Supplementary problems.

(1) Let $X_1, X_2 \dots X_n$ be a sequance of independent random variables each X_i having symmetric distribution, that is

$$P(\xi \in A) = P(X_i \in -A)$$

for any Borel set $A \subset \mathbb{R}$. Assume that $E(X_i^4) < \infty$ for $i = 1 \dots n$. Consider the sums $S_n = \sum_{j=1}^n X_j$. Show that

$$P(\max_{k<n} |S_k| > t) \leq \frac{\mathbb{E}(S_n^4)}{t^4}.$$

(2) Let $X_1, X_2 \dots X_n \dots$ be independent random variables such that for large n

$$P(X_n = 1) = \frac{1}{n^\alpha}, \quad P\left(X_n = \frac{1}{n^\beta}\right) = \frac{1}{2}, \quad P\left(X_n = -\frac{1}{n^\beta}\right) = \frac{1}{2} - \frac{1}{n^\alpha}.$$

For which α and β does $\sum_{n=1}^\infty X_n$ converge?

(3) Let $X_1, X_2 \dots X_n \dots$ be iid having Gaussian distribution with zero mean and variance σ^2. Let $S_n = \sum_{j=1}^n X_j$. Compute

$$I(a) = \lim_{n\to\infty} \frac{\ln P(S_n \geq an)}{n}.$$

(4) Let $X_1, X_2 \dots X_n \dots$ be iid such that $M_X(t) = \mathbb{E}(e^{tX}) < \infty$ for all $t \in \mathbb{R}$. Set $S_n = \sum_{j=1}^n X_j$,

$$I(a) = \lim_{n\to\infty} \frac{\ln P(S_n \geq an)}{n}.$$

Show that $I(a) \to -\infty$ as $a \to +\infty$.

(5) Let X be a bounded random variable. Define

$$\operatorname{ess\,sup}(X) = \{\sup x : P(X > x) > 0\}, \quad \operatorname{ess\,inf}(X) = \{\inf x : P(X < x) > 0\}.$$

Consider the moment generating function $M_X(t) = \mathbb{E}(e^{tX})$. Show that

$$\lim_{t\to\infty} \frac{\ln M_X(t)}{t} = \operatorname{ess\,sup}(X), \quad \lim_{t\to-\infty} \frac{\ln M_X(t)}{t} = \operatorname{ess\,inf}(X).$$

(6) Let (X_1, X_2, X_3, X_4) be a Gaussian random vector with zero mean and covariance matrix

$$\begin{pmatrix} 5 & 1 & 2 & 2 \\ 1 & 10 & 1 & 2 \\ 2 & 1 & 8 & 2 \\ 2 & 2 & 2 & 10 \end{pmatrix}.$$

Compute $P(X_1, X_2 | X_3, X_4)$.

(7) A fair coin is tossed 12 times. Let M be the number of heads during the first 6 tosses and N be the total number of heads. Compute $P(M | N = 8)$.