Relative Gromov-Witten Invariants and the Degeneration Formula

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1 Introduction

Let X be a smooth projective variety. Let g, n be integers and let $\beta \in H_2(X)$. Then we can form the moduli space $\overline{\mathcal{M}}_{g,n}(X,\beta)$ of stable maps from n-pointed genus-g nodal curves to X representing β. We have the virtual dimension

$$\mathrm{vdim}(\overline{\mathcal{M}}_{g,n}(X,\beta)) = -K_X \cdot \beta + (1-g)(\dim X - 3) + n$$

and the virtual fundamental class

$$[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}} \in H_{2\mathrm{vdim}}(\overline{\mathcal{M}}_{g,n}(X,\beta)).$$

If we have classes $\gamma_i \in H^*(X)$, we can define the Gromov-Witten (GW) invariant

$$\Psi^X_{g,n,\beta}(\gamma_1 \cdots \gamma_n) = \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}} \mathrm{ev}_1^*(\gamma_1) \cup \cdots \cup \mathrm{ev}_n^*(\gamma_n),$$

where $\mathrm{ev}_i \colon \overline{\mathcal{M}}_{g,n}(X,\beta) \to X$ is the evaluation map at the i-th marked point.

Now consider a projective family $W \to \mathbb{A}^1$ such that the total space is smooth, the fibers W_t are smooth for $t \neq 0$, and W_0 is the union of two smooth varieties Y_1 and Y_2 that intersect transversally. As GW invariants are deformation-invariant, we know that the GW theories for all W_t with $t \neq 0$ must be the same. This indicates that the GW theory of W_0 must also be equal to that of W_t in some way. This is made precise by relative GW invariants [Li01] and the degeneration formula [Li02].

2 Relative GW-Invariants

Let $D \subset Y$ be a smooth divisor in a smooth variety. Then we say

$$f\colon (X, p_1, \ldots, p_n, q_1, \ldots, q_r) \to Y^{\mathrm{rel}} = (Y, D)$$

is a relative map if the p_i's are ordinary marked points and $f(q_j) \in D$ for all j.

We have that $\Delta = \mathbb{P}_D(N_{D/Y} \oplus \mathcal{O}_D)$ is a $\mathbb{P}^1$-bundle over D. Let $\mathbb{P}(N_{D/Y})$ and $\mathbb{P}(\mathcal{O}_D)$ be the zero and infinity sections respectively. We then can glue n copies of Δ by gluing the infinity section of the j-th copy to the zero section of the $(j+1)$-st copy. Then by gluing the zero section of the first Δ to $D \subset Y$, we get a new space Y_n. Let D_n be the infinity section of the last Δ. Then we can define relative stable morphisms to be maps $X \to Y_n$ such that $f^{-1}(D_n) = \sum \mu_i q_i$ as Cartier divisors (in addition to satisfying some stability and "admissibility" conditions to ensure that the moduli space of relative stable morphisms of some fixed topological type Γ is a proper, separated Deligne-Mumford stack $\overline{\mathcal{M}}_\Gamma(Y, D)$).

The stack $\overline{\mathcal{M}}_\Gamma(Y,D)$ admits a perfect obstruction theory and so it also has a well-defined virtual class. This allows us to define the relative GW invariant

$$\Psi_\Gamma^{Y^{\mathrm{rel}}}(a,b) = \int_{[\overline{\mathcal{M}}_\Gamma(Y,D)]^{\mathrm{vir}}} \mathrm{ev}_Y^*(a) \cup \mathrm{ev}_D^*(b),$$

where $\mathrm{ev}_Y \colon \overline{\mathcal{M}}_\Gamma(Y,D) \to Y^n$ and $\mathrm{ev}_D \colon \overline{\mathcal{M}}_\Gamma(Y,D) \to D^r$ are the ordinary and relative evaluation maps respectively.

3 Degeneration Formula

Let $\pi \colon W \to \mathbb{A}^1$ be a family as before. Take $\alpha \in H^0(R^*\pi_*\mathbb{Q}_W)^{\times n}$, where $\mathbb{Q}_W$ is the sheaf of locally constant functions on W, and let $\alpha(t) \in H^*(W_t)^{\times n}$ be its image in the fiber. Then the degeneration formula [Li02] states that

$$\Psi_{g,n,\beta}^Y(\alpha(t)) = \sum_\gamma \frac{m(\gamma)}{|\operatorname{Aut}(\gamma)|} \left[\Psi_{\Gamma_1}^{Y_1^{\mathrm{rel}}}(j_1^*\alpha(0), b) \cdot \Psi_{\Gamma_2}^{Y_2^{\mathrm{rel}}}(j_2^*\alpha(0), b^*)\right].$$

Here, $\gamma = (\Gamma_1, \Gamma_2)$ are the "admissible" triples that glue to a ordinary stable morphism of type (g,n,β); $m(\gamma)$ is the product of the "weights" of γ; $\operatorname{Aut}(\gamma)$ are the automorphisms of γ; and b^* is the dual of b.

4 Application to the Caporaso-Harris Formula

In [CH96], Caporaso and Harris proved a recursive formula for the number of certain curves in $\mathbb{P}^2$. Precisely, let $\alpha = (\alpha_i)$ and $\beta = (\beta_j)$ be two finite sequences of positive integers, and fix a line $L \subset \mathbb{P}^2$. Let $N^{d,g}(\alpha,\beta)$ be the number of degree d genus g nodal curves that

- have contact order i at α_i fixed points of L;
- have contact order j at β_j arbitrary points of L; and
- pass through $n = 2d + g + |\beta| - 1$ additional points of $\mathbb{P}^2$, where $|\beta| = \sum \beta_j$.

This number can be regarded as a relative GW invariant [Li04]: indeed, we have

$$N^{d,g}(\alpha,\beta) = \int_{[\overline{\mathcal{M}}]^{\mathrm{vir}}} \mathrm{ev}_{\mathbb{P}^2}^*(a) \cup \mathrm{ev}_L^*(b),$$

where $a \in H^4((\mathbb{P}^2)^n)$ is the product of point classes and $b \in H^*(L^{|\alpha|})$ is the product of $|\alpha|$ copies of the point class in $H^2(L)$ and $|\beta|$ copies of $1 \in H^0(L)$. By using the degeneration formula, we can recover the original formula of Caporaso-Harris.

5 Further Developments

In [AF11], D. Abramovich and B. Fantechi provide a new approach to relative GW invariants that simplifies the obstruction theory and extends the degeneration formula to orbifolds. Another generalization of relative GW invariants is given by logarithmic GW invariants [GS11], which allows $D \subset Y$ to be a normal crossings divisor, as opposed to a smooth divisor in Jun Li's theory. This was further extended in [ACGS20] to admit negative contact orders. Finally, there are also degeneration techniques for rank 1 Donaldson-Thomas Theory given by J. Li and B. Wu [LW11] for smooth divisors and by D. Maulik and D. Ranganathan [MR20] for simple normal crossings divisors.

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