}.
\end{aligned}
\tag{4.5}
$$

The coefficients of $\omega^\pm_{j\pm\frac{1}{2},k}, \omega^\pm_{j\mp\frac{1}{2},k}, \omega^\pm_{j,k\pm\frac{1}{2}}, \omega^\pm_{j,k\mp\frac{1}{2}}$ in (4.5) are positive under the CFL condition (4.3). In addition, due to the discrete incompressibility relation (3.8), these coefficients sum up to 1. Hence, $\bar{\omega}^{n+1}_{j,k}$ in (4.5) is expressed as a convex combination of $\omega^\pm_{j\pm\frac{1}{2},k}, \omega^\pm_{j\mp\frac{1}{2},k}, \omega^\pm_{j,k\pm\frac{1}{2}}, \omega^\pm_{j,k\mp\frac{1}{2}}$. Due to the properties of the MinMod limiter, the interface values $\omega^\pm$ lie in the convex hull of the neighbouring cell-averages, and this concludes the proof. □

REMARKS

1. Obtaining a maximum principle on the scheme by writing the new values at time t^{n+1} as a convex combination of the values at the previous time-steps is a method that was used in related works (see Jiang & Tadmor, 1998, for a fully discrete central scheme for 2D hyperbolic conservation laws, Levy & Tadmor, 1997, for a fully discrete central scheme for the 2D incompressible Euler equations and Kurganov & Tadmor, 2000, for a semi-discrete central scheme for hyperbolic conservation laws).
2. The only property of the specific (second-order) MinMod reconstruction that was used in the proof is that the interface values $\omega^\pm$ lie in the convex hull of the neighbouring cell-averages. This implies that the theorem is valid for any reconstruction that satisfies this property regardless of its order of accuracy.
3. Since we refer to Kurganov & Tadmor (2000), we note in passing that there were several typos in Kurganov & Tadmor (2000). The expressions that are given for some of the reconstructed variables in (4.18) in Kurganov & Tadmor (2000) are wrong. There are also a couple of wrong signs in equation (5.5) in Kurganov & Tadmor (2000). In addition, the statement of Theorem 5.1 in Kurganov & Tadmor (2000) is not optimal. Once the new value at time t^{n+1} is written as a convex combination of the point-values on the interfaces at time t^n, the MinMod limiter allows for a local bound instead of the global bound used in Kurganov & Tadmor (2000). This observation motivated the local form of the maximum principle that was stated in this paper. It is straightforward to write a local maximum theorem also for the case of hyperbolic conservation laws instead of the global Theorem 5.1 in Kurganov & Tadmor (2000).
4. To avoid confusion we would like to emphasize that the fully discrete version of our semi-discrete scheme (4.1)–(4.2) is not the fully discrete scheme of Levy & Tadmor (1997). The main difference between the schemes is that the semi-discrete numerical flux has a significantly reduced numerical dissipation (compared with the fully discrete flux). In addition, staggering is not present in the schemes that originate from the semi-discrete formulation.

TABLE 1 *Multi-level methods*

	η	c_0	c_s
Second-order time differencing			
4-level method ($s = 2$)	$\frac{3}{4}$	2	0
5-level method ($s = 3$)	$\frac{8}{9}$	$\frac{3}{2}$	0
Third-order time differencing			
5-level method ($s = 3$)	$\frac{16}{27}$	3	$\frac{12}{11}$
6-level method ($s = 4$)	$\frac{25}{32}$	2	$\frac{10}{7}$
7-level method ($s = 5$)	$\frac{108}{125}$	$\frac{5}{3}$	$\frac{30}{17}$

Following Kurganov & Tadmor (2000) we can now extend Theorem 4.1 to higher-order discretizations in time. Such extensions are straightforward due to the observations of Osher and Shu in Shu (1988) and Shu & Osher (1988) in which certain Runge–Kutta solvers can be written as a convex combination of forward Euler steps.

COROLLARY 4.1 Let $\mathcal{R}$ denote the RHS of (3.4). Consider the following Runge–Kutta methods

$$\begin{aligned} \omega^{(1)} &= \bar{\omega}^n + \Delta t \mathcal{R}[\omega^n], \\ \omega^{(l+1)} &= \eta_l \omega^n + (1-\eta_l)(\omega^{(l)} + \Delta t \mathcal{R}[\omega^{(l)}]), \ l = 1, 2, \ldots, s-1, \\ \bar{\omega}^{n+1} &= \omega^{(s)}, \end{aligned} \tag{4.6}$$

where for the second-order scheme ($s = 2$) $\eta_1 = 1/2$ and for the third-order scheme ($s = 3$) $\eta_1 = 3/4$, $\eta_2 = 1/3$. Assume that the point-values of the vorticity on the interfaces, $\omega^{\pm}_{j+\frac{1}{2},k}$, $\omega^{\pm}_{j,k+\frac{1}{2}}$, are given by (3.3) with a MinMod limiter approximation of the derivative (2.3), and that the discrete velocities satisfy (3.8). Finally, assume that the CFL condition (4.3) holds. Then the scheme (4.6) satisfies the local maximum principle (4.4).

A similar result holds for multi-level strong stability preserving Runge–Kutta methods (SSP-RK) (Gottlieb *et al.*, 2001), as stated in the following corollary.

COROLLARY 4.2 Let $\mathcal{R}$ denote the RHS of (3.4). Consider the multi-level methods of the form

$$\omega^{n+1} = \eta(\omega^n + c_0 \Delta t \mathcal{R}[\omega^n]) + (1-\eta)(\omega^{n-s} + c_s \Delta t \mathcal{R}[\omega^{n-s}]), \tag{4.7}$$

with coefficients that are given in Table 1. Assume that the point-values of the vorticity on the interfaces, $\omega^{\pm}_{j+\frac{1}{2},k}$, $\omega^{\pm}_{j,k+\frac{1}{2}}$, are given by (3.3) with a MinMod limiter approximation of the derivative (2.3), and that the discrete velocities satisfy (3.8). Finally, assume that the following CFL condition is satisfied:

$$\max\left(\lambda \max_{j,k} \left|u_{j+\frac{1}{2},k}\right|, \mu \max_{j,k} \left|v_{j,k+\frac{1}{2}}\right|\right) \leqslant \min_{c_k} \frac{1}{4c_k}. \tag{4.8}$$

Then the scheme (4.7) satisfies the local maximum principle (4.4).

5. Comments on convergence

In this section we follow the ideas of Lopes Filho *et al.* (2000), and show how the convexity argument that is used in the proof of Theorem 4.1 implies the H^{-1}_{loc}-stability of our scheme. We will then comment on how this result can be potentially used to prove the convergence of a first-order version of the scheme to a weak solution of the Euler equations. We start with the notion of H^{-1}_{loc}-stability. We denote by $\mathbb{A}^2$ the 2×2 anti-symmetric matrices with real entries.

DEFINITION 5.1 The sequence $\{u^\epsilon\}$ is called H^{-1}_{loc}-stable if $\{\omega^\epsilon = \nabla \times u^\epsilon\}$ is a precompact subset of $C((0, T); H^{-1}_{\text{loc}}(\mathbb{R}^2, \mathbb{A}^2))$.

The main result of Lopes Filho *et al.* (2000) was that H^{-1}_{loc}-stability is a criterion that excludes the phenomena of concentrations (i.e. a loss of energy to the small scales of the flow through concentrations of energy). One of the goals of Lopes Filho *et al.* (2000) was to identify regularity spaces of initial vorticities which give rise to weak solutions of the Euler equations with no concentrations. In the context of rearrangement invariant Banach spaces, such regularity results were obtained for compactly supported initial vorticities in Lebesgue spaces, L^p, $p > 1$, Orlicz spaces, $L(\log L)^\alpha$, $\alpha \geqslant 1/2$, and Lorenz spaces, $L^{(1,q)}$, $1 \leqslant q < 2$. Certain extensions were made to spaces that are not rearrangement invariant. If we denote the Biot–Savart kernel by K, i.e. $K(\xi) = \xi^\perp/(2\pi|\xi|^2)$, then the following theorem applies to rearrangement invariant Banach spaces:

THEOREM 5.1 (Lopes Filho *et al.*, 2000, Corollary 2.2) Let X be a rearrangement invariant Banach space such that C is dense in X_{loc} which in turn is compactly embedded in $H^{-1}_{\text{loc}}(\mathbb{R}^2)$. Let $\{u^\epsilon\}$ be the family of approximate solutions associated with the mollified initial vorticity $\omega_0^\epsilon = \eta_\epsilon * \omega_0 \in X$. Then $\{u^\epsilon\}$ is strongly compact in $L^\infty([0, T]; L^2_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2))$ and hence it has a strong limit, $u(\cdot, t)$, which is a weak solution associated with the initial velocity $u_0 = K * \omega_0$ without concentrations.

In this work, the class of spaces we focus on is the Orlicz spaces $L(\log L)^\alpha$ with $\alpha \geqslant 1/2$. We recall that an Orlicz space, L^ϕ, consists of all measurable functions f such that $\int \phi(|f(x)|)\,\mathrm{d}x < \infty$, where ϕ is any convex function for which $\lim_{s \to l} \phi(s)/s = l$, $l \in \{0, \infty\}$ (e.g. see Bennett & Sharpley, 1988). When applied to Orlicz spaces, Theorem 5.1 becomes

THEOREM 5.2 (Lopes Filho *et al.*, 2000, Theorem 2.2) Let $\omega_0 \in L(\log L)^\alpha_c(\mathbb{R}^2)$ with $\alpha \geqslant 1/2$. Then there exists a weak solution of the incompressible 2D Euler equations, $u(\cdot, t)$, subject to the initial condition $u_0 = K * \omega_0$, with no concentrations.

We note in passing that the borderline case of $\alpha = 1/2$ refers to sequences of approximate vorticities that correspond to mollified initial $L(\log L)^{1/2}$ vorticities, but not to general $L(\log L)^{1/2}$ initial vorticities as $L(\log L)^{1/2}$ is not compactly embedded in $H^{-1}_{\text{loc}}(\mathbb{R}^2)$. See Lopes Filho *et al.* (2000) for a counterexample.

Equipped with Theorem 5.2, we are now ready to investigate the H^{-1}_{loc}-stability of our scheme. We demonstrate our arguments with the first-order case. Hence, we replace the piecewise-linear reconstruction (2.2) by a piecewise-constant reconstruction, i.e. $\omega^-_{j+\frac{1}{2},k} = \omega^+_{j-\frac{1}{2},k} = \bar{\omega}^n_{j,k}$ for all k and $\omega^-_{j,k+\frac{1}{2}} = \omega^+_{j,k-\frac{1}{2}} = \bar{\omega}^n_{j,k}$ for all j (i.e. all the discrete derivatives in the reconstruction are set to zero). A first-order fully discrete scheme (with a forward Euler discretization in time) is obtained by making the necessary adjustments in (4.5), i.e.

$$\begin{aligned}
\bar{\omega}_{j,k}^{n+1} &= \frac{\lambda}{2}\left(\left|u_{j+\frac{1}{2},k}\right| - u_{j+\frac{1}{2},k}\right)\bar{\omega}_{j+1,k}^{n} + \frac{\lambda}{2}\left(\left|u_{j-\frac{1}{2},k}\right| + u_{j-\frac{1}{2},k}\right)\bar{\omega}_{j-1,k}^{n} \\
&\quad + \left[\frac{1}{2} - \frac{\lambda}{2}\left(u_{j-\frac{1}{2},k} + \left|u_{j-\frac{1}{2},k}\right| - u_{j+\frac{1}{2},k} + \left|u_{j+\frac{1}{2},k}\right|\right)\right]\bar{\omega}_{j,k}^{n} \\
&\quad + \frac{\mu}{2}\left(\left|v_{j,k+\frac{1}{2}}\right| - v_{j,k+\frac{1}{2}}\right)\bar{\omega}_{j,k+1}^{n} + \frac{\mu}{2}\left(\left|v_{j,k-\frac{1}{2}}\right| + v_{j,k-\frac{1}{2}}\right)\bar{\omega}_{j,k-1}^{n} \\
&\quad + \left[\frac{1}{2} - \frac{\mu}{2}\left(v_{j,k-\frac{1}{2}} + \left|v_{j,k-\frac{1}{2}}\right| - v_{j,k+\frac{1}{2}} + \left|v_{j,k+\frac{1}{2}}\right|\right)\right]\bar{\omega}_{j,k}^{n} \\
&= \frac{\lambda}{2}\left(\left|u_{j+\frac{1}{2},k}\right| - u_{j+\frac{1}{2},k}\right)\bar{\omega}_{j+1,k}^{n} + \frac{\lambda}{2}\left(\left|u_{j-\frac{1}{2},k}\right| + u_{j-\frac{1}{2},k}\right)\bar{\omega}_{j-1,k}^{n} \\
&\quad + \left[1 - \frac{\lambda}{2}\left(\left|u_{j-\frac{1}{2},k}\right| + \left|u_{j+\frac{1}{2},k}\right|\right) - \frac{\mu}{2}\left(\left|v_{j,k-\frac{1}{2}}\right| + \left|v_{j,k+\frac{1}{2}}\right|\right)\right]\bar{\omega}_{j,k}^{n} \\
&\quad + \frac{\mu}{2}\left(\left|v_{j,k+\frac{1}{2}}\right| - v_{j,k+\frac{1}{2}}\right)\bar{\omega}_{j,k+1}^{n} + \frac{\mu}{2}\left(\left|v_{j,k-\frac{1}{2}}\right| + v_{j,k-\frac{1}{2}}\right)\bar{\omega}_{j,k-1}^{n}. \qquad (5.1)
\end{aligned}$$

The last equality holds due to the discrete incompressibility relation (3.8). Note that $\bar{\omega}_{j,k}^{n+1}$ is written as a convex combination of $\bar{\omega}_{j,k}^{n}, \bar{\omega}_{j\pm1,k}^{n}, \bar{\omega}_{j,k\pm1}^{n}$, as long as the following CFL condition is satisfied (compare with (4.3)):

$$\lambda \max_{j,k}\left|u_{j+\frac{1}{2},k}\right| + \mu \max_{j,k}\left|v_{j,k+\frac{1}{2}}\right| \leqslant 1. \qquad (5.2)$$

Hence, for any convex ϕ, (5.1) implies that

$$\begin{aligned}
\phi(\bar{\omega}_{j,k}^{n+1}) \leqslant \phi(\bar{\omega}_{j,k}^{n}) &+ \frac{\lambda}{2}\left(\left|u_{j+\frac{1}{2},k}\right| - u_{j+\frac{1}{2},k}\right)\phi(\bar{\omega}_{j+1,k}^{n}) + \frac{\lambda}{2}\left(\left|u_{j-\frac{1}{2},k}\right| + u_{j-\frac{1}{2},k}\right)\phi(\bar{\omega}_{j-1,k}^{n}) \\
&- \frac{\lambda}{2}\left(\left|u_{j-\frac{1}{2},k}\right| + \left|u_{j+\frac{1}{2},k}\right|\right)\phi(\bar{\omega}_{j,k}^{n}) - \frac{\mu}{2}\left(\left|v_{j,k-\frac{1}{2}}\right| + \left|v_{j,k+\frac{1}{2}}\right|\right)\phi(\bar{\omega}_{j,k}^{n}) \\
&+ \frac{\mu}{2}\left(\left|v_{j,k+\frac{1}{2}}\right| - v_{j,k+\frac{1}{2}}\right)\phi(\bar{\omega}_{j,k+1}^{n}) + \frac{\mu}{2}\left(\left|v_{j,k-\frac{1}{2}}\right| + v_{j,k-\frac{1}{2}}\right)\phi(\bar{\omega}_{j,k-1}^{n}). \qquad (5.3)
\end{aligned}$$

Summing over all j, k, in (5.3), and once again making use of the discrete incompressibility relation (3.8), we have

$$\sum_{j,k}\phi(\bar{\omega}_{j,k}^{n+1}) \leqslant \sum_{j,k}\phi(\bar{\omega}_{j,k}^{n}). \qquad (5.4)$$

Hence, the total mass of the piecewise-constant approximate solution is non-increasing in time for any convex ϕ. This means that for the first-order scheme (5.1), the evolution mapping $\omega(\cdot, 0) \mapsto \omega(\cdot, t^n)$ maps any Orlicz space into itself, i.e. the method is H_{loc}^{-1}-stable. We summarize with the following theorem:

THEOREM 5.3 Consider the first-order central scheme (5.1) that is complemented with the streamfunction computation of the velocity field (3.6)–(3.7). Assume that the CFL condition (5.2) is fulfilled. Then

1. For any convex ϕ, (5.4) holds.
2. The evolution mapping $\omega(\cdot, 0) \mapsto \omega(\cdot, t^n)$ maps any Orlicz space into itself.

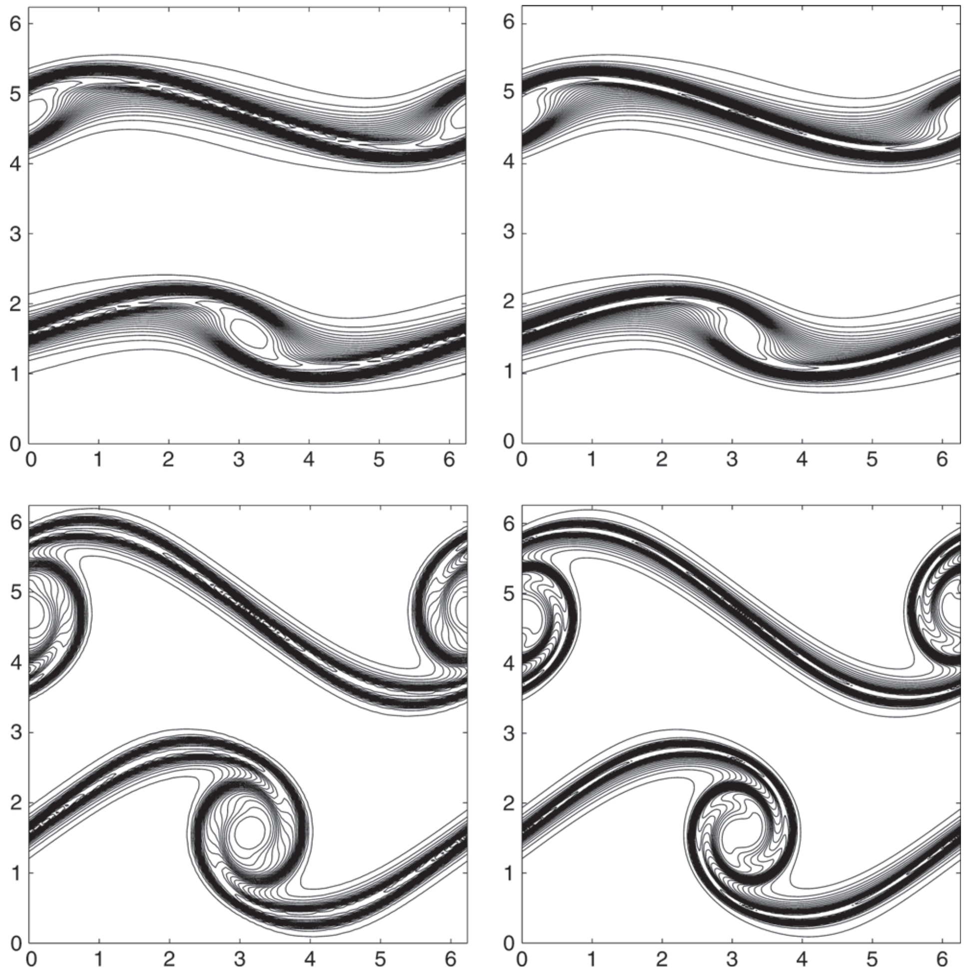

FIG. 3. A double shear-layer problem. Left: 128 × 128. Right: 256 × 256. Up: $T = 4$. Bottom: $T = 6$.

It is important to note that the result of Theorem 5.3 is insufficient for claiming that the scheme converges to a weak solution of the Euler equations. Passing from the discrete framework that is provided by the approximation scheme to a suitable continuum framework is an intricate task. In particular, the convergence theory of Lopes Filho *et al.* (2000) cannot be applied as is to the velocity field that is defined according to the relations (3.6)–(3.7) since there are no div–curl relations at the differential levels.

Instead of considering the velocity field that is defined by (3.6)–(3.7), an alternative approach would be to start from the discrete vorticity that is provided by the scheme and reconstruct a piecewise-constant vorticity (for the first-order equation) of the form

$$\omega_{\Delta x,\Delta y}(x, y, t^n) = \sum_{j,k} \bar{\omega}^n_{j,k}\chi_{j,k}(x, y).$$

This vorticity can be then used to define a global velocity field by $(U, V) = \nabla^{\perp}\psi$ with $\Delta\psi = -\omega_{\Delta x,\Delta y}$. In such a way, the resulting field is guaranteed to be incompressible, while $\nabla\times(U, V) = \omega$. Theorem 5.3

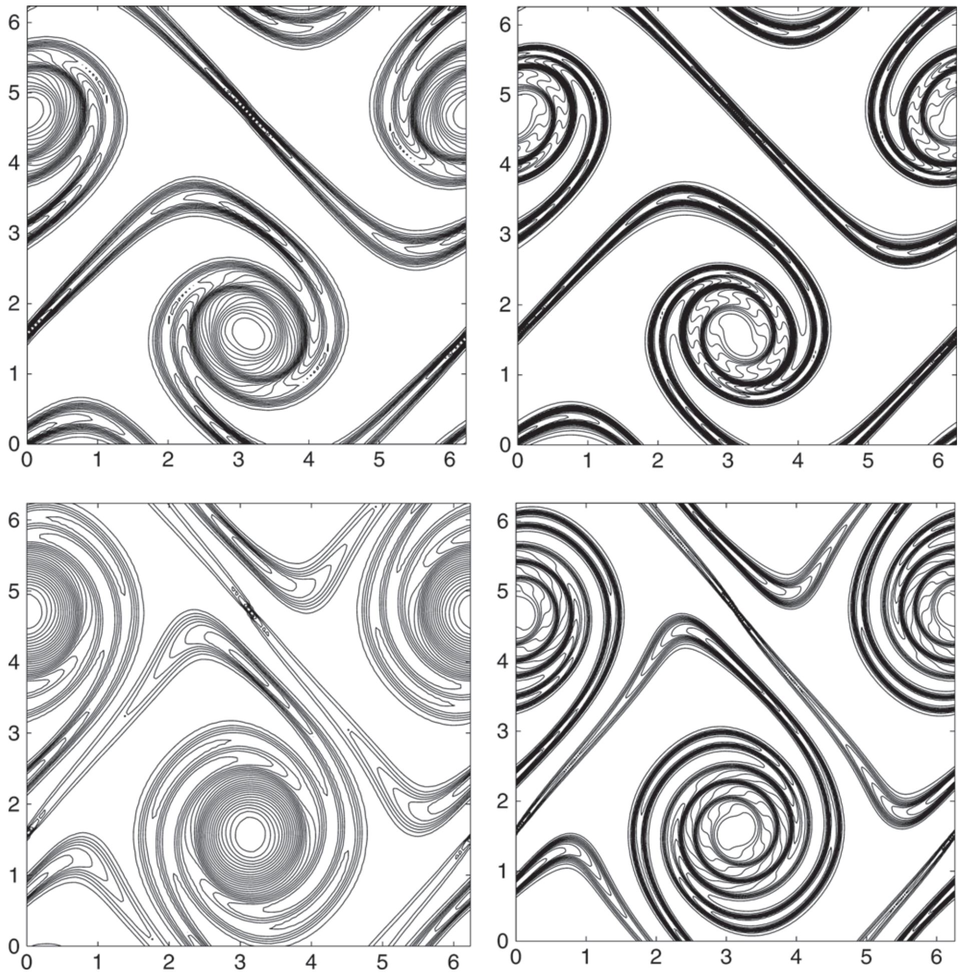

FIG. 4. A double shear-layer problem. Left: 128×128. Right: 256×256. Up: $T = 8$. Bottom: $T = 10$.

then guarantees that this velocity field converges. We would like to stress that this velocity field is the second velocity field we need to use (simultaneously). We keep the discrete incompressible velocity field (3.6)–(3.7) in the intermediate steps of the scheme in order to guarantee the vorticity bounds. Once we know that the velocity field (U, V) converges, it still remains to determine the limit. In order to obtain convergence of this velocity field to a weak solution of the Euler equations, one has to show that it satisfies the Euler equations in their velocity formulation modulo vanishing H^{-1}-errors. While this issue seems to be tractable in the present framework, it is beyond the scope of this paper and it is left for future work.

REMARK The maximum principle that was proved for the second-order scheme in Section 4 was based on a convexity argument. Hence, it is straightforward to extend Theorem 5.3 to the second-order case. The convergence arguments are less trivial and may not hold in the second-order case.

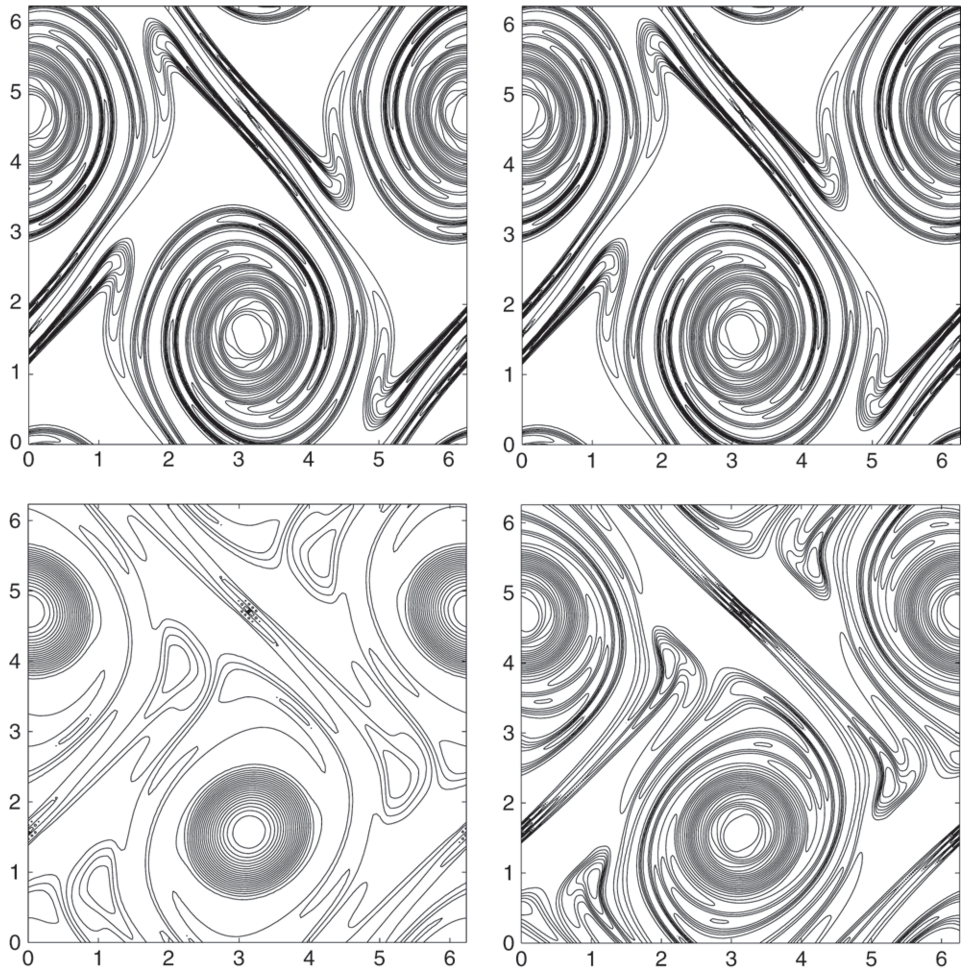

FIG. 5. A double shear-layer problem. Left: 128 × 128. Right: 256 × 256. Up: $T = 12$. Bottom: $T = 14$.

6. Numerical results

In this section we use our second-order scheme (4.1), (4.2), (3.7), (3.3) and (2.3) to approximate solutions of the incompressible Euler equations (1.1) in a doubly periodic box $[0, 2\pi) \times [0, 2\pi)$. The test case we consider is of a double shear-layer model taken from Bell *et al.* (1989). The initial velocity field is given by

$$u(x, y, 0) = \begin{cases} \tanh\left(\frac{1}{\rho}\left(y - \frac{\pi}{2}\right)\right), & y < \pi, \\ \tanh\left(\frac{1}{\rho}\left(\frac{3\pi}{2} - y\right)\right), & y \geqslant \pi, \end{cases} \qquad v(x, y, 0) = \delta \sin(x). \tag{6.1}$$

In our simulations we set the "thick" shear-layer parameter as $\rho = \pi/15$, and the perturbation parameter is taken as $\delta = 0.05$.

For the time marching method we use a second-order SSP-RK method. The derivatives are computed with the MinMod limiter (2.3), where $\theta = 1.3$. In Figs 3–5, we show the results obtained at time $T = 4, 6, 8, 10, 12, 14$ on a uniform grid with $N = 128$ and $N = 256$ grid-points in the x- and y-directions. These figures are in agreement with the results we obtained with our third-order semi-discrete scheme (Kurganov & Levy, 2000) and with the second-order fully discrete scheme (Levy & Tadmor, 1997).

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