THE VISITS TO ZERO OF A RANDOM WALK DRIVEN BY AN IRRATIONAL ROTATION

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Abstract. We give a detailed analysis of the returns to zero of the "deterministic random walk" $S_n(x) = \sum_{k=0}^{n-1} f(x+k\alpha)$ where α is a quadratic irrational, $f(x) = 1_{[\frac{1}{2},1)}(\{x\}) - 1_{[0,\frac{1}{2})}(\{x\})$, and x is sampled uniformly in $[0,1]$.

The method is to find the asymptotic behavior of the ergodic sums of L^1 functions for linear flows on the infinite staircase surface.

Our methods also provide a new proof of J. Beck's central limit theorem for $S_n(0)$ where $n \in \{1, \dots, N\}$ is uniform and $N \to \infty$, and they allow us to determine the generic points for certain infinite measure preserving skew products ("cylinder maps").

1. Introduction and overview

The simple random walk (SRW) can be generated from a dynamical system as follows. Pick x in $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ uniformly, and iterate the *angle doubling map*

$$\tau : \mathbb{T} \to \mathbb{T}, \quad \tau(x) = 2x \mod 1.$$

Place a "walker" at $0 \in \mathbb{Z}$. At time step k ($k \geq 0$), ask the walker to make one step to the left if $\tau^k(x) \in [0, \frac{1}{2})$, and one step to the right if $\tau^k(x) \in [\frac{1}{2}, 1)$. This procedure generates the simple random walk, because the steps are $s_k := (-1)^{x_k+1}$ where $0.x_0x_1x_2\cdots$ is the binary expansion of x, and if $x \in \mathbb{T}$ is chosen uniformly, then s_k are independent random variables, equal to $+1$ or -1 with probability $\frac{1}{2}$.

The angle doubling map is a standard example of a "chaotic" map: It is mixing, it has positive entropy, and it has countable Lebesgue spectrum. It is natural to ask what happens if we replace τ by an "non-chaotic" ergodic map, such as the *irrational rotation*

$$R_\alpha : \mathbb{T} \to \mathbb{T}, \quad R_\alpha(x) = x + \alpha \mod 1 \quad (\alpha \in \mathbb{R} \setminus \mathbb{Q} \text{ fixed}).$$

R_α is not mixing, it has zero entropy, and its spectrum is discrete, properties associated with "determinism" (see [**Pet**] page 245).

If we replace τ by R_α, then we obtain a stochastic process called the *deterministic random walk* [**AK**]. To define it formally, let $\mathscr{B}$ denote the Borel σ–algebra of $\mathbb{T}$, let $m_{\mathbb{T}}$ be the normalized Lebesgue measure on $\mathbb{T}$ thought of as the unit interval mod 1, and define $f : \mathbb{T} \to \mathbb{Z}$ by

$$f(x) = \begin{cases} -1 & x \in [0, \frac{1}{2}) \\ +1 & x \in [\frac{1}{2}, 1). \end{cases}$$

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The *deterministic random walk (DRW)* with angle $\alpha \in \mathbb{R}\setminus\mathbb{Q}$ is the sequence $S_0 = 0$, $S_n = \sum_{k=0}^{n-1} f \circ R_\alpha^k$ $(n \geq 1)$ on the probability space $(\mathbb{T}, \mathscr{B}, m_{\mathbb{T}})$. The following table compares it with the simple random walk:

	Simple RW	**Deterministic RW**
Recurrence	A.e. orbit returns to zero. The set of exceptions has full Hausdorff dimension [**BS**]	A.e. orbit returns to zero [**At**]. The set of exceptions is finite [**Ra**].
Trace $\{S_n(x) : n \geq 0\}$	$\mathbb{Z}$ for a.e. x, but not for all x.	$\mathbb{Z}$ for a.e. x [**C**],[**CK**],[**Sch**], but not for all x [**BR**],[**Pe**].
Drift $\lim_{n\to\infty} S_n(x)/n$	A.e. orbit has zero drift. The set of exceptions has full Hausdorff dimension [**BS**]	All orbits have zero drift by Weyl's Theorem ([**KN**, chapter 1])
Central Limit Theorem	Yes	No, by the Denjoy–Koksma Ineq. [**Her**, page 73]. Other choices of f may have CLT [**BD**],[**V**]. See also [**Hu**],[**B1**]

In this paper we contribute to the study of the visits to zero of the deterministic random walk: $N_n(x) = N_n(x;\alpha) := \#\{0 \leq k \leq n-1 : S_k(x) = 0\}$.

For the simple random walk, if the number of visits to zero up to time n is $\widehat{N}_n$, then $\mathbb{E}(\widehat{N}_n) \sim \sqrt{2n/\pi}$ (de Moivre–Laplace Theorem), and $\frac{1}{\sqrt{n}}\widehat{N}_n \xrightarrow[n\to\infty]{\text{dist}} \Theta(1)$, where $\Theta(t)$ is Brownian local time [**Bor**]. The deterministic random walk behaves differently. Aaronson and Keane showed in [**AK**] that if α is a quadratic irrational, then there are constants $c_1, c_2 > 0$ s.t. $c_1(\frac{n}{\sqrt{\ln n}}) \leq \mathbb{E}(N_n) \leq c_2(\frac{n}{\sqrt{\ln n}})$.

We show, among other things, that if α is a quadratic irrational, then there is a positive constant $c(\alpha)$ with the following properties:

(1) $\mathbb{E}(N_n) \sim c(\alpha)(\frac{n}{\sqrt{\ln n}}) =: a_n(\alpha)$ as $n \to \infty$.

(2) $\frac{1}{a_n(\alpha)} N_n \xrightarrow[n\to\infty]{\text{dist}} \sqrt{2}\exp[-\frac{1}{2}\chi^2]$, where χ has the standard normal distribution.

(3) $\frac{1}{\ln\ln m}\sum_{n=0}^{m-1}\frac{1}{n\ln n}\left(\frac{1}{a_n(\alpha)}N_n\right) \xrightarrow[m\to\infty]{} 1$ almost surely.

(4) $c(\alpha) = \sqrt{\frac{|\ln\lambda|}{4\pi\sigma^2}}$ where $\lambda, \sigma^2 \in \mathbb{Q}[\alpha]$ can be calculated explicitly. For example, $c(\sqrt{2}) = \sqrt{\frac{\sqrt{2}}{3\pi}\ln(17+12\sqrt{2})}$.

These results should be contrasted with Kesten's work [**Kes**] which says that if α is also randomized (i.e. (x,α) chosen uniformly in $[0,1]^2$), then the right scaling for $N_n(x;\alpha)$ is $n/\ln n$.

Our main tool is the *cylinder map* $T_\alpha(x,\xi) = (x+\alpha \mod 1, \xi + f(x))$ on $\mathbb{T}\times\mathbb{Z}$, together with the (infinite) invariant measure $m_{\mathbb{T}\times\mathbb{Z}} := m_{\mathbb{T}} \times m_{\mathbb{Z}}$ ($m_{\mathbb{T}}$=normalized Lebesgue measure on $\mathbb{T}$, $m_{\mathbb{Z}}$ =counting measure on $\mathbb{Z}$). To see the connection to the DRW, write $T_\alpha^n := T_\alpha \circ \cdots \circ T_\alpha$ (n times) and observe by direct calculation that

- $S_n(x)$ is the second coordinate of $T_\alpha^n(x,0)$, and
- $N_n(x) = \sum_{k=0}^{n-1} 1_{\mathbb{T}\times\{0\}}(T_\alpha^k(x,0))$, where 1_E is the indicator function of E.

We will analyze the asymptotic behavior of $S_nG := \sum_{k=0}^{n-1} G\circ T_\alpha^k$ for general non-negative functions $G \in L^1(\mathbb{T}\times\mathbb{Z})$. The results for N_n follow from the special case $G = 1_{\mathbb{T}\times\{0\}}$.

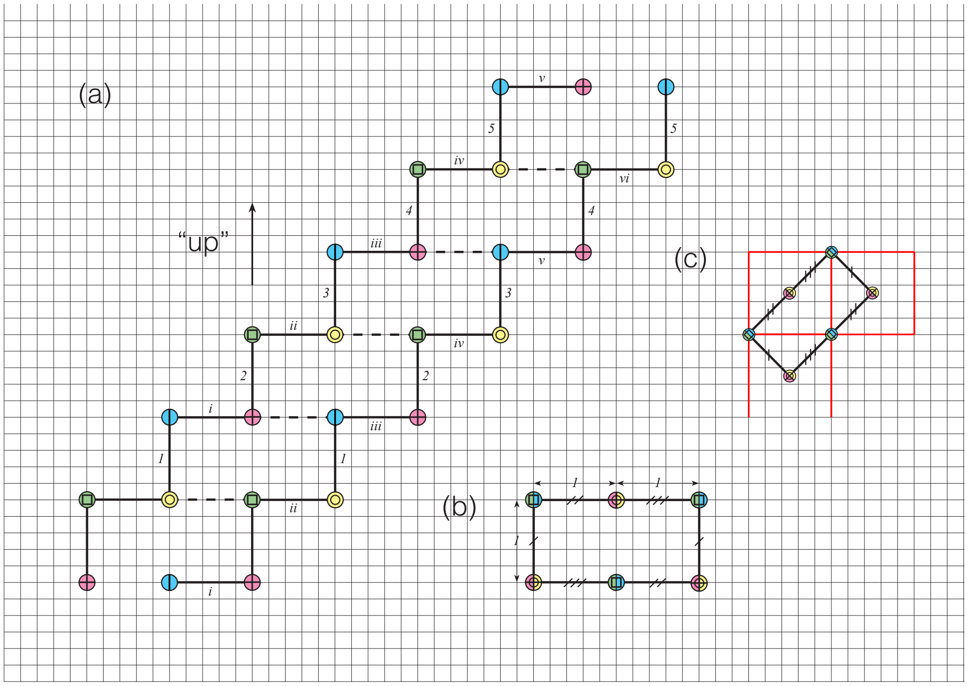

FIGURE 1. (a) The infinite staircase St; (b) The translation surface St_0 it covers; (c) St_0 is a punctured torus

T_α is ergodic and measure preserving [**CK**]. But since $m_{\mathbb{T}\times\mathbb{Z}}$ is an infinite measure, if $G \in L^1$ has non-zero integral, then there is no sequence of numbers a_n s.t. $\frac{1}{a_n}(S_nG)$ converges a.e. to a finite non-zero limit [**A1**]. Nevertheless one can hope to show that $(S_nG)(x,\xi) \sim a_n \int Gdm \times F_n(x,\xi)$ a.e. where a_n is deterministic, and $F_n(x,\xi)$ is a *fluctuating* term independent of G which converges in distribution on $\mathbb{T}\times\{k\}$ (see e.g. [**A2**], [**AS**]). Further study of $F_n(x,\xi)$ can hopefully also lead to a summability method which kills the fluctuations almost surely, and results in a "higher order" pointwise ergodic theorem as in [**ADF**],[**Fi1**],[**Fi2**],[**LS1**],[**LS2**]. This is what we shall do (Theorems 3.1 and 4.3).

The asymptotic expansion we obtain holds for an explicit set of full measure of (x,ξ). This allows us to characterize the generic points of T_α, which partially answers a question in [**Sa**]. See §7 for precise statements.

Our methods also allow us to give a new proof of a result of J. Beck on the central limit theorem for $\sum_{k=0}^{n-1} f(\{k\alpha\})$ where n is chosen randomly uniformly in $\{1,\dots,N\}$, and $N\to\infty$ [**B1**, **B2**]. See §5 for precise statement.

To study the cylinder map, we use a remarkable geometric construction due to Pat Hooper, Pascal Hubert & Barak Weiss [**HHW**] (see also [**HW**]). They constructed the *infinite staircase surface*, St, described in figure 1. The rectangles are 2×1 rectangles with the short side in the direction of the positive y–axis ("up"). Edges with identical labels are identified by translations.

The vertices split into four infinite classes of identified points, called the *singularities* of St. We let $\mathrm{St}^* := \mathrm{St}\setminus\{\text{singularities}\}$, and think of the singularities of

St as of *punctures* in St^*. Each singularity is the meeting point of infinitely many rectangles, and the angle around it is infinite.

The *linear flow at direction* $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ is the flow φ_θ^t which moves a point on St^* in the direction $\binom{\sin\theta}{\cos\theta}$ t units of distance respecting identifications ($\theta = 0$ is moving "up"). The definition makes sense for the set of full measure of points p whose orbit does not hit a singularity.

The connection to the cylinder map is explained by the following observation from [**HHW**]. Recall that a *Poincaré section* for φ_θ is a set $\mathfrak{S} \subset \mathrm{St}$ s.t. for a.e. $p \in \mathrm{St}$ there is a minimal positive time $r(p) > 0$ s.t. $\varphi_\theta^{r(p)}(p) \in \mathfrak{S}$, and $\inf_{p\in\mathfrak{S}} r(p) > 0$. The function $r : \mathfrak{S} \to (0, \infty)$ and the map $T : \mathfrak{S} \to \mathfrak{S}$, $T(p) = \varphi_\theta^{r(p)}(p)$, are called the *roof function* and *Poincaré map* of $\mathfrak{S}$.

Lemma 1.1. *For $\theta \neq \pm\frac{\pi}{2} + 2\pi k$, the union of the horizontal sides of the horizontal rectangles in figure* 1 *is a Poincaré section for φ_θ with constant roof function. The Poincaré map is isomorphic to the cylinder map T_α where $\alpha = \frac{1}{2}\tan\theta + \frac{1}{2}$.*

The isomorphism is very simple: Divide St into horizontal rectangles, call one of them "rectangle zero" and tag the remaining rectangles by $\xi \in \mathbb{Z}$ in such a way that the rectangle directly above rectangle ξ is rectangle $\xi + 1$. The point $(x, \xi) \in \mathbb{T} \times \mathbb{Z}$ corresponds to the point $\omega(x)$ on the top horizontal side of rectangle ξ, and located $2x$ units of distance away from the left end.

Since T_α is a Poincaré map for φ_θ with constant roof function, there is a standard way to reduce the study of the Birkhoff sums of T_α to the analysis of the Birkhoff integrals of φ_θ. This is what we will do.

The gain in the reduction to the infinite staircase model is that St has many symmetries which are hidden for the DRW: For special directions θ, it is possible to find a "nice" automorphism $\psi : \mathrm{St} \to \mathrm{St}$ s.t. for some $0 < \lambda < 1$

$$\psi \circ \varphi_\theta^t = \varphi_\theta^{\lambda t} \circ \psi. \qquad (*)$$

This is what happens for the θ whose corresponding α is a quadratic irrational. $(*)$ is the key to the asymptotic behavior of the Birkhoff sums of φ_θ and T_α, and therefore also to the asymptotic behavior of the visits to zero of the DRW.

2. The infinite staircase and its automorphisms

$\mathbb{Z}$–cover. St^* is a regular $\mathbb{Z}$–cover of a finite area surface St_0^* (figure 1(b)). Let

$$\pi : \mathrm{St}^* \to \mathrm{St}_0^*$$

be the covering map. St_0^* is a twice punctured torus (Figure 1(c)). Let St_0 denote the completion of St_0^* with respect to the natural metric. St_0 is a torus, and $\pi : \mathrm{St}^* \to \mathrm{St}_0^*$ extends continuously to a map $\pi : \mathrm{St} \to \mathrm{St}_0$. The extension is two-to-one on the singularities of St and infinite-to-one elsewhere.

The group of deck transformations of the covering is generated by an obvious translation. We denote it by

$$D : \mathrm{St} \to \mathrm{St}.$$

D^2 fixes the singularities.

$\mathbb{Z}$–coordinate. Choose a bounded connected $R \subset \mathrm{St}^*$ s.t. $\mathrm{St}^* = \biguplus_{k\in\mathbb{Z}} D^k(R)$ ($\biguplus$ =disjoint union), e.g. one of the horizontal rectangles in figure 1 minus the vertices and bottom horizontal side. The *$\mathbb{Z}$–coordinate of* $p \in \mathrm{St}^*$ (relative to R) is

$$\xi(p) := \text{unique } k \text{ s.t. } p \in D^k(R).$$

Notice that $\xi \circ D = \xi + 1$. This definition depends on the choice of R. We will refer to this as "choosing a $\mathbb{Z}$–coordinate."

Homogeneous automorphisms. St has an obvious atlas of charts whose change of coordinates transformations are euclidean translations. This allows us to identify the tangent spaces of St at different points with $\mathbb{R}^2$ (and therefore with each other) consistently. We will use the convention that direction "up" in figure 1 is $\binom{0}{1} \in \mathbb{R}^2$.

Once we have identified the tangent spaces at different points with $\mathbb{R}^2$, we can view the differential $d\psi_p : T_p(\mathrm{St}) \to T_{\psi(p)}(\mathrm{St})$ of a smooth map $\psi : \mathrm{St} \to \mathrm{St}$ ($p \in \mathrm{St}^*$) as a linear map $\mathbb{R}^2 \to \mathbb{R}^2$. The matrix representing this map is called the *derivative* at p.

A map $\psi : \mathrm{St} \to \mathrm{St}$ is an *orientation preserving affine automorphism*, if ψ is a homeomorphism; $\psi(\mathrm{St}^*) = \mathrm{St}^*$; $\psi : \mathrm{St}^* \to \mathrm{St}^*$ is differentiable; $\psi : \mathrm{St}^* \to \mathrm{St}^*$ is orientation preserving; and ψ has constant derivative. In what follows orientation preserving affine automorphisms will be simply called *automorphisms.*

An automorphism $\psi : \mathrm{St} \to \mathrm{St}$ is called *homogeneous*, if it commutes with D and preserves the D–orbits of the singularities of St. Necessarily ψ^2 fixes the singularities of St^*. The homogeneous automorphisms form a subgroup of finite index in the group of automorphisms [**HW**].

If ψ is homogeneous, then $\psi : \mathrm{St} \to \mathrm{St}$ is the lift of a toral automorphism $\psi_0 : \mathrm{St}_0 \to \mathrm{St}_0$, called the *projection* of ψ. To define ψ_0, let

$$\psi_0(p) = \pi[\psi(\widetilde{p})] \text{ for some (any) } \widetilde{p} \in \pi^{-1}(p).$$

The definition is proper, because $\pi^{-1}(p)$ is an orbit of D, $\psi \circ D = D \circ \psi$ and $\pi \circ D = \pi$. It is easy to see that ψ_0 is invertible, ψ_0 has constant derivative, and ψ_0 fixes the singularities of St_0.

Proposition 2.1. *All homogeneous automorphisms are area preserving.*

Proof. Let J, J_0 denote the Jacobian functions of a homogeneous automorphism ψ and its projection ψ_0 . Since $\pi \circ \psi = \psi_0 \circ \pi$ and π is a local isometry, $J = J_0 \circ \pi$. The Jacobian of ψ is constant (ψ has constant derivative), therefore the Jacobian of ψ_0 is constant. Since ψ_0 is a self-bijection of a surface of finite area, this constant equals one. So $J = J_0 \circ \pi \equiv 1$, and ψ is area preserving. □

Frobenius functions and drifts. Fix some $\mathbb{Z}$–coordinate $\xi : \mathrm{St} \to \mathbb{Z}$. The *Frobenius function* of a homogeneous automorphism $\psi : \mathrm{St} \to \mathrm{St}$ is

$$F_\psi : \mathrm{St}_0^* \to \mathbb{Z}\ ,\ F_\psi(p) = \xi[\psi(\widetilde{p})] - \xi[\widetilde{p}] \text{ for some (any) } \widetilde{p} \in \pi^{-1}(p).$$

The definition is proper because $\pi^{-1}(p) = \{D^n(p) : n \in \mathbb{Z}\}$, $\psi \circ D = D \circ \psi$, and $\xi \circ D = \xi + 1$. F_ψ depends on the choice of the $\mathbb{Z}$–coordinate. If we change the $\mathbb{Z}$–coordinate, F_ψ changes by a coboundary of ψ_0, see below.

The *average drift* (or just *drift*) of a homogenous automorphism $\psi : \mathrm{St} \to \mathrm{St}$ is

$$\delta(\psi) := \frac{1}{\mathrm{area}(\mathrm{St}_0)} \int_{\mathrm{St}_0} F_\psi(p)dp,\ \ (dp = \text{area measure}).$$

We will see later that $\delta(\psi)$ is the drift of a certain random walk associated to ψ.

Lemma 2.2. *The average drift is independent of the choice of the $\mathbb{Z}$–coordinate, and $\delta(\psi\circ\phi)=\delta(\psi)+\delta(\phi)$ for any homogeneous automorphisms ψ,ϕ.*

Proof. Let $\psi_0:\mathrm{St}_0\to\mathrm{St}_0$ be the projection of ψ, and suppose ξ,η are two choices of $\mathbb{Z}$–coordinates with Frobenius functions F^ξ_ψ, F^η_ψ. We claim that $\int F^\xi_\psi=\int F^\eta_\psi$.

Define $\Delta:\mathrm{St}_0\to\mathbb{Z}$, $\Delta(p)=\xi(\widetilde{p})-\eta(\widetilde{p})$ for some (any) $\widetilde{p}\in\pi^{-1}(p)$. The definition is proper since $\pi^{-1}(p)$ is a D–orbit, and $(\xi-\eta)\circ D=(\xi+1)-(\eta+1)=\xi-\eta$. A simple calculation shows that $F^\xi_\psi-F^\eta_\psi=\Delta\circ\psi_0-\Delta$. Since ψ_0 is measure preserving, $\int(F^\xi_\psi-F^\eta_\psi)=\int(\Delta\circ\psi_0-\Delta)=0$, and $\int F^\xi_\psi=\int F^\eta_\psi$.

Next suppose ψ,ϕ are two homogeneous automorphisms. It is easy to see that $\psi\circ\phi$ is a homogeneous automorphism, and for every $p\in\mathrm{St}_0$ and $\widetilde{p}\in\pi^{-1}(p)$,

$$\begin{aligned}F_{\psi\circ\phi}(p)&=\xi[\psi(\phi(\widetilde{p}))]-\xi[\widetilde{p}]=\xi[\psi(\phi(\widetilde{p}))]-\xi[\phi(\widetilde{p})]+\xi[\phi(\widetilde{p})]-\xi[\widetilde{p}]\\&=(F_\psi\circ\phi_0)(p)+F_\phi(p),\ \text{where } \phi_0 \text{ is the projection of } \phi.\end{aligned}$$

Since ϕ_0 is area preserving, $\delta(\psi\circ\phi)=\int F_\psi\circ\phi_0+\int F_\phi=\delta(\psi)+\delta(\phi)$. ☐

By [**HHW**] the set of derivatives of homogeneous automorphisms equals

$$\Gamma=\{A\in\mathrm{SL}(2,\mathbb{Z}):A=\begin{pmatrix}1&0\\0&1\end{pmatrix}\text{ or }\begin{pmatrix}0&1\\-1&0\end{pmatrix}\mod 2\}$$

Here is a refinement of this statement. The proof is in the appendix.

Proposition 2.3 (Classification of homogeneous automorphisms)**.**

(1) *If $A\in\mathrm{SL}(2,\mathbb{Z})$, $A=\begin{pmatrix}1&0\\0&1\end{pmatrix}\mathrm{mod}\,2$ and $\delta_0\in\mathbb{Z}$, then there is a unique homogeneous automorphism with derivative A and drift δ_0.*
(2) *If $A\in\mathrm{SL}(2,\mathbb{Z})$, $A=\begin{pmatrix}0&1\\-1&0\end{pmatrix}\mathrm{mod}\,2$ and $\delta_0\in\frac12+\mathbb{Z}$, then there is a unique homogeneous automorphism with derivative A and drift δ_0.*
(3) *No other homogeneous automorphisms exist.*

Renormalizing hyperbolic automorphisms. A homogeneous automorphism of St is called *hyperbolic* if its derivative matrix has two real eigenvalues, λ,λ^{-1}, where $0<|\lambda|<1$.

Definition 2.4. *A hyperbolic homogeneous automorphism ψ* renormalizes *$\alpha\in\mathbb{R}$, if $\alpha=\frac12+\frac12\tan\theta(\mathrm{mod}\,1)$ where $\binom{\sin\theta}{\cos\theta}$ is an eigenvector of the derivative of ψ. In this case we say that α is* renormalized by *ψ.*

The motivation is that if $\alpha=\frac12+\frac12\tan\theta(\mathrm{mod}\,1)$, then T_α is the Poincaré map of the linear flow in direction θ, $\varphi_\theta:\mathrm{St}\to\mathrm{St}$, and

$$\psi\circ\varphi^t_\theta=\varphi^{\lambda t}_\theta\circ\psi$$

where λ is the eigenvalue of $\binom{\sin\theta}{\cos\theta}$.

There is no loss of generality in assuming that (a) the eigenvalues are positive, (b) ψ fixes the singularities of St, (c) ψ has zero drift, and (d) $0<\lambda<1$: We saw above that any homogeneous automorphism ψ has drift in $\frac12\mathbb{Z}$, so $2\delta(\psi)$ is always an integer. One of the automorphisms $D^{-4\delta(\psi)}\psi^4$, $D^{4\delta(\psi)}\psi^{-4}$ satisfies (a),(b),(c),(d).

We characterize the irrational numbers α which possess renormalizing automorphisms. Recall that a *quadratic irrational* is an irrational α s.t. $a\alpha^2+b\alpha+c=0$ for some $a,b,c\in\mathbb{Z}$ not all equal to zero.

Proposition 2.5. *α is renormalized by a hyperbolic homogeneous automorphism iff it is a quadratic irrational.*

Proof. The derivative of a hyperbolic homogeneous automorphism belongs to $\mathrm{SL}(2,\mathbb{Z})$. The eigenvalues of such matrices are quadratic irrationals, and the slopes of the eigenvectors of such matrices are quadratic irrationals. It follows that all irrationals with renormalizing hyperbolic automorphisms are quadratic.

For the converse suppose that α is a quadratic irrational. We prove that a renormalizing automorphism exists. Let $\alpha' := 1/(2\alpha - 1)$. This is also a quadratic irrational.

By Lagrange's Theorem, the continued fraction expansion of α' is eventually periodic. So there is a map $\varphi(z) = \frac{a'z+b'}{c'z+d'}$ with $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$ s.t. the continued fraction expansion of $\varphi(\alpha')$ is (completely) periodic:

$$\varphi(\alpha') = [a_0, \dots, a_{n-1}, a_0, \dots, a_{n-1}, \cdots]. \tag{2.1}$$

Let $\frac{p_k}{q_k}$ denote the principal convergents of $\beta := \varphi(\alpha')$. By the theory of continued fractions, $\det\begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} = (-1)^{n+1}$, and β is a fixed point of $\psi(z) = \frac{p_n z + p_{n-1}}{q_n z + q_{n-1}}$. So $\psi[\varphi(\alpha')] = \varphi(\alpha')$, whence $(\varphi^{-1}\psi\varphi)(\alpha') = \alpha'$.

Let $\phi := \varphi^{-1}\psi\varphi$, then $\phi^N(\alpha') = \alpha'$ for all N. We claim that for some N, ϕ^N is a Möbius transformation with matrix belonging to

$$\Gamma(2) := \{A \in \mathrm{SL}(2,\mathbb{Z}) : A = \mathrm{Id} \mod 2\}.$$

Let A be the matrix which represents ϕ^2. Obviously, $\phi^2 \in \mathrm{SL}(2,\mathbb{Z})$. Let $[A]_2 \in \mathrm{SL}(2,\mathbb{Z}_2)$ denote the residue class of $A \mod 2$. The group $\mathrm{SL}(2,\mathbb{Z}_2)$ is finite, therefore $[A^N]_2 = ([A]_2)^N = \mathrm{Id}$ for some N. So ϕ^{2N} is represented by a matrix in $\Gamma(2)$, proving the claim.

Write $\phi^{2N}(z) = \frac{c+dz}{a+bz}$ for $\begin{pmatrix} d & c \\ b & a \end{pmatrix} \in \Gamma(2)$, then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ \alpha' \end{pmatrix} = \begin{pmatrix} a + b\alpha' \\ c + d\alpha' \end{pmatrix} = (a + b\alpha')\begin{pmatrix} 1 \\ \phi^{2N}(\alpha') \end{pmatrix} = (a + b\alpha')\begin{pmatrix} 1 \\ \alpha' \end{pmatrix}, \tag{2.2}$$

proving that $\binom{1}{\alpha'}$ is an eigenvector of $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(2)$. This matrix is hyperbolic, because its trace is bigger than two: $a + d = \mathrm{tr}\left[\phi^{2N}\right] = \mathrm{tr}\left[\begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix}^{2N}\right]$, and every 2×2 matrix with determinant one and all of whose entries are positive integers, has trace bigger than two.

By (2.2), the homogeneous automorphism with zero drift and derivative $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ renormalizes $\alpha = \frac{1}{2} + \frac{1}{2}(\frac{1}{\alpha'})$. □

The previous proof is constructive, but it does not provide a convenient tool for calculating renormalizing automorphisms. This is the purpose of the next result.

Proposition 2.6. *Any quadratic irrational α equals $\frac{1}{2} + \frac{k+\sqrt{q(q+1)}}{2n} \pmod 1$ for some $k, q, n \in \mathbb{Z}$ satisfying $q(q+1) \neq 0$ and $n | k^2 - q(q+1)$. In this case there is a renormalizing homogeneous automorphism ψ with zero drift and derivative*

$$d\psi = \begin{pmatrix} 2(q-k)+1 & 2 \cdot \frac{k^2 - q(q+1)}{n} \\ -2n & 2(q+k)+1 \end{pmatrix}. \tag{2.3}$$

Example: For $\alpha = \sqrt{2}$, we can take $k = n = 3, q = 8$, and get the homogeneous automorphism with zero drift and derivative $\begin{pmatrix} 11 & -42 \\ -6 & 23 \end{pmatrix}$.

Similar formulas can be obtained for $\sqrt{3}$ $(k = n = 1, q = 3)$, $\sqrt{5}$ $(k = n = 1, q = 4)$, $\sqrt{7}$ $(k = n = 12, q = 63)$ etc.

Proof. Since α is a quadratic irrational, it has a hyperbolic renormalizing automorphism with zero drift. Let A be the derivative. By proposition 2.3, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with a, d odd and b, c even, and $ad - bc = 1$.

We claim that $\mathrm{tr}(A) = 2 \,(\mathrm{mod}\, 4)$. Since a, d are odd, they are equal to $\pm 1 \,(\mathrm{mod}\, 4)$. Write $a = 4\alpha + \varepsilon, d = 4\beta + \eta, b = 2\gamma, c = 2\delta$ with $\alpha, \beta, \gamma, \delta \in \mathbb{Z}$ and $\varepsilon, \eta = \pm 1$. Since $1 = \det(A) = \varepsilon\eta (\mathrm{mod}\, 4)$, $\varepsilon = \eta$. It follows that $a = d (\mathrm{mod}\, 4)$ and $\mathrm{tr}\, A = a + d = 4(\alpha + \beta) \pm 2 \in 4\mathbb{Z} + 2$.

Write $\mathrm{tr}(A) = 4q + 2$ with some $q \in \mathbb{Z}$. Since a, d are odd and $a + d = 4q + 2$, we can put a, d in the form $a = 2(q - k) + 1$ and $d = 2(q + k) + 1$ with $k \in \mathbb{Z}$.

Since c is even, $c = -2n$ with some $n \in \mathbb{Z}$. Since $ad - bc = 1$, either $n = 0$ and $A = \mathrm{Id}$, or $n \neq 0$ and $b = 2 \cdot \frac{k^2 - q(q+1)}{n}$. So $A = \begin{pmatrix} 2(q-k)+1 & 2 \cdot \frac{k^2 - q(q+1)}{n} \\ -2n & 2(q+k)+1 \end{pmatrix}$ with $q, n, k \in \mathbb{Z}$ s.t. $n \neq 0$ and $n | k^2 - q(q+1)$. Such choice of k, q, n determines a matrix in $\mathrm{SL}(2, \mathbb{Z})$ equal to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ mod 2.

The characteristic polynomial of A is $x^2 - x\, \mathrm{tr}\, A + \det A = x^2 - (4q+2)x + 1$. The eigenvalues are $(2q+1) \pm 2\sqrt{q(q+1)}$. A is hyperbolic iff $q(q+1) \neq 0$. The eigenvectors are proportional to $(\frac{k \pm \sqrt{q(q+1)}}{n}, 1)$, so the automorphism with derivative A renormalizes $\alpha := \frac{1}{2} + \frac{k \pm \sqrt{q(q+1)}}{n}$. Playing with the signs of k, n we see that there is no loss in taking $\alpha := \frac{1}{2} + \frac{k + \sqrt{q(q+1)}}{n}$. □

Markov partitions and symbolic dynamics. Every hyperbolic homogeneous automorphism $\psi : \mathrm{St} \to \mathrm{St}$ covers a hyperbolic toral automorphism $\psi_0 : \mathrm{St}_0 \to \mathrm{St}_0$. Adler and Weiss introduced in [**AW**] a technique for coding $\psi_0 : \mathrm{St}_0 \to \mathrm{St}_0$ as the action of the left shift map on the collection of two sided infinite paths on a finite directed graph. This is done using *Markov partitions*. The purpose of this section is to describe this method.

The original work of Adler & Weiss applies to general hyperbolic automorphisms. It is important for our purposes to apply the Adler–Weiss construction in a way which respects that fact that ψ_0 fixes the punctures of St_0 and has derivative matrix

$$A \in \Gamma(2) := \{A \in \mathrm{SL}(2, \mathbb{Z}) : A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod 2\}.$$

We will assume for simplicity that A has positive eigenvalues, $0 < \lambda < 1$ and $\lambda^{-1} > 1$. Then there are vectors $\underline{v} = \binom{1}{v}$, $\underline{w} = \binom{1}{w}$ such that $A\underline{v} = \lambda^{-1}\underline{v}$ and $A\underline{w} = \lambda\underline{w}$. Since $A \in \Gamma(2)$, v, w are irrational. We call $\underline{w}$ the *stable* direction and $\underline{v}$ the *unstable* direction (of ψ_0).

The first step in the Adler–Weiss construction is to divide the torus into two parallelograms Q_1, Q_2 with sides parallel to $\underline{v}, \underline{w}$. They cut the torus along two line segments emanating from a single fixed point. We prefer to use one segment passing through the first puncture, and the other passing through the second puncture: this simplifies the analysis of the coded Frobenius function, see §6 below.

Suppose first that $-1 < w < 0$, $v > 1$ (case 1), or $0 < w < 1$, $v > 1$ (case 2). Then Q_1, Q_2 can be constructed as in Figure 2. One of the parallelograms, which we call Q_1, does not contain any punctures in its top or bottom sides. The other, which we call Q_2, does.

The general case can be reduced to case 1 or 2 by working with $\theta \circ \psi_0 \circ \theta^{-1}$ or $\theta \circ \psi_0^{-1} \circ \theta^{-1}$ for a suitable toral automorphism $\theta : \mathrm{St}_0 \to \mathrm{St}_0$ which fixes

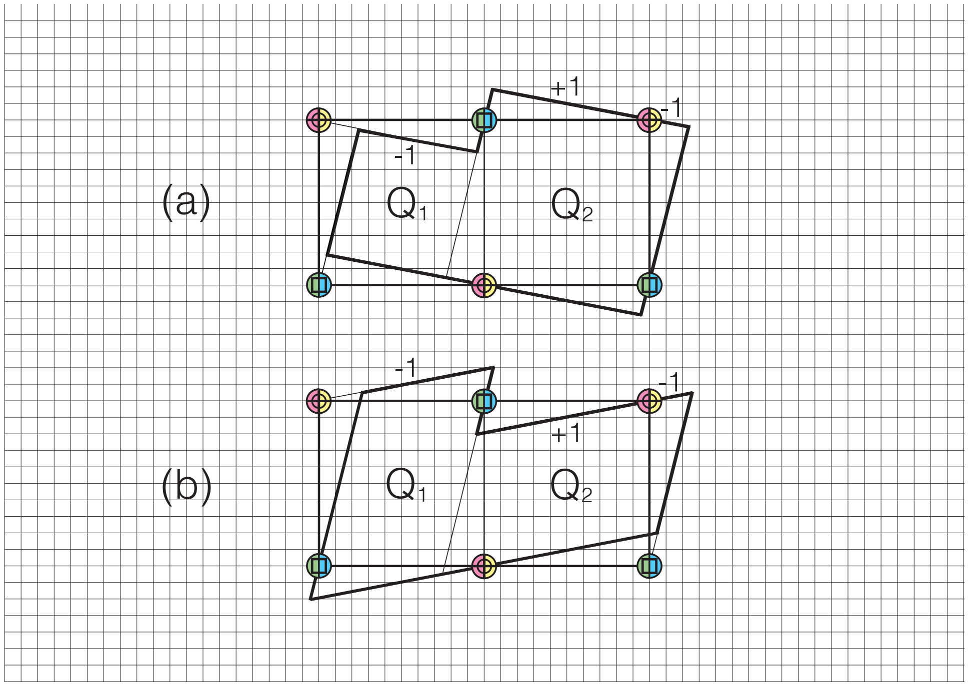

FIGURE 2. Partition of the torus

the punctures. The derivative matrix of θ is produced from the following lemma, applied to the irrational numbers $\xi = v^{-1}, \eta = w^{-1}$ (see the appendix for proof):

Lemma 2.7. *For every* $\xi, \eta \in \mathbb{R} \setminus \mathbb{Q}$ *s.t.* $\xi \neq \eta$ *there exists* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(2)$ *such that* $s_1 := \frac{a\xi+b}{c\xi+d}, s_2 := \frac{a\eta+b}{c\eta+d}$ *satisfy one of the following: One of* s_1, s_2 *is in* $(0,1)$ *and the other is in* $(1,\infty)$*; Or one of* s_1, s_2 *is in* $(-1,0)$ *and the other is in* $(1,\infty)$.

θ itself can be produced using Proposition 2.3 by projecting the homogeneous automorphism with zero drift and derivative $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ to St_0. Homogeneity guarantees that θ fixes the punctures.

We call $R := Q_1 \cup Q_2$ the *fundamental polygon* of ψ_0. The sides of R in direction $\underline{w}$ (resp. $\underline{v}$) are called *stable* (resp. *unstable*). Let $\partial_s R$:=union of stable sides of R and $\partial_u R$:=union of unstable sides of R. Since $\partial_s R, \partial_u R$ are linear segments containing fixed points of ψ_0 and in the direction of eigenvectors of $d\psi_0$, we have $\psi_0(\partial_s R) \subset \partial_s R$ and $\psi_0^{-1}(\partial_u R) \subset \partial_u R$.

A *u–fibre* is a linear segment in direction $\underline{v}$ with endpoints in $\partial_s R$. Since $\psi_0(\partial_s R) \subset \partial_s R$, $A\underline{v} = \lambda^{-1}\underline{v}$, and $0 < \lambda < 1$, the ψ_0–image of a u–fibre is a finite union of u–fibres. Similarly, an *s-fibre* is a linear segment in direction $\underline{w}$ and endpoints in $\partial_u R$. The ψ_0–image of an s–fibre is a subset of an s–fibre. We orient u/s–fibres in the direction of $\underline{v}, \underline{w}$.

Thus $\psi_0(Q_i)$ is a finite union of non-overlapping parallelograms $Q_{i1}, \ldots, Q_{iN_i} \subset R$ with sides in the stable and unstable directions, and with s–sides contained in $\partial_s R$. We use the following convention for the order $Q_{i1}, \ldots, Q_{iN_i}$ $(i = 1, 2)$: Recall that u–fibres are oriented in the direction of $\underline{v}$, then every parallelogram Q_{ij} has a

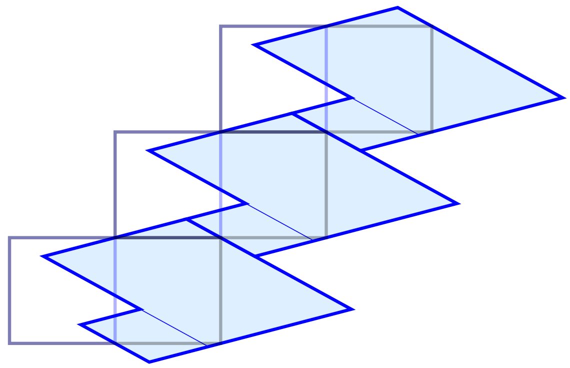

FIGURE 3. The $\mathbb{Z}$–coordinate associated to the canonical renormalizing automorphism of $\sqrt{2}$

bottom s–side, and a *top* s–side. The ordering is done so that the top side of Q_{ij} is identified with the bottom side of $Q_{i,j+1}$ ($j = 1, \ldots, N_i - 1$).

The interior of Q_{ij} is completely contained in the interiors of Q_k for $k = 1$ or 2. Otherwise, $\psi_0(\mathrm{int}(Q_i))$ intersects $\partial_u R$, in contradiction to $\psi_0^{-1}(\partial_u R) \subset \partial_u R$.

Let $\mathfrak{P} := \{Q_{ij} : i = 1, 2; 1 \leq j \leq N_i\}$. Since ψ_0 is bijective, $\mathfrak{P}$ is a partition of St_0. By the previous paragraph, $\mathfrak{P}$ is a refinement of $\{Q_1, Q_2\}$. $\mathfrak{P}$ is the *Adler–Weiss Markov partition*.

The *dynamical graph* of $\mathfrak{P}$ is the directed graph $\mathscr{G}$ with set of vertices $\mathfrak{P}$ and edges $P_i \to P_j$ for any pair of $P_i, P_j \in \mathfrak{P}$ s.t. $\mathrm{int}(P_i \cap \psi_0^{-1}(P_j)) \neq \varnothing$. Let $\Sigma(\mathscr{G})$ denote the collection of bi-infinite paths on $\mathscr{G}$:

$$\Sigma(\mathscr{G}) := \{(P_k)_{k\in\mathbb{Z}} \in \mathfrak{P}^{\mathbb{Z}} : P_k \to P_{k+1} \text{ for every } k \in \mathbb{Z}\}.$$

Equip $\Sigma(\mathscr{G})$ with the metric $d(\underline{x}, \underline{y}) := \exp(-\min\{|k| : x_k \neq y_k\})$. Let $\sigma : \Sigma(\mathscr{G}) \to \Sigma(\mathscr{G})$ denote the *left shift map*, $\sigma : (x_i)_{i\in\mathbb{Z}} \mapsto (x_{i+1})_{i\in\mathbb{Z}}$.

Theorem 2.8 (Adler and Weiss)**.** *For every $(P_i)_{i\in\mathbb{Z}} \in \Sigma(\mathscr{G})$, there is a unique point $\pi_0[(P_i)_{i\in\mathbb{Z}}] \in \bigcap_{i\in\mathbb{Z}} \psi_0^{-i}(\overline{P_i})$, and $\pi_0 : \Sigma(\mathscr{G}) \to \mathrm{St}_0$ has the following properties:*

(1) *$\pi_0 : \Sigma(\mathscr{G}) \to \mathrm{St}_0$ is onto and $|\pi_0^{-1}(p)| = 1$ for Lebesgue almost every $p \in \mathrm{St}_0$.*
(2) *π_0 is Hölder continuous and $\pi_0 \circ \sigma = \psi_0 \circ \pi_0$.*
(3) *Let m_0 denote the normalized Lebesgue measure on St_0, then $m_0 = \widehat{m}_0 \circ \pi_0^{-1}$ where $\widehat{m}_0$ is a mixing stationary Markov measure on Σ.*
(4) *$\widehat{m}_0$ is the measure of maximal entropy for $\sigma : \Sigma(\mathscr{G}) \to \Sigma(\mathscr{G})$.*

See [**AW**] for proof. Additional information on the combinatorial structure of $\mathscr{G}$ can be found in §6.

Let $\widetilde{R}$ denote a connected lift of the fundamental polygon $Q_1 \cup Q_2$ to St. The corresponding $\mathbb{Z}$–coordinate $\xi : \mathrm{St} \to \mathbb{Z}$ is called the *$\mathbb{Z}$–coordinate associated to the automorphism* ψ, see figure 3.

The main advantage of the associated $\mathbb{Z}$–coordinate is the following fact, whose proof we defer for reasons of exposition to §6 (Lemma 6.8): If F_ψ is the Frobenius function of ψ with respect to the associated $\mathbb{Z}$–coordinate of ψ, then

$$F_\psi \text{ is } \mathfrak{P} \vee \psi_0^{-1}(\mathfrak{P})\text{–measurable or } \mathfrak{P} \vee \psi_0(\mathfrak{P})\text{–measurable.} \tag{2.4}$$

This means there exists a function $g : \mathfrak{P} \times \mathfrak{P} \to \mathbb{Z}$ s.t. the *coded Frobenius function*

$$F := F_\psi \circ \pi_0 : \Sigma(\mathscr{G}) \to \mathbb{Z}$$

takes the form $F[\underline{P}] = g(P_0, P_1)$ or $F[\underline{P}] = g(P_{-1}, P_0)$, where $\underline{P} = (P_i)_{i \in \mathbb{Z}} \in \Sigma(\mathscr{G})$.

The following additional property of F is proved in the appendix.

Lemma 2.9 (Aperiodicity Lemma)**.** *If $e^{itF} = zh/h \circ \sigma$ where $|z| = 1$, $t \in \mathbb{R}$, and $h : \Sigma(\mathscr{G}) \to \mathbb{C}$ is continuous , then $z = 1$, $t \in 2\pi\mathbb{Z}$, and $h = \text{const}$.*

This is called the *aperiodicity condition* in [**GH**], and should be viewed as a strong way of saying that F does not take values in a set of the form $a + b\mathbb{Z}$ "up to a coboundary." The aperiodicity condition is used in §3, to show that $\sigma \neq 0$.

The twist at a singularity. The contents of this section are only used in §5.

Suppose ψ is a homogeneous hyperbolic automorphism of the infinite staircase, and let p denote one of the four singularities of St. Recall that $D^2(p) = p$ and $\psi^2(p) = p$.

Let $\underline{w}$ be some non-zero vector. There are infinitely many rays emanating from p in direction $\underline{w}$: one for each horizontal rectangle with vertex congruent to p such that the vector $\underline{w}$ based at p points into the rectangle. Let $L_i(p, \underline{w})$ denote the ray which starts at horizontal rectangle number i. So $D(L_i(p, \underline{w})) = L_{i+1}(D(p), \underline{w})$.

Now suppose $\underline{w}$ is an eigenvector of $d\psi^n$ for some n. Then $d\psi^{2n}(\underline{w}) = \lambda \underline{w}$ with $\lambda > 0$, and $\psi^{2n}(p) = p$. It follows that $\psi^{2n}[L_i(p, \underline{w})] = L_j(p, \underline{w})$ for some $j = j(i)$. It is not difficult to see that $(j - i)/2n$ is independent of the choice of i and n.

Definition 2.10. *The* twist of $\underline{w}$ at p *is* $\tau_\psi(p, \underline{w}) := \frac{1}{2n}(j - i)$.

Lemma 2.11. *$\tau_\psi(p, \underline{w}) \in \frac{1}{2}\mathbb{Z}$. If ψ is hyperbolic with positive eigenvalues, then $\tau_\psi(p, \underline{w}) \in \mathbb{Z}$.*

Proof. Every eigenvector of $d\psi^n$ is an eigenvector of $d\psi$, so we can take $n = 1$, whence $\tau_\psi(p, \underline{w}) \in \frac{1}{2}\mathbb{Z}$. Now suppose in addition that $\lambda > 0$. If $\psi(p) = p$, then $\psi[L_i(p, \underline{w})] = L_{i+k}(p, \underline{w})$ for some integer k, and therefore $\psi^2[L_i(p, \underline{w})] = L_{i+2k}(p, \underline{w})$ and $\tau_\psi(p, \underline{w}) = k \in \mathbb{Z}$. If $\psi(p) \neq p$, then by homogeneity, $\phi := D \circ \psi$ fixes p, and by the previous line $\tau_\phi(p, \underline{w}) \in \mathbb{Z}$. So $\tau_\psi(p, \underline{w}) = \tau_\phi(p, \underline{w}) - 1 \in \mathbb{Z}$. □

Example 1. Let ψ be the homogeneous automorphism with derivative $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and drift $\frac{1}{2}$ (see figure 5). Let p :=lower left corner of horizontal rectangle #0. For any vector $\underline{w}$ with positive coordinates, $\psi^2[L_i(p, \underline{w})] = L_{i+1}(p, -\underline{w})$. So $\tau_\psi(p, \underline{w}) = \frac{1}{2}$.

Example 2. Let ψ denote the renormalizing automorphism of $\sqrt{2}$ with zero drift and derivative $\begin{pmatrix} 11 & -42 \\ -6 & 23 \end{pmatrix}$, and let $\underline{w} := \binom{1+2\sqrt{2}}{1}$ be its contracted eigenvector. Let p be one of the singularities at the bottom left corner of one of the horizontal rectangles, say rectangle #0. We show below (Theorem 6.3) that $\tau_\psi(p, \underline{w}) = 1$.

Lemma 2.12. *Suppose ψ is a hyperbolic homogeneous automorphism with zero drift and positive eigenvalues. Let p be a singularity, and $\underline{w}$ an eigenvector of $d\psi$, then $\tau_\psi(p, \underline{w})$ =minus the drift of ϕ, where ϕ is the unique homogeneous automorphism which fixes $L_0(p, \underline{w})$, and which has the same derivative as ψ.*

Proof. As in the proof of the previous lemma, there exist $k \in \mathbb{Z}$ and $\ell = 0,1$ s.t. $\psi[L_i(p,\underline{w})] = D^{\ell}[L_{i+k}(p,\underline{w})]$. Let $\phi := D^{-(k+\ell)} \circ \psi$, then ϕ fixes $L_i(p,\underline{w})$ and has the same derivative as ψ. This determines ϕ uniquely, because every other homogeneous automorphism with the same derivative has the form $D^n \circ \phi$ with $n \neq 0$. By the definition of k and ℓ, $\tau_\psi(p,\underline{w}) = k+\ell$. By the definition of ϕ and Proposition 2.2, $\delta(\phi) = \delta(\psi) - (k+\ell) = -(k+\ell)$. ☐

3. Estimates of Birkhoff sums

In this section we find pointwise asymptotic estimates for the Birkhoff sums of the cylinder map $T_\alpha : \mathbb{T}\times\mathbb{Z} \to \mathbb{T}\times\mathbb{Z}$

$$T_\alpha(x,t) = (x+\alpha (\text{mod } 1), t + f(x)),$$

where α is a quadratic irrational, and $f = 1_{[\frac{1}{2},1)} - 1_{[0,\frac{1}{2})}$.

By proposition 2.5, there is a hyperbolic homogeneous automorphism ψ with zero drift s.t. $\alpha = \frac{1}{2} + \frac{1}{2}\tan\theta (\text{mod } 1)$, where $\binom{\sin\theta}{\cos\theta}$ is an eigenvector of the derivative of ψ, with eigenvalue $0<\lambda<1$.

Recall that the infinite staircase is made from a $\mathbb{Z}$–array of 2×1 horizontal rectangles. Declare one of these rectangles to be "rectangle zero" and let $\omega : \mathbb{T} \to \mathrm{St}$ be the function which associates to $\omega(x)$ the unique point on the top horizontal side of rectangle zero, located $2x$ units of distance away from its left corner. In what follows $\log^* := \log_{\lambda^{-1}}$, $\xi : \mathrm{St} \to \mathbb{Z}$ is some (any) $\mathbb{Z}$–coordinate on the infinite staircase, and $C_c(Y) := \{$real continuous functions with compact support on $Y\}$.

Theorem 3.1. *There exists $\sigma > 0$ such that for every $(x,\ell) \in \mathbb{T}\times\mathbb{Z}$ for which $\frac{1}{k}\xi[\psi^k(\omega(x))] \xrightarrow[k\to\infty]{} 0$, for every non–negative $G \in C_c(\mathbb{T}\times\mathbb{Z})$,*

$$\sum_{i=0}^{n-1}(G\circ T_\alpha^i)(x,\ell) = \frac{[1+o(1)]n\int G dm_{\mathbb{T}\times\mathbb{Z}}}{2\sigma\sqrt{\pi\log^* n}}\cdot\sqrt{2}\exp\left[-\frac{1+o(1)}{2\sigma^2}\left(\frac{\xi[\psi^{[\log^* n]}(\omega(x))]}{\sqrt{\log^* n}}\right)^2\right].$$

The following uniformity holds: $\forall\varepsilon>0\ \exists\delta, N>0$ (which depend on G but not x) s.t. if $|\frac{1}{[\log^* n]}\xi[\psi^{[\log^* n]}(\omega(x))]|<\delta$ and $n>N$, then the $o(1)$ terms are in $[-\varepsilon,\varepsilon]$.

We will see in §4 that the condition $\frac{1}{k}\xi[\psi^k(\omega(x))] \xrightarrow[k\to\infty]{} 0$ holds almost everywhere. Thus Theorem (3.1) describes the almost sure behavior of Birkhoff sums for non-negative $G \in C_c(\mathbb{T}\times\mathbb{Z})$. By the ratio ergodic theorem, this is the almost sure behavior of every L^1 function with non-zero integral.

We will also see in §4 that $\frac{\xi[\psi^k(\omega(x))]}{\sqrt{k}} \xrightarrow[k\to\infty]{\text{dist}} N(0,\sigma^2)$ on $\mathbb{T}\times\{k\}$ $(k\in\mathbb{Z})$. Thus $\sum_{i=0}^{n-1} G\circ T_\alpha^i$ grows a.e. like a constant times $\frac{n}{\sqrt{\log n}}\int G$, but if we normalize by this growth rate, then we get fluctuating non-convergent behavior. The fluctuations are driven by the renormalizing automorphism, and happen on an exponential time scale. They are independent of G. Similar results were proved for horocycle flows on $\mathbb{Z}^d$ covers of hyperbolic surfaces of finite area in [**LS1**], [**LS2**], and for Hajian-Ito-Kakutani skew products in [**AS**].

We will obtain Theorem 3.1 from a study of the Birkhoff integrals of the linear flow in direction θ on the infinite staircase. Denote this flow by φ_θ. We will show:

Theorem 3.2. *There exists $\sigma > 0$ s.t. for every $\omega \in \mathrm{St}$ s.t. $\frac{1}{k}\xi[\psi^k(\omega)] \xrightarrow[k\to\infty]{} 0$, and for every $G \in C_c(\mathrm{St})$ such that $\int G dm > 0$ (m =non-normalized area measure),*

$$\int_0^n G[\varphi_\theta^t(\omega)]dt = \frac{1}{2} \cdot \frac{[1+o(1)]n\int Gdm}{2\sigma\sqrt{\pi \log^* n}} \cdot \sqrt{2}\exp\left[-\frac{1+o(1)}{2\sigma^2}\left(\frac{\xi[\psi^{[\log^* n]}(\omega)]}{\sqrt{\log^* n}}\right)^2\right].$$

The following uniformity holds: $\forall\varepsilon > 0\ \exists\delta, N > 0$ (which depend on G but not ω) s.t. if $|\frac{1}{[\log^* n]}\xi[\psi^{[\log^* n]}(\omega)]| < \delta$ and $n > N$, then the $o(1)$ terms are in $[-\varepsilon, \varepsilon]$.

The extra $\frac{1}{2}$ in the asymptotic expansion for $\int_0^n G\circ\varphi_\theta^t dt$ is because $\frac{m([0,2]\times[0,1])}{m_{\mathbb{T}\times\mathbb{Z}}(\mathbb{T}\times\{0\})} = 2$.

Notation. Let $\psi_0 : \mathrm{St}_0 \to \mathrm{St}_0$ denote the projection of $\psi : \mathrm{St} \to \mathrm{St}$ to the covered torus St_0, and let $\mathfrak{P}$ denote the Adler–Weiss Markov partition of ψ_0.

Let $\underline{v} = \binom{1}{v}$ and $\underline{w} = \binom{1}{w}$ denote eigenvectors of the derivative of ψ with eigenvalues λ^{-1} and λ. They define the *unstable* and *stable* directions.

Linear segments in St or St_0 in the direction of $\pm\underline{w}$ will be called *stable*. For example, $\{\varphi_\theta^t(\omega_0) : a < t < b\}$ is a stable linear segment in St.

The following is a particularly useful way to generate stable linear segments. Let $\pi_0 : \Sigma(\mathscr{G}) \to \mathrm{St}_0$ denote the Adler-Weiss coding map of Theorem 2.8. Let $\Sigma^+ := \{(x_0, x_1, \dots) \in \mathfrak{P}^{\mathbb{N}} : x_i \to x_{i+1} \text{ for all } i \geq 0\}$. For every $\underline{x} \in \Sigma^+$, let

$$W^s(\underline{x}) := \pi_0\{\underline{y} \in \Sigma(\mathscr{G}) : y_i = x_i \quad (i \geq 0)\}.$$

Lemma 3.3. *$W^s(\underline{x})$ is a stable linear segment. It is the s–fibre through $\pi_0(\underline{x})$ in rectangle $x_0 \in \mathfrak{P}$. Let $h(\underline{x}) := \ell^s(x_0)$ be its length, then $\sum_{\sigma(\underline{y})=\underline{x}} h(\underline{y}) = \lambda^{-1}h(\underline{x})$, where the sum ranges over $\underline{y} \in \Sigma^+$ and $\sigma(y_0, y_1, \dots) := (y_1, y_2, \dots)$.*

Proof. By the Markov property of ψ_0, $C_n := x_0 \cap \psi_0^{-1}(x_1) \cap \cdots \cap \psi_0^{-(n-1)}(x_{n-1})$ is a decreasing intersection of compact parallelograms with s–side of length $\ell^s(x_0)$ and u–side of length $O(\lambda^n)$. So $\bigcap_{n\geq 0} C_n$ is an s–fibre in x_0, which passes through $\pi_0(\underline{x})$. This is a stable linear segment.

The Markov property also implies that $\psi_0^{-1}[W^s(\underline{x})] = \bigcup_{\sigma(\underline{y})=\underline{x}} W^s(\underline{y})$. Since ψ_0 contracts s–fibres linearly by factor λ, $\sum_{\sigma(\underline{y})=\underline{x}} h(\underline{y}) = \lambda^{-1}h(\underline{x})$. □

Let $\xi : \mathrm{St} \to \mathbb{Z}$ be the $\mathbb{Z}$–coordinate associated to ψ (Figure 3). We write $[\xi = k] := \xi^{-1}\{k\}$ and $W^s(\underline{x}, k)$:=lift of $W^s(\underline{x})$ to St so that $\pi_0(\underline{x})$ lifts to a point in $[\xi = k]$. This is a stable segment in St, and it has length $h(x_0)$. $W^s(\underline{x}, k) \subseteq [\xi = k]$, because $W^s(\underline{x})$ lies completely inside an element of $\mathfrak{P}$, and such sets lift in their entirety to subsets of $D^i(F)(i \in \mathbb{Z})$ where F is the fundamental polygon of ψ_0.

Proof of Theorem 3.2. We begin with some reductions.

Any two $\mathbb{Z}$–coordinates are within uniformly bounded distance from one another, therefore if the theorem holds with one choice of a $\mathbb{Z}$–coordinate, then it holds with all other possible choices. We will work with the $\mathbb{Z}$–coordinate associated to ψ.

With this choice of ξ, the Frobenius function F_ψ is either $\mathfrak{P}\vee\psi_0^{-1}(\mathfrak{P})$–measurable, or $\mathfrak{P} \vee \psi_0(\mathfrak{P})$–measurable. We will carry out the proof in the first case, and leave to the reader the (routine) modifications needed for the second case.

A *symbolic cylinder* is a set of the form ${}_\ell[P_\ell, \dots, P_{\ell'}] := \bigcap_{i=\ell}^{\ell'} \psi_0^{-i}(P_i)$, where $P_i \in \mathfrak{P}$. This is a parallelogram with sides parallel to $\underline{v}$ and $\underline{w}$. Symbolic cylinders are subsets of St_0. They are *not* necessarily cylinders in the geometric sense.

Instead of working with $G \in C_c(\mathrm{St})$, we will work with indicators of lifts of symbolic cylinders to St_0. Any non–negative continuous function with compact support can be sandwiched between linear combinations of such functions, so this suffices for our purposes.

Here is the precise definition of the sets which we will work with:

$$ {}_{\ell}[P_\ell, \ldots, P_{\ell'}]^k := \text{lift to } \{\xi = k\} \text{ of } {}_{\ell}[P_\ell, \ldots, P_{\ell'}] := \bigcap_{i=\ell}^{\ell'} \psi_0^{-i}(P_i) \ . $$

Here $\ell' > \ell$ and $P_{\ell'}, \ldots, P_\ell \in \mathfrak{P}$ are arbitrary.

Most of our calculations will be done in the special case $\ell = k = 0$. This is enough, because $\exists i, j$ s.t. ${}_{\ell}[P_\ell, \ldots, P_{\ell'}]^k = (D^i \circ \psi^j)({}_0[P_\ell, \ldots, P_{\ell'}]^0)$ where D is a deck transformation. Since $D^i \circ \psi^j$ preserves the area measure and does not affect the asymptotic drift $\lim \xi[\psi^n(\omega)]/n$, whatever works for the special case $\ell = k = 0$ works in general.

Similarly we may assume without loss of generality that $\xi(\omega) = 0$. From now on, fix $\omega \in \mathrm{St}$ s.t. $\xi(\omega) = 0$ and $\xi(\psi^n(\omega))/n \to 0$, and let

$$ E := {}_0[P_0, \ldots, P_{\ell-1}]^0. $$

Our aim is to find the asymptotic behavior of $\int_0^n 1_E[\varphi_\theta^t(\omega)]dt$ as $n \to \infty$.

In what follows $\ell[\cdot]$ is the euclidean length measure, and $n_0 \in \mathbb{N}$ is a free parameter that will be calibrated at the end of the proof. For every n, let

$$ n^* := \lceil \log^*(n/n_0) \rceil. $$

Notice that $\lambda^{n^*} \cdot n \in [\lambda n_0, n_0]$.

Let $A_n(\omega) := \{\varphi_\theta^t(\omega) : 0 < t < n\}$. This a stable linear segment, and we are interested in $\int_0^n 1_E[\varphi_\theta^t(\omega)]dt = \ell[A_n(\omega) \cap E]$.

Let $B_n(\omega) := \psi^{n^*}[A_n(\omega)]$. Since ψ contracts stable linear segments by factor λ, $B_n(\omega)$ is a stable linear segment with length $\ell[B_n(\omega)] \in [\lambda n_0, n_0]$. Break $B_n(\omega)$ into a finite union of lifted s–fibres $W^s(\underline{x}^{(1)}, \xi_1^*), \ldots, W^s(\underline{x}^{(n_1)}, \xi_{n_1}^*)$ plus two pieces of stable fibres $W^s(\underline{x}^{(0)}, \xi_0^*)$, $W^s(\underline{x}^{(n_1+1)}, \xi_{n_1+1}^*)$ to take care of edge effects:

$$ \biguplus_{i=1}^{n_1} W^s(\underline{x}^{(i)}, \xi_i^*) \subseteq B_n(\omega) \subseteq \biguplus_{i=0}^{n_1+1} W^s(\underline{x}^{(i)}, \xi_i^*). \tag{3.1} $$

Even though $n_1, \underline{x}^{(i)}$ and ξ_i^* depend on n, some uniformities are observed:

(1) $\frac{\lambda n_0}{\max h} - 2 \leq n_1 \leq \frac{n_0}{\lambda \min h}$ ($\because \lambda n_0 \leq \ell[B_n] \leq n_0$, $\ell[W^s(\underline{x}^{(i)}, \xi_i^*)] = h(x_0^{(i)})$).

(2) $|\xi_i^* - \xi(\psi^{n^*}(\omega))| < \frac{n_0}{\min h}$ for all i, because $\xi_0^*, \ldots, \xi_{n_1+1}^*, \xi(\psi^{n^*}(\omega))$ are $\mathbb{Z}$ coordinates of points in $B_n(\omega)$, $\ell[B_n(\omega)] \leq n_0$, and because it takes at least $\min h$ units of distance to cross the fundamental polygon of ψ when moving in the stable direction.

By the definition of $B_n(\omega)$, $\int_0^n 1_E(\varphi_\theta^t(\omega))dt = \ell[E \cap \psi^{-n^*}(B_n(\omega))]$, so (3.1) translates to

$$ \sum_{i=1}^{n_1} J_n(\underline{x}^{(i)}, \xi_i^*) \leq \int_0^n 1_E[\varphi_\theta^t(\omega)]dt \leq \sum_{i=0}^{n_1+1} J_n(\underline{x}^{(i)}, \xi_i^*), \tag{3.2} $$

where $J_n(\underline{x}^{(i)}, \xi_i^*) := \ell[E \cap \psi^{-n^*}(W^s(\underline{x}^{(i)}, \xi_i^*))]$. The remainder of the proof is an analysis of $J_n(\underline{x}^{(i)}, \xi_i^*)$.

We start by asking when does a point $\omega' \in W^s(\underline{x}^{(i)}, \xi_i^*)$ belong to $\psi^{n^*}(E)$. We claim that $\psi^{-n^*}(\omega') \in E$ iff $\psi_0^{-n^*}[\pi(\omega')] \in \pi(E)$ and $\big(\sum_{j=0}^{n^*-1} F_\psi \circ \psi_0^j\big)[\psi_0^{-n^*}(\pi(\omega'))] = \xi_i^*$, where π is the covering $\mathrm{St} \to \mathrm{St}_0$.

Explanation: By the definition of the Frobenius function F_ψ, if $\omega' \in W^s(\underline{x}^{(i)}, \xi_i^*)$, then $\xi_i^* - \xi[\psi^{-n^*}(\omega')] = \xi(\omega') - \xi[\psi^{-n^*}(\omega')] \equiv F_\psi[\psi_0^{-n^*}(\pi(\omega'))] + \cdots + F_\psi[\psi_0^{-1}(\pi(\omega'))]$. It follows that $\xi[\psi^{-n^*}(\omega')] = 0 \Leftrightarrow \big(\sum_{j=0}^{n^*-1} F_\psi \circ \psi_0^j\big)[\psi_0^{-n^*}(\pi(\omega'))] = \xi_i^*$.

Writing $\omega'' := \psi_0^{-n^*}[\pi(\omega')]$ (a point in St_0), we see that

$$\begin{aligned} J_n(\underline{x}^{(i)}, \xi_i^*) = \ell\{\omega'' \in {}_0[P_0, \ldots, P_{\ell-1}] : \psi_0^{n^*}(\omega'') \in W^s(\underline{x}^{(i)}), \\ \text{and } \sum_{j=0}^{n^*-1} F_\psi[\psi_0^j(\omega'')] = \xi_i^*\}. \end{aligned} \tag{3.3}$$

We write this in more convenient form. Let $\sigma : \Sigma^+ \to \Sigma^+$ denote the one–sided shift defined before Lemma 3.3. The assumption that F_ψ is $\mathfrak{P} \vee \psi_0^{-1}\mathfrak{P}$–measurable allows us to view $F := F_\psi \circ \pi_0$ as a function on Σ^+, $F(\underline{x}) = g(x_0, x_1)$. By the Markov property, $\psi_0^{-n^*}[W^s(\underline{x}^{(i)})] = \biguplus_{\sigma^{n^*}(\underline{y}) = \underline{x}^{(i)}} W^s(\underline{y})$ mod Lebesgue, and since $F(\underline{y}) = g(y_0, y_1)$, $F_{n^*}(\underline{y}) := F(\underline{y}) + F(\sigma(\underline{y})) + \cdots + F(\sigma^{n^*-1}(\underline{y}))$ is constant on $W^s(\underline{y})$. It follows that

$$J_n(\underline{x}^{(i)}, \xi_i^*) = \sum_{\sigma^{n^*}(\underline{y}) = \underline{x}^{(i)}} h(y_0) 1_{[\underline{P}]}(\underline{y}) \delta_0(F_{n^*}(\underline{y}) - \xi_i^*).$$

Here $h(y_0)$ is the length of the stable side of the parallelogram y_0, $1_{[\underline{P}]}(\underline{y})$ equals one when $(y_0, \ldots, y_{\ell-1}) = (P_0, \ldots, P_{\ell-1})$ and zero otherwise, and $\delta_0(k)$ equals one if $k = 0$ and zero otherwise.

We will use the methods of Babillot & Ledrappier [**BL1**],[**BL2**] to estimate this sum. Given $w \in \mathbb{R}/2\pi\mathbb{Z}, u \in \mathbb{R}$, let $(L_{u+iw}\varphi)(\underline{x}) = \sum_{\sigma(\underline{y}) = \underline{x}} e^{(u+iw)F(\underline{y})}\varphi(\underline{y})$. This is an operator on $\mathscr{L} := \{\varphi : \Sigma^+ \to \mathbb{C} : \|\varphi\| := \|\varphi\|_\infty + \mathrm{Lip}(\varphi) < \infty\}$, where $\mathrm{Lip}(\varphi)$ is the best Lipschitz constant of φ. For all u,

$$\begin{aligned} J_n(\underline{x}^{(i)}, \xi_i^*) &= h(P_0) \sum_{\sigma^n(\underline{y}) = \underline{x}^{(i)}} 1_{[\underline{P}]}(\underline{y}) \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{(u+iw)(F_{n^*}(\underline{y}) - \xi_i^*)} dw \\ &= \frac{h(p_0)}{2\pi} \int_{-\pi}^{\pi} e^{-(u+iw)\xi_i^*} (L_{u+iw}^{n^*} 1_{[\underline{P}]})(\underline{x}^{(i)}) dw. \end{aligned} \tag{3.4}$$

The parameter u does not affect the value of the integral, but a judicious choice $u = u(\xi_i^*, n^*)$ will facilitate the analysis of the integrand.

$L_z : \mathscr{L} \to \mathscr{L}$ ($z = u + iw$) has the following properties ([**PP**] chapter 4):

(1) L_0 has leading eigenvalue λ^{-1}, with eigenprojection $P\varphi = h\nu(\varphi)$ where h is given by Lemma 3.3, and ν satisfies $h d\nu = \widehat{m}_0$ (cf. Theorem 2.8).
(2) The eigenvalue λ^{-1} is simple and isolated. All other eigenvalues have strictly smaller absolute value.
(3) For all u real, L_u has spectral radius $\exp p(u)$ where

$$p(u) = P_{top}(uF) := \sup\left\{ h_\mu(\sigma) + u \int F d\mu : \begin{array}{l} \mu \text{ is a } \sigma\text{–invariant} \\ \text{probability measure} \end{array} \right\}.$$

(4) For all u, w real, $w \notin 2\pi\mathbb{Z}$, L_{u+iw} has spectral radius strictly smaller than $\exp p(u)$. This uses the Aperiodicity Lemma (Lemma 2.9).
(5) There is $\varepsilon_{pert} > 0$ such that for every $|z| < \varepsilon_{pert}$, $L_z = \lambda(z)[P(z) + N(z)]$ where $\lambda(z) \in \mathbb{C}$, $P(z)$ is a projection with one–dimensional image, $N(z)$ is an operator with spectral radius strictly less than one s.t. $PN = NP = 0$, and $z \mapsto \lambda(z), P(z), N(z)$ are analytic on $\{z : |z| < \varepsilon_{pert}\}$.
(6) $p(z) := \log \lambda(z)$ is an analytic extension of $p(u)$ to $U = \{z : |z| < \varepsilon_{pert}\}$. On U, $p(z) = -\log \lambda + \frac{1}{2}\sigma^2 z^2 + o(z^2)$, where $\sigma > 0$. σ does not vanish because of the Aperiodicity Lemma, see [**PP**, Prop. 4.12].

Part (6) implies that the image of $p'(\cdot)$ is a neighborhood of zero. Suppose $\frac{\xi_i^*}{n^*}$ belongs to this neighborhood, and choose u s.t. $p'(u) = \frac{\xi_i^*}{n^*}$. The closer $\frac{\xi_i^*}{n^*}$ is to zero, the closer u is to zero. Since, by construction, $\xi_i^* = \xi(\psi^{n^*}(\omega)) + O(1)$, there exists $\varepsilon_0 > 0$ so small and n_0 so large that for all $n^* > n_0$

$$\left|\frac{\xi(\psi^{n^*}(\omega))}{n^*}\right| < \varepsilon_0 \Longrightarrow |u| < \varepsilon_{pert}.$$

The condition will be satisfied for all n large enough, because of the assumption that $\xi[\psi^k(\omega)]/k \xrightarrow[k\to\infty]{} 0$. Henceforth we assume that $\left|\frac{\xi(\psi^{n^*}(\omega))}{n^*}\right| < \varepsilon_0$ and take $|u| < \varepsilon_{pert}$ s.t. $p'(u) = \frac{\xi_i^*}{n^*}$.

Let $\rho(L_{u+iw})$ denote the spectral radius of L_{u+iw}. Since $u + iw \mapsto L_{u+iw}$ is continuous, $u + iw \mapsto \rho(u + iw)$ is upper semi-continuous. Therefore, by part (4), there exists $0 < \kappa < 1$ s.t. $\sup\{e^{-p(u)}\rho(L_{u+iw}) : \mathrm{dist}(w, 2\pi\mathbb{Z}) > \varepsilon_{pert}\} < \kappa$.

Similar reasoning gives (perhaps for a slightly larger $0 < \kappa < 1$)

$$\sup\{|e^{-p(u)}\rho(N(u + iw))| : |u + iw| \le \varepsilon_{pert}\} < \kappa.$$

It is not difficult to see, using the spectral radius formula and the continuity of $z \mapsto L_z$, that $\|L_{u+iw}^{n^*} 1_{[\underline{p}]}\| = O(e^{n^* p(u)}\kappa^{n^*})$ uniformly on $\{w \in (-\pi, \pi) : |w| \ge \varepsilon_{pert}\}$, and $\|N(u + iw)^{n^*} 1_{[\underline{p}]}\| = O(e^{n^* p(u)}\kappa^{n^*})$ uniformly on $(-\varepsilon_{pert}, \varepsilon_{pert})$.

If we split the domain of integration in (3.4) into $(-\varepsilon_{pert}, \varepsilon_{pert})$ and its complement and then substitute $L = \lambda(P + N)$ into the first piece, then we get the following (where $J_n = J_n(\underline{x}^{(i)}, \xi_i^*)$, $\underline{x} = \underline{x}^{(i)}$, $\xi^* = \xi_i^*$):

$$J_n = \frac{h(P_0)}{2\pi} \int_{-\varepsilon_{pert}}^{\varepsilon_{pert}} e^{-(u+iw)\xi^*} \left[\lambda(u + iw)^{n^*} (P(u + iw)1_{[\underline{P}]})(\underline{x})\right] dw \\ + O(e^{n^* p(u) - u\xi^*}\kappa^{n^*}).$$

The error bound can be simplified using the Legendre transform. Let $H(v)$ denote minus the Legendre transform of $p(u)$: $H(v) := p(u) - up'(u)$ for the $u = u(v)$ s.t. $p'(u) = v$. By the choice of u, $n^* p(u) - u\xi^* = n^* H(\xi^*/n^*)$, whence

$$J_n = \frac{h(P_0)}{2\pi} \int_{-\varepsilon_{pert}}^{\varepsilon_{pert}} e^{-(u+iw)\xi^* + n^* p(u+iw)} (P(u + iw)1_{[\underline{P}]})(\underline{x}) dw + O(e^{n^* H(\frac{\xi^*}{n^*})}\kappa^{n^*}).$$

The next step is to use the Taylor expansion of $p(z)$ at $z = u$ to see that the exponential term in the integrand equals

$$e^{n^*[p(u) - u\frac{\xi^*}{n^*}]} \cdot e^{in^* w[p'(u) - \frac{\xi^*}{n^*}]} \cdot e^{n^*[-\frac{1}{2}p''(u)w^2 + O(w^3)]}.$$

The first term is $\exp[n^* H(\frac{\xi^*}{n^*})]$, and the second term is 1 by the choice of u. So

$$\begin{aligned} J_n &= \frac{e^{n^* H(\frac{\xi^*}{n^*})} h(P_0)}{2\pi} \left[\int_{-\varepsilon_{pert}}^{\varepsilon_{pert}} e^{-n^*[\frac{1}{2}p''(u)w^2 + O(w^3)]} (P(u+iw)1_{[\underline{P}]})(\underline{x}) dw + O(\kappa^{n^*}) \right] \\ &= \frac{e^{n^* H(\frac{\xi^*}{n^*})} h(P_0)}{2\pi} \left[\int_{-\varepsilon_{pert}\sqrt{n^*}}^{\varepsilon_{pert}\sqrt{n^*}} e^{-\frac{1}{2}p''(u)v^2 + O(\frac{v^3}{\sqrt{n^*}})} (P(u + \frac{iv}{\sqrt{n^*}})1_{[\underline{P}]})(\underline{x}) \frac{dv}{\sqrt{n^*}} + O(\kappa^{n^*}) \right]. \end{aligned}$$

We discuss the asymptotic behavior of this expression as $n^* \to \infty$, subject to the assumption that $\frac{1}{n^*}\xi[\psi^{n^*}(\omega)] \to 0$. Since $\xi^* \equiv \xi_i^* = \xi[\psi^{n^*}(\omega)] + O(1)$,

$$\frac{\xi^*}{n^*} \xrightarrow[n^* \to \infty]{} 0, \text{ and therefore } u \xrightarrow[n^* \to \infty]{} 0.$$

Recall the definition of the eigenprojections P, $P(z)$ of L_0, L_z. Since $\|P(z) - P\| \xrightarrow[|z| \to 0]{} 0$ and $P(0)1_{[\underline{P}]} = P1_{[\underline{P}]} = h\nu[\underline{P}]$ is bounded away from zero,

$$(P(u+i\tfrac{v}{\sqrt{n^*}})1_{[\underline{P}]})(\underline{x}) = [1+o(1)]h(\underline{x})\nu[\underline{P}] = [1+o(1)]\ell[W^s(\underline{x})]\nu[\underline{P}] \text{ unif. as } n^* \to \infty.$$

(But caution! $\underline{x} = \underline{x}^{(i)}$ varies as $n^* \to \infty$ so the term on the right side fluctuates.)

If ε_{pert} and $|u|$ are small enough then $|p''(u)| > \frac{1}{2}p''(0) = \frac{1}{2}\sigma^2$ and $|O(w^3)| \le \frac{1}{8}\sigma^2|w|^2$ for $|w| < \varepsilon_{pert}$. We see that the exponential term is bounded by const $\cdot e^{-\frac{1}{8}v^2}$. By the dominated convergence theorem,

$$\begin{aligned} J_n &= [1+o(1)] \frac{e^{n^* H(\frac{\xi^*}{n^*})}}{2\pi} h(P_0)\nu[\underline{P}]\ell[W^s(\underline{x}^i)] \left[\frac{1}{\sqrt{n^*}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\sigma^2 v^2} dv + O(\kappa^{n^*}) \right] \\ &= [1+o(1)] \frac{e^{n^* H(\frac{\xi^*}{n^*})}}{\sqrt{2\pi\sigma^2 n^*}} h(P_0)\nu[\underline{P}]\ell[W^s(\underline{x}^i)] = [1+o(1)] \frac{e^{n^* H(\frac{\xi^*}{n^*})}}{\sqrt{2\pi\sigma^2 n^*}} m(E)\ell[W^s(\underline{x}^i)]. \end{aligned}$$

Notice that $h(P_0)\nu[\underline{P}] = \widehat{m}_0[\underline{P}] = m_0(\pi(E)) = \frac{1}{2}m(E)$, where m_0 is the normalized area measure on St_0 and m is the non–normalized area measure on St.

We analyze $n^* H(\frac{\xi^*}{n^*})$. Since $H(\cdot)$ is minus the Legendre transform of $p(\cdot)$ and $p(z) = -\log\lambda + \frac{1}{2}\sigma^2 z^2 + o(z^2)$, $H(v) = -\log\lambda - \frac{v^2}{2\sigma^2} + o(v^2)$. In particular $H'(0) = 0$ and $H''(0) = -\frac{1}{\sigma^2}$. Recalling that $\xi^* = \xi[\psi^{n^*}(\omega)] + O(1)$ and expanding $H(u)$ around $u_0 = \frac{\xi[\psi^{n^*}(\omega)]}{n^*}$, we obtain

$$\begin{aligned} n^* H(\tfrac{\xi^*}{n^*}) &= n^* \left[H(\tfrac{\xi[\psi^{n^*}(\omega)]}{n^*}) + H'(\tfrac{\xi[\psi^{n^*}(\omega)]}{n^*}) \tfrac{\xi^* - \xi[\psi^{n^*}(\omega)]}{n^*} + o(\tfrac{\xi^* - \xi[\psi^{n^*}(\omega)]}{n^*}) \right] \\ &= n^* H(\tfrac{\xi[\psi^{n^*}(\omega)]}{n^*}) + n^*[H'(0) + o(1)] \frac{O(1)}{n^*} + n^* o\left(\frac{O(1)}{n^*}\right) \quad (\because \tfrac{\xi[\psi^{n^*}(\omega)]}{n^*} \to 0) \\ &= n^* H(\tfrac{\xi[\psi^{n^*}(\omega)]}{n^*}) + o(1) \quad (\because H'(0) = 0). \end{aligned}$$

Now we expand H around zero to see that

$$n^* H(\tfrac{\xi^*}{n^*}) = -n^* \log\lambda - \frac{1}{2\sigma^2}[1+o(1)] \left(\frac{\xi[\psi^{n^*}(\omega)]}{\sqrt{n^*}} \right)^2 + o(1).$$

This and the definition of n^* give

$$J_n(\underline{x}^{(i)},\xi_i^*) = [1+o(1)]\frac{\lambda^{-n^*}m(E)\ell[W^s(\underline{x}^i)]}{2\sqrt{\log_{\lambda^{-1}} n}}\times$$
$$\times\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left[-\frac{1}{2\sigma^2}[1+o(1)]\left(\frac{\xi[\psi^{n^*}(\omega)]}{\sqrt{n^*}}\right)^2\right].$$

By (3.2), the sum of these expressions over $i=1,\ldots,n_1$ gives a lower bound for $\int_0^n 1_E[\varphi_\theta^t(\omega)]dt$, and the sum over $0,\ldots,n_1+1$ gives an upper bound. The only term which depends on i is $\ell[W^s(\underline{x}^{(i)})]$. Since by (3.1),

$$\ell[B_n(\omega)] - 2\max h \le \sum_{i=1}^{n_1}\ell[W^s(\underline{x}^{(i)})] \le \sum_{i=0}^{n_1+1}\ell[W^s(\underline{x}^{(i)})] \le \ell[B_n(\omega)] + 2\max h,$$

and since both sides are $\ell[B_n(\omega)][1+O(\frac{1}{n_0})] = \lambda^{n^*}n[1+O(\frac{1}{n_0})]$, we have

$$\int_0^n 1_E[\varphi_\theta^t(\omega)]dt \le \frac{n(1+o(1))(1+O(\frac{1}{n_0}))}{2\sqrt{\log_{\lambda^{-1}} n}}m(E)\cdot\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{1+o(1)}{2\sigma^2}\left(\frac{\xi[\psi^{n^*}(\omega)]}{\sqrt{n^*}}\right)^2},$$
$$\int_0^n 1_E[\varphi_\theta^t(\omega)]dt \ge \frac{n(1+o(1))(1+O(\frac{1}{n_0}))}{2\sqrt{\log_{\lambda^{-1}} n}}m(E)\cdot\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{1+o(1)}{2\sigma^2}\left(\frac{\xi[\psi^{n^*}(\omega)]}{\sqrt{n^*}}\right)^2}.$$

We now remember that n_0 is a free parameter, and can be chosen arbitrarily large. The asymptotic expansion of the theorem follows in the case $G = 1_E$.

The case $G \in C_c(S)$ is treated by decomposing $G = G^+ - G^-$ with $G^\pm \in C_c(S)$ non–negative, and approximating $G^\pm$ from above and below by linear combinations of indicators of symbolic cylinders. □

Proof of Theorem 3.1. It is enough to prove the asymptotic statement in case $G(x,k) = \gamma(x)1_{\mathbb{T}\times\{0\}}(x,k)$ with $\gamma \in C(\mathbb{T})$, $\int\gamma(t)dt > 0$. The case $G(x,k) = \gamma(x)1_{\mathbb{T}\times\{k_0\}}(x,k)$ for $k_0 \neq 0$ is similar, and the general case $G \in C_c(\mathbb{T}\times\mathbb{Z})$ follows by linear combinations.

The infinite staircase can be decomposed into an infinite collection of horizontal 2×1 rectangles. Fix one of them, calling it "rectangle zero", and identify it with $[0,2]\times[0,1]$. Define $\widetilde{G}$ on rectangle zero by

$$\widetilde{G}(x',y') = \pi\cos\theta\cdot\gamma(\tfrac{1}{2}(x'-y'\tan\theta))\cdot\sin(\pi y'),$$

then $\int_0^{1/\cos\theta}(\widetilde{G}\circ\varphi_\theta^t)(\omega(x))dt = G(x,0)$. The upper limit $1/\cos\theta$ is the time it takes $\varphi_\theta^t(\omega(x))$ to reach the upper side of $[0,2]\times[0,1]$.

Extend $\widetilde{G}$ to the rest of the infinite staircase surface by setting it equal to zero outside rectangle zero. Since $\widetilde{G}(x'+2,y') = \widetilde{G}(x',y')$ and $\widetilde{G}(*,0) = \widetilde{G}(*,1) = 0$, this is a continuous function. A calculation shows that $\int\widetilde{G}dm = 2\cos\theta\int_{\mathbb{T}\times\mathbb{Z}}Gdm_{\mathbb{T}\times\mathbb{Z}}$, where m is the non-normalized area measure on St.

The orbit $\{\varphi_\theta^t(\omega(x)) : 0 < t < n/\cos\theta\}$ can be split into segments of length $1/\cos\theta$ which go across horizontal rectangles. The j–th segment enters the bottom side of rectangle $\sum_{i=0}^{j-1}f(x+i\alpha)$ at distance $2x+2j\alpha \mod 2$ from the left endpoint. Only the segments s.t. $\sum_{i=0}^{j-1}f(x+i\alpha) = 0$ contribute to $\int_0^{n/\cos\theta}\widetilde{G}[\varphi_\theta^t(\omega(x))]dt$. The contribution is $G(x+j\alpha,0) = (G\circ T_\alpha^j)(x,0)$.

It follows that $\int_0^{n/\cos\theta} \widetilde{G}(\varphi_\theta^t(\omega(x)))dt = \sum_{j=0}^{n-1} G\big(x + j\alpha, \sum_{i=1}^{j-1} f(x + i\alpha)\big) = \sum_{j=0}^{n-1}(G \circ T^j)(x, 0)$. The theorem now follows from Theorem 3.2. □

4. Stochastic properties of Birkhoff sums

Theorem 3.1 expresses the Birkhoff sums of the cylinder map T_α asymptotically in terms of $\frac{1}{\sqrt{k}}(\Xi_k(x))$ where $\Xi_k(x) := \xi[\psi^k(\omega(x))]$, ψ is a renormalizing automorphism of α with zero drift, ξ is its associated $\mathbb{Z}$–coordinate, and $\omega : \mathbb{T} \to \mathrm{St}$ is the map which associates to $x \in \mathbb{T}$ the point on the top side of a (fixed) horizontal rectangle at distance $2x$ from its left endpoint.

Thus the stochastic behavior of the Birkhoff sums of the cylinder map is determined by the stochastic process $\{\Xi_k(x)\}_{k\geq 1}$, when x is chosen uniformly in $[0, 1]$. In this section we prove the following.

Theorem 4.1. *Choose $x \in [0, 1]$ uniformly, then*

(1) $\Xi_k/k \xrightarrow[k\to\infty]{} 0$ *a.e.*

(2) $\forall\varepsilon > 0\ \exists I(\varepsilon) > 0$ *s.t.* $\mathbb{P}\big[|\Xi_k/k| > \varepsilon\big] = O(e^{-kI(\varepsilon)})$ $(k \to \infty)$.

(3) $\Xi_k/\sqrt{k} \xrightarrow[k\to\infty]{\text{dist}} N(0, \sigma^2)$. *Moreover, there is a probability space $(\Omega, \mathscr{F}, \mu)$ equipped with two continuous time stochastic processes $\widetilde{\Xi}_t, \widetilde{B}_t : \Omega \to \mathbb{R}$ s.t. $\{\widetilde{\Xi}_n\}_{n\geq 1} \overset{dist}{=} \{\Xi_n\}_{n\geq 1}$, $\{\widetilde{B}_t\}_{t\geq 0} \overset{dist}{=}$ standard Brownian motion, and for some $0 < \delta < \frac{1}{2}$, $|\widetilde{\Xi}_t - \sigma\widetilde{B}_t| = o(t^\delta)$ a.s. as $t \to \infty$.*

(4) *If $f, \widehat{f} \in L^1(\mathbb{R})$, then $\lim\limits_{n\to\infty} \frac{1}{\ln n} \sum_{k=1}^n \frac{1}{k} f\big(\Xi_k/\sqrt{k}\big) = \mathbb{E}[f(N)]$ almost surely, where N is the standard gaussian, and $\widehat{f}$ is the Fourier transform of f.*

Lemma 4.2. *There are a stationary mixing Markov chain $\{X_i\}_{i=1}^\infty$ with finite set of states S, $g : S \times S \to \mathbb{R}$ s.t. $\mathbb{E}[g(X_0, X_1)] = 0$, and a uniformly bounded sequence of random variables ε_k s.t. $\Xi_k \overset{dist}{=} g(X_0, X_1) + \cdots + g(X_{k-1}, X_k) + \varepsilon_k$ (equality of stochastic processes). There is no Borel function H s.t. $g(x_0, x_1) = H - H \circ \sigma + \mathrm{const}$.*

Proof. Define $\xi_k : \mathrm{St}_0 \to \mathbb{Z}$ as follows: given $p \in \mathrm{St}_0$,

$$\xi_k(p) := \xi[\psi^k(\widetilde{p})] - \xi(\widetilde{p}) \text{ for some (all) } \widetilde{p} \in \pi^{-1}(p).$$

This can be easily seen to be independent of the choice of $\widetilde{p}$.

Next define $x : \mathrm{St}_0 \to [0, 1]$ as follows: given $p \in \mathrm{St}_0$, lift p to a point $\widetilde{p} \in \mathrm{St}$ in rectangle #0, and project $\widetilde{p}$ to the top side of this rectangle in the stable direction. The result has the form $\omega(x)$ for some unique $x = x(p) \in [0, 1]$.

Claim. *If p is chosen uniformly in St_0, then $x(p)$ is distributed uniformly in $[0, 1]$, and $\varepsilon_k(p) := \xi_k(p) - \xi[\psi^k(\omega(x(p)))]$ are uniformly bounded on St_0.*

The first statement is because rectangle zero is congruent to the parallelogram with a horizontal side of length 2 and a side in the stable direction. The second statement is because $\widetilde{p} - \omega(x) \propto \underline{w}$ where $\underline{w}$ is in the stable direction of the derivative of ψ, so $\mathrm{dist}(\psi^k(\widetilde{p}), \psi^k[\omega(x)]) \leq \lambda^k\sqrt{1 + \tan^2\theta} \leq 1/\cos\theta$.

It follows that $\xi[\psi^k(\omega(x))] \overset{\text{dist}}{=} \xi_k + \varepsilon_k$, where $|\varepsilon_k| \leq 1/\cos\theta$ and ξ_k is the stochastic process

$$\xi_k(p) := \xi[\psi^k(p)], \text{ where } p \text{ is distributed uniformly in } \mathrm{St}_0.$$

We will use the Adler–Weiss Theorem to represent ξ_k as a random walk driven by a Markov chain.

Let $\mathfrak{P}$ denote the Adler–Weiss Markov partition, and $\mathscr{G}$ the dynamical graph of $\mathfrak{P}$, see §2. Let $\pi_0 : \Sigma(\mathscr{G}) \to \mathrm{St}_0$ denote the symbolic coding of the projected automorphism ψ_0, given by Theorem 2.8, then $m_0 = \widehat{m}_0 \circ \pi_0^{-1}$ where $\widehat{m}_0$ is a mixing shift invariant Markov measure. So $X_k : \Sigma(\mathscr{G}) \to \mathfrak{P}$, $X_k[\{P_i\}_{i\in\mathbb{Z}}] = P_k$ with the joint distribution induced by $\widehat{m}_0$ is a finite state mixing stationary Markov chain.

By the definition of the Frobenius function,

$$\begin{aligned}\xi_k(p) &= \xi[\psi^k(\widetilde{p})] - \xi(\widetilde{p}) \text{ for some (all) } \widetilde{p} \in \pi^{-1}(p)\\ &= \sum_{j=0}^{k-1} \xi[\psi^{j+1}(\widetilde{p})] - \xi[\psi^j(\widetilde{p})] = \sum_{j=0}^{k-1} \xi[\psi(\widetilde{p}_j)] - \xi[\widetilde{p}_j], \text{ where } \widetilde{p}_j \in \pi^{-1}[\psi_0^j(p)].\end{aligned}$$

So $\xi_k(p) = \sum_{j=0}^{k-1}(F_\psi \circ \psi_0^j)(p)$.

Recall that $F := F_\psi \circ \pi_0$ can be expressed in the form $g(X_0, X_1)$ or $g(X_{-1}, X_0)$ for some function $g : \mathfrak{P} \times \mathfrak{P} \to \mathbb{Z}$. Since $\widehat{m}_0 \circ \pi_0^{-1} = m_0$,

$$\sum_{j=0}^{k-1} F_\psi \circ \psi_0^j \overset{\text{dist}}{=} \sum_{j=0}^{k-1} F \circ \sigma^j = \sum_{j=0}^{k-1} g(X_j, X_{j-1}) \text{ or } \sum_{j=0}^{k-1} g(X_{j-1}, X_j).$$

Since $\{X_j\}_{j\in\mathbb{Z}}$ is stationary, $\xi_k(p) \overset{\text{dist}}{=} g(X_0, X_1) + \cdots + g(X_{k-1}, X_k)$ as required.

$\mathbb{E}[g(X_0, X_1)] = \int F_\psi dm_0 = 0$, because ψ has zero drift. There is no function $H : \mathfrak{P} \to \mathbb{R}$ s.t. $g(X_0, X_1) = H(X_0) - H(X_1) + \text{const}$, because of Lemma 2.9. □

Proof of Theorem 4.1. Let $S_k g := g(X_0, X_1) + \cdots + g(X_{k-1}, X_k)$.

(1) By the ergodic theorem, $S_k g/k \xrightarrow[k\to\infty]{} \mathbb{E}[g(X_0, X_1)] = 0$ a.s.

(2) By the Gärtner–Ellis Theorem, $\mathbb{P}\big[|S_k g/k| > \varepsilon\big] = O(e^{-kI(\varepsilon)})$ as $k \to \infty$ where $I(\cdot)$ is the Legendre transform of $\lim\limits_{n\to\infty} \frac{1}{n} \log \mathbb{E}[\exp(uS_n g)] = p(u) =$ topological pressure of uF. Since $g \neq H(X_0) - H(X_1) + \text{const}$, $p(t)$ is analytic and strictly convex. So $I(\varepsilon)$ is strictly convex. Since $p'(0) = \mathbb{E}(g) = 0$, $I(\varepsilon) > 0$ for all $\varepsilon > 0$. See §6 for a calculation of $p(u)$ in a special case.

(3) By the central limit theorem for finite state Markov chains, $\frac{1}{\sqrt{k}} S_k g \xrightarrow[k\to\infty]{\text{dist}} N(0, \sigma_0^2)$ for $\sigma_0^2 := \lim\limits_{n\to\infty} \frac{1}{n}\mathrm{Var}[S_n g]$. Since $g \neq H(X_0) - H(X_1) + \text{const}$, $\sigma_0 \neq 0$ (Leonov's Theorem). By Philipp & Stout's Almost Sure Invariance Principle ([**PS**], chapter 4), there is a probability space $(\Omega, \mathscr{F}, \mu)$ equipped with two continuous time stochastic processes $\widetilde{\Xi}_t, \widetilde{B}_t : \Omega \to \mathbb{R}$ s.t. $\{\widetilde{\Xi}_n\}_{n\geq 1} \overset{dist}{=} \{S_n g + \varepsilon_n\}_{n\geq 1}$, $\{\widetilde{B}_t\}_{t\geq 0} \overset{dist}{=}$ standard Brownian motion, such that for some $0 < \delta < \frac{1}{2}$, $|\widetilde{\Xi}_t - \sigma_0 \widetilde{B}_t| = o(t^\delta)$ a.s. as $t \to \infty$.

By Theorem 4.13 in [**PP**], $\sigma_0^2 = p''(0)$. It follows that $\sigma_0 = \sigma$ where σ is the constant appearing in Theorems 3.1 and 3.2.

(4) If $f, \widehat{f} \in L^1(\mathbb{R})$, then $\lim\limits_{n\to\infty} \frac{1}{\ln n} \sum_{k=1}^n \frac{1}{k} f\big(S_k g/\sqrt{k}\big) = \mathbb{E}[f(N)]$ almost surely, where N is the standard gaussian, and $\widehat{f}$ is the Fourier transform of f. This follows from (3) as in Lemma 2 in [**LS1**] (see also [**Fi1**]).

The theorem follows, since $\{\Xi_k\}_{k\geq 1} \overset{dist}{=} \{S_k g + \varepsilon_k\}_{k\geq 1}$ with $\varepsilon_k = O(1)$. □

Application to the Cylinder map. Theorems 3.1 and 4.1 combine to give the following statement. Let χ be a standard gaussian random variable.

Theorem 4.3. *Suppose α is a quadratic irrational. There are $\sigma^2 > 0$ and $0 < \lambda < 1$ s.t. if $a_n := \sqrt{\frac{|\ln\lambda|}{4\pi\sigma^2}}\big(\frac{n}{\sqrt{\ln n}}\big)$, then for every $G \in L^1(\mathbb{T}\times\mathbb{Z})$ s.t. $\int G dm_{\mathbb{T}\times\mathbb{Z}} = 1$:*

(1) $\frac{1}{a_n}\sum_{k=0}^{n-1} G\circ T_\alpha^k \xrightarrow[n\to\infty]{dist} \sqrt{2}\exp(-\frac{1}{2}\chi^2)$.

(2) $\lim_{N\to\infty} \frac{1}{\ln\ln N}\sum_{n=2}^{N}\frac{1}{n\ln\ln n}\big(\frac{1}{a_n}\sum_{k=1}^{n} G\circ T_\alpha^k\big) = 1$ *a.e.*

(3) *If in addition $G \in C_c(\mathbb{T}\times\mathbb{Z})$, then* $\int_{\mathbb{T}\times\{0\}}\big(\sum_{j=0}^{n-1} G\circ T_\alpha^j\big)dm_{\mathbb{T}\times\mathbb{Z}} = [1+o(1)]a_n$.

Part 2 of the theorem is a "higher order ergodic theorem" in the sense of A. Fisher [**Fi1**],[**Fi2**],[**ADF**].

Proof. (1) and (2) are immediate.

For (3) let $A_\delta(n) := \{(x,0) : |\Xi_n(x)/n| \le \delta\}$, $B_\delta(n) := \{(x,0) : |\Xi_n(x)/n| > \delta\}$. We break the integral into the main part $\int_{A_\delta(n)}$ and the remainder $\int_{B_\delta(n)}$. The remainder is $O(ne^{-nI(\delta)}) = o(a_n)$, because of Theorem 4.1(2) and the boundedness of G. The main term is sandwiched between two bounds of the form

$$(1+\varepsilon(\delta))a_n\cdot\sqrt{2}\mathbb{E}[\exp(-\tfrac{1-\varepsilon(\delta)}{2\sigma^2}(\Xi_{[\log^* n]}/[\log^* n])^2]$$
$$(1-\varepsilon(\delta))a_n\cdot\sqrt{2}\mathbb{E}[\exp(-\tfrac{1+\varepsilon(\delta)}{2\sigma^2}(\Xi_{[\log^* n]}/[\log^* n])^2]$$

with $\varepsilon(\delta)\xrightarrow[\delta\to 0^+]{} 0$ (this is a consequence of the uniformity in x in Theorem 3.1). Since $\Xi_k/\sqrt{k}\xrightarrow[k\to\infty]{\text{dist}} N(0,\sigma^2)$, these bounds converge to $(1\pm\varepsilon(\delta))\sqrt{2}\mathbb{E}[e^{-\frac{1\mp\varepsilon(\delta)}{2}\chi^2}]$ as $n\to\infty$. Since $\mathbb{E}[e^{-\frac{1\mp\varepsilon(\delta)}{2}\chi^2}]\xrightarrow[\delta\to 0]{}\mathbb{E}(e^{-\frac{1}{2}\chi^2}) = 2^{-\frac{1}{2}}$, the main term is $[1+o(1)]a_n$. Part (3) follows. □

Application to the deterministic random walk.

Theorem 4.4. *Suppose α is a quadratic irrational, and N_n is the number of visits of the DRW to zero up to time $n-1$, then*

(1) $\mathbb{E}(N_n) = [1+o(1)]a_n$, *where* $a_n = \sqrt{\frac{|\ln\lambda|}{4\pi\sigma^2}}\big(\frac{n}{\sqrt{\ln n}}\big)$.

(2) $\frac{1}{a_n}N_n \xrightarrow[n\to\infty]{dist}\sqrt{2}\exp(-\frac{1}{2}\chi^2)$, *where χ is a standard gaussian.*

(3) $\lim_{N\to\infty}\frac{1}{\ln\ln N}\sum_{n=2}^{N}\frac{1}{n\ln\ln n}\big(\frac{1}{a_n}N_n\big) = 1$ *a.s.*

(4) *λ is an eigenvalue of the renormalizing automorphism ψ, and σ^2 is the asymptotic variance in* $\frac{1}{\sqrt{k}}\Xi_k\xrightarrow[k\to\infty]{dist} N(0,\sigma^2)$.

This follows from the previous theorem and the identity $N_n = \sum_{k=0}^{n-1} 1_{\mathbb{T}\times\{0\}}\circ T_\alpha^k$.

Stochastic interpretation of twists. Theorem 4.1 and Lemma 4.2 extend trivially to automorphisms ψ with non-zero drift. One just needs to replace Ξ_k by $\Xi_k - k\delta(\psi)$ where $\delta(\psi)$ is the drift of ψ. The Markov chain and the function g Lemma 4.2 are defined as before, except that now $\mathbb{E}(g) = \delta(\psi) \neq 0$.

We can use this simple observation to calculate twists. Suppose ψ is a hyperbolic homogeneous automorphism with positive eigenvalues, and let $\underline{w}$ be an eigenvector of its derivative. Recall from Lemma 2.12 that there is a unique homogeneous automorphism ϕ with the same derivative as ψ, and which fixes the rays $L_i(p, \underline{w})$. The drift of ϕ equals minus $\tau_\psi(p, \underline{w})$. Consequently,

Corollary 4.5. *Let* $\widehat{\Xi}_k := \xi[\phi^k(z)]$*, where* z *is distributed uniformly in horizontal rectangle zero, then* $\frac{1}{n}\widehat{\Xi}_n \to -\tau_\phi(p, \underline{w})$ *a.s., and* $\frac{1}{\sqrt{n}}\big(\widehat{\Xi}_n + n\tau_\phi(p, \underline{w})\big) \xrightarrow[n\to\infty]{dist} N(0, \sigma^2)$.

5. Application to a result of J. Beck

In this section we explain how to use the machinery developed in sections 2 and 4 to prove the following theorem of J. Beck [**B1**, **B2**]. Fix an irrational α and let

$$Z^*_\alpha(n) := \#\{1 \le k \le n : \{k\alpha\} \in [0, \frac{1}{2})\} - \frac{1}{2}n \equiv -\frac{1}{2}\sum_{k=1}^{n} f(\{k\alpha\}).$$

$Z^*_\alpha(n)$ appears in the theory of uniform distribution, see [**RS**] and references therein.

Theorem 5.1 (Beck)**.** *If* α *is a quadratic irrational, then there are (explicit) constants* C_1, C_2 *depending on* α *s.t. for all* $a < b$ *real*

$$\frac{1}{N}\#\{1 \le n \le N : \frac{Z^*_\alpha(n) - C_1 \ln N}{C_2\sqrt{\ln N}} \in [a, b]\} \xrightarrow[N\to\infty]{} \frac{1}{\sqrt{2\pi}}\int_a^b e^{-u^2/2}du.$$

This is Theorem 1.1 in [**B1**] in the special case of the interval $[0, \frac{1}{2})$. The constants are calculated in [**B2**] using algebraic number theory and harmonic analysis. We will give a different proof, which sheds additional light on C_1, C_2.

First we explain how to translate Beck's theorem into a statement on linear flows on the infinite staircase.

In what follows ξ denotes a $\mathbb{Z}$–coordinate induced by the natural partition of the infinite staircase into horizontal rectangles, and p_0 denotes the singularity at the bottom left corner of rectangle zero.

We wish to define $\varphi^t_\theta(p_0)$ for $t > 0$ for an irrational direction θ. There is an element of choice here, because p_0 is a singularity, and there are infinitely many rays in direction θ emanating from p_0, one for each horizontal cylinder C such that the vector $\binom{\sin\theta}{\cos\theta}$ based in p points inside R.

We define $\varphi^t_\theta(p_0)$ to be the movement at unit speed along the ray $L_0(p_0, \binom{\sin\theta}{\cos\theta})$ emanating from p_0 in direction θ which begins at rectangle zero.

Lemma 5.2. *Let* $\theta = \tan^{-1}(2\alpha - 1)$ *and* $c := \sqrt{1 + \tan^2\theta}$*, then*

$$Z^*_\alpha(n) = \frac{1}{2}\xi(\varphi^t_\theta(p_0)) \text{ for all } cn < t < c(n+1), \tag{5.1}$$

Proof. The constant c is exactly the time it takes φ_θ to cross a horizontal cylinder in the vertical direction, so $\varphi^{cn}_\theta(p_0)$ lies on the bottom horizontal side of a unique horizontal rectangle R_n, and $\xi[\varphi^t_\theta(p_0)] = \text{const}$ for $cn < t < c(n+1)$.

Let ξ_n denote the $\mathbb{Z}$–coordinate of R_n, and let x_n denote the distance of $\varphi^{cn}_\theta(p_0)$ from the bottom left corner of R_n. We show by induction that $x_n = 2n\alpha \mod 2$ and $\xi[\varphi^t_\theta(p_0)] = 2Z^*_n(n)$ for $cn < t < c(n+1)$.

At time zero, the flow is at p_0, so $x_0 = 0$, and by the definition of $\varphi^t_\theta(p_0)$, $\xi[\varphi^t_\theta(p_0)] = 0 = 2Z^*_\alpha(0)$ for all $0 < t < c$.

Suppose by induction that $x_n = 2n\alpha \mod 2$ and $\xi[\varphi_\theta^t(p)] = 2Z_\alpha^*(n)$ for $cn < t < c(n+1)$. By the definition of St,

- if $x_n + \tan\theta \in [0,1) + 2\mathbb{Z}$, then $\xi_{n+1} = \xi_n - 1$ and $x_{n+1} = x_n + \tan\theta + 1 \mod 2$,
- if $x_n + \tan\theta \in [1,2) + 2\mathbb{Z}$, then $\xi_{n+1} = \xi_n + 1$ and $x_{n+1} = x_n + \tan\theta - 1 \mod 2$.

We see that $x_{n+1} = x_n + 2\alpha \mod 2 = 2(n+1)\alpha \mod 2$, and

$$\begin{aligned}\xi_{n+1} &= \xi_n + 1_{[1,2)+2\mathbb{Z}}(x_n + \tan\theta) - 1_{[0,1)+2\mathbb{Z}}(x_n + \tan\theta)\\ &= \xi_n + 1_{[0,1)+2\mathbb{Z}}(x_n + 2\alpha) - 1_{[1,2)+2\mathbb{Z}}(x_n + 2\alpha)\\ &= 2Z_n^*(n) + 1_{[0,\frac{1}{2})}(\{(n+1)\alpha\}) - 1_{[\frac{1}{2},1)}(\{(n+1)\alpha\})\\ &= 2\left(Z_n^*(n) + 1_{[0,\frac{1}{2})}(\{(n+1)\alpha\}) - \frac{1}{2}\right) = 2Z_{n+1}^*(n+1). \qquad \square\end{aligned}$$

Proof of Beck's Theorem. Let ψ denote a hyperbolic homogeneous automorphism which renormalizes α, has zero drift, and with the property that the eigenvalues of $d\psi$ are positive. Let $0 < \lambda < 1$ denote the contracting eigenvalue, and let $\underline{w}$ denote a contracted eigenvector in direction $\theta = \tan^{-1}(2\alpha - 1)$. Let

$$C_1 := \frac{\tau_\psi(p_0, \underline{w})}{2|\ln\lambda|} \text{ and } C_2 := \frac{\sigma}{2\sqrt{|\ln\lambda|}} \tag{5.2}$$

where $\tau_\psi(p_0, \underline{w})$ is the twist (Definition 2.10), and σ^2 is the asymptotic variance mentioned in the previous sections. We will show that

$$D_N(a,b) := \frac{1}{N}\#\{1 \le n \le N : \frac{Z_\alpha^*(n) - C_1\ln N}{C_2\sqrt{\ln N}} \in [a,b]\} \xrightarrow[N\to\infty]{} \frac{1}{\sqrt{2\pi}}\int_a^b e^{-u^2/2}du.$$

Let $\Gamma_N := \{\varphi_\theta^t(p_0) : c < t < c(N+1)\}$, and $\ell_{\Gamma_N}(\cdot)$ denote the length (Lebesgue) measure on Γ_N. By (5.1),

$$D_N(a,b) = \frac{1}{\ell_{\Gamma_N}(\Gamma_N)}\ell\{q \in \Gamma_N : \frac{\xi(q) - 2C_1\ln N}{2C_2\sqrt{\ln N}} \in [a,b]\} \tag{5.3}$$

Let $N^* := \lfloor \log_{\lambda^{-1}} N \rfloor$, and $\gamma_N := \psi^{N^*}(\Gamma_N)$. Since $\psi \circ \varphi_\theta^t = \varphi_\theta^{\lambda t} \circ \psi$, γ_N is a linear segment with bounded length in direction θ.

By the definition of the twist, $\psi^{N^*}(\Gamma_N) \subset D^{k_{N^*}}[L_0(p_0, \underline{w})]$, where $k_{N^*} = N^*\tau_\psi(p_0, \underline{w}) + O(1) = 2C_1\ln N + O(1)$. So $\frac{1}{2}\xi(\cdot) = C_1\ln N + O(1)$ uniformly on γ_N. By (5.3) and the identity $\ell_{\Gamma_N} = \lambda^{-N^*}\ell_{\gamma_N} \circ \psi^{N^*}|_{\Gamma_N}$ where ℓ_{γ_N} =Lebesgue measure on γ_N, $D_N(a,b) = \frac{1}{\lambda^{N^*}\ell_{\Gamma_N}(\Gamma_N)}\ell_{\gamma_N}\{\psi^{N^*}(q) : q \in \Gamma_N, \frac{\xi(q) - 2C_1\ln N}{2C_2\sqrt{\ln N}} \in [a,b]\}$. From now on we set $\ell := \ell_{\gamma_N}$, and $z = \psi^{-N^*}(q)$, then

$$\begin{aligned}D_N(a,b) &= \frac{1}{\ell(\gamma_N)}\ell\{z \in \gamma_N : \frac{\xi(\psi^{-N^*}(z)) - 2C_1\ln N}{2C_2\sqrt{\ln N}} \in [a,b]\}\\ &= \frac{1}{\ell(\gamma_N)}\ell\{z \in \gamma_N : \frac{\xi(\psi^{-N^*}(z)) - \xi(z) + O(1)}{\sigma\sqrt{N^*} + O(1)} \in [a,b]\}\\ &= \frac{1}{\ell(\gamma_N)}\ell\{z \in \gamma_N : \frac{\xi(\psi^{-N^*}(z)) - \xi(z)}{\sigma\sqrt{N^*}} \in [a + O(\tfrac{1}{\sqrt{N^*}}), b + O(\tfrac{1}{\sqrt{N^*}})]\}\\ &= \frac{1}{\ell(\gamma_N)}\ell_{\widehat{\gamma}_N}\{z \in \widehat{\gamma}_N : \frac{\xi(\psi^{-N^*}(z))}{\sigma\sqrt{N^*}} \in [a + O(\tfrac{1}{\sqrt{N^*}}), b + O(\tfrac{1}{\sqrt{N^*}})]\},\end{aligned}$$

where $\widehat{\gamma}_N := D^{-k_{N^*}}(\gamma_N)$, and $\ell_{\widehat{\gamma}_N}$ is the Lebesgue measure on $\widehat{\gamma}_N$.

The advantage in passing to $\widehat{\gamma}_N$, apart from canceling $\xi(z)$ up to bounded error, is that the family $\{\widehat{\gamma}_N\}_{N\geq 1}$ is precompact. This is because the beginning point of $\widehat{\gamma}_N$ is at distance $c\lambda^{-N^*}$ from p_0 on $L_0(p_0,\underline{w})$, and $\ell(\widehat{\gamma}_N)$ is bounded away from zero and infinity. It follows that every sequence has a subsequence $N_k \uparrow \infty$ along which $\widehat{\gamma}_{N_k} \xrightarrow[k\to\infty]{} \widehat{\gamma}$, where $\widehat{\gamma}$ is a bounded linear segment in direction θ, emanating from p_0, and beginning in rectangle zero. It is enough to prove that $D_{N_k}(a,b) \to \frac{1}{\sqrt{2\pi}}\int_a^b e^{-u^2/2}du$ along such sequences.

Suppose $N_k \uparrow \infty$ and $\widehat{\gamma}_{N_k} \to \widehat{\gamma}$ as above. Let c_0 :=length of $\widehat{\gamma}$. Fix ε much smaller than c_0, so small that $\frac{c_0+\varepsilon}{c_0-\varepsilon} \in [e^{-\varepsilon}, e^{\varepsilon}]$. Let $\widehat{\gamma}^-$ and $\widehat{\gamma}^+$ denote two linear segments in rectangle zero, in direction θ, emanating from p_0, and with lengths $c_0(1-\varepsilon)$ and $c_0(1+\varepsilon)$ respectively. Then $\widehat{\gamma}^- \subset \widehat{\gamma} \subset \widehat{\gamma}^+$, and $D_{N_k}(a,b)$ is sandwiched between $D^+_{N_k}(a,b), D^-_{N_k}(a,b)$, where

$$D^+_N(a,b) := \frac{e^{\varepsilon}}{\ell(\widehat{\gamma}^+)}\ell\{z\in\widehat{\gamma}^+ : \frac{\xi(\psi^{-N^*}(z))}{\sigma\sqrt{N^*}} \in [a+O(\tfrac{1}{\sqrt{N^*}}), b+O(\tfrac{1}{\sqrt{N^*}})]\}$$
$$D^-_N(a,b) := \frac{1}{e^{\varepsilon}\ell(\widehat{\gamma}^-)}\ell\{z\in\widehat{\gamma}^- : \frac{\xi(\psi^{-N^*}(z))}{\sigma\sqrt{N^*}} \in [a+O(\tfrac{1}{\sqrt{N^*}}), b+O(\tfrac{1}{\sqrt{N^*}})]\},$$

The linear segments $\widehat{\gamma}^{\pm}$ are in the unstable (expanding) direction of ψ^{-1}. Let $Q^{\pm}$ denote a thickening of these segments in the stable direction (the inside of a parallelogram with one side equal to $\widehat{\gamma}^{\pm}$ and the other side a segment in the stable (contracting) direction of ψ^{-1}). For the same reasons explained in the proof of Lemma 4.2,

$$D^+_N(a,b) = \frac{e^{\varepsilon}}{m(Q^+)}m\{z\in Q^+ : \frac{\xi(\psi^{-N^*}(z))}{\sigma\sqrt{N^*}} \in [a+O(\tfrac{1}{N^*}), b+O(\tfrac{1}{N^*})]\} + o(1)$$
$$D^-_N(a,b) := \frac{e^{-\varepsilon}}{m(Q^-)}m\{z\in Q^- : \frac{\xi(\psi^{-N^*}(z))}{\sigma\sqrt{N^*}} \in [a+O(\tfrac{1}{N^*}), b+O(\tfrac{1}{N^*})]\} + o(1),$$

where m is the area measure.

We saw in the previous section that if z is chosen uniformly in rectangle number zero, then $\frac{\xi(\psi^{-N^*}(z))}{\sigma\sqrt{N^*}} \xrightarrow[N^*\to\infty]{dist} N(0,1)$, because of the central limit theorem for finite state mixing Markov chains. The same is true for obvious reasons when z is sampled uniformly in a finite union of such rectangles. In $D^{\pm}_{N_k}$ we are sampling z from a finite union of rectangles with respect to an absolutely continuous measure (Lebesgue times the density function $\frac{1}{m(Q^{\pm})}1_{Q^{\pm}}$). By Eagleson's Theorem [**E**], the central limit theorem still holds, whence $D^{\pm}_N(a,b) \xrightarrow[N\to\infty]{} \frac{e^{\pm\varepsilon}}{\sqrt{2\pi}}\int_a^b e^{-u^2/2}du$. It follows that $D_N(a,b) \xrightarrow[N\to\infty]{} \frac{1}{\sqrt{2\pi}}\int_a^b e^{-u^2/2}du$. □

The proof shows that C_1 and C_2 in Beck's theorem are given by (5.2). For example, if $\alpha=\sqrt{2}$ then the calculations done in the next section give for a suitable automorphism $\lambda = 17-12\sqrt{2} = (1+\sqrt{2})^{-4}$, $\tau_\psi(p_0,\underline{w})=1$, and $\sigma^2=\frac{3}{8}\sqrt{2}$. Thus $C_1 = \frac{1}{8\ln(1+\sqrt{2})}$ and $C_2 = \frac{1}{8}(\frac{3}{\sqrt{2}\ln(1+\sqrt{2})})^{\frac{1}{2}}$, in agreement with [**B1**],[**B2**].

6. Calculation of constants

The purpose of this section is to prove:

Theorem 6.1. *If α is a quadratic irrational with renormalizing automorphism ψ, then $\sigma^2 \in \mathbb{Q}[\alpha]$ and the twists of eigenvectors at singularities are in $\frac{1}{2}\mathbb{Z}$.*

Theorem 6.2. *In the special case when $\alpha = \sqrt{2}$ and ψ is a renormalizing automorphism with derivative $\left(\begin{smallmatrix} 11 & -42 \\ -6 & 23 \end{smallmatrix}\right)$ and zero drift, $\sigma^2 = \frac{3}{8}\sqrt{2}$.*

Theorem 6.3. *Let ψ be the automorphism in the previous theorem. If p_0 is one of the singularities in the bottom left corner of a horizontal rectangle, and $\underline{w}$ is the contracted eigenvector, then $\tau_\psi(p_0, \underline{w}) = 1$.*

To prove these results we recall that at the end of section 4 we showed the existence of a stationary mixing finite Markov chain $\{X_i\}_{i\in\mathbb{Z}}$ and a function g s.t.

$$\mathbb{E}[g(X_0, X_1)] = -\tau_\phi(p_0, \underline{w}) \ , \ \frac{\sum_{i=0}^{k-1} g(X_i, X_{i+1}) + k\tau_\psi(p_0, \underline{w})}{\sqrt{k}} \xrightarrow[k\to\infty]{\text{dist}} N(0, \sigma^2). \quad (6.1)$$

We will calculate $\{X_k\}_{k\in\mathbb{Z}}$ and g explicitly, and then find $\mathbb{E}[g(X_0, X_1)]$ and σ^2 using the theory of Markov chains.

Along the way we will prove (2.4), as promised in §2.

In what follows ψ is a hyperbolic homogeneous automorphism *with zero drift* which renormalizes α. We assume without loss of generality that ψ has positive eigenvalues and that ψ fixes the singularities of St (otherwise we pass to ψ^2, and note that $\sigma^2(\psi^2) = 2\sigma^2(\psi)$ and $\tau_{\psi^2}(p_0, \underline{w}) = 2\tau_{\psi^2}(p_0, \underline{w})$).

Next we assume that ϕ is the unique hyperbolic homogeneous automorphism with the same derivative as ψ and which fixes the rays $L_i(p_0, \underline{w})$, see Lemma 2.12. The drift of ϕ equals $-\tau_\psi(p_0, \underline{w})$.

Both ψ and ϕ project to the same toral automorphism, which we denote by $\psi_0 : \mathrm{St}_0 \to \mathrm{St}_0$.

The Markov chain $\{X_k\}_{k\in\mathbb{Z}}$. Let $\mathfrak{P}$ denote the Adler–Weiss Markov partition of ψ_0, with dynamical graph $\mathscr{G}$. As always, A is the derivative of ψ, λ is the eigenvalue of A in $(0,1)$, and $\underline{w}, \underline{v}$ are eigenvectors of λ, λ^{-1}.

Recall that $\{X_k\}_{k\in\mathbb{Z}}$ is the Markov chain with the set of states $\mathfrak{P}$, allowed transitions $P \to Q$ iff $\psi_0(\mathrm{int}(P)) \cap \mathrm{int}(Q) \neq \varnothing$, and the transition matrix and stationary probability vector which generates of the measure of maximal entropy on $\Sigma(\mathscr{G})$. We calculate this data in terms of the fundamental polygon of ψ.

We begin with the cardinality of $\mathfrak{P}$. Recall that

$$\mathfrak{P} = \{Q_{ij} : i = 1, 2; j = 1, \ldots, N_i\}$$

where $\psi_0(Q_i) = \bigcup_{j=1}^{N_i} Q_{ij}$. Q_{ij} are ordered so that the top s–side of $Q_{i,j+1}$ is identified with the bottom s–side of $Q_{i,j+1}$. Since $\mathfrak{P}$ is a refinement of $\{Q_1, Q_2\}$, some of the Q_{ij} are contained in Q_1 and some are contained in Q_2. Let

$$N_{ik} := \#\{1 \le j \le N_i : Q_{ij} \subset Q_k\},$$

then $N_i = N_{i1} + N_{i2}$ and $|\mathfrak{P}| = \sum_{i,j} N_{ij}$. The following lemma determines N_{ij}:

Lemma 6.4. *Let $\ell^u(Q_i)$ denote the length of the unstable fibres in Q_i, $i = 1, 2$, then $(N_{ij})_{2\times 2}$ is the unique solution in $\mathbb{Z}$ to*

$$\begin{aligned} N_{11}\ell^u(Q_1) + N_{12}\ell^u(Q_2) &= \lambda^{-1}\ell^u(Q_1) \\ N_{21}\ell^u(Q_1) + N_{22}\ell^u(Q_2) &= \lambda^{-1}\ell^u(Q_2) \end{aligned} \quad (6.2)$$

Proof. If W^u is a u-fibre in Q_i, then $\psi_0(W^u)$ can be partitioned into N_{i1} u–fibres in Q_1 and N_{i2} u–fibres in Q_2 (one for each Q_{ij}). The sum of the lengths of these u–fibres must equal $\ell[\psi_0(W^u)] = \lambda^{-1}\ell(W^u) = \lambda^{-1}\ell^u(Q_i)$, so N_{ij} solve (6.2).

The existence of a solution of (6.2) in $\mathbb{Z}$ implies that $\ell^u(Q_1), \ell^u(Q_2)$ are linearly independent over $\mathbb{Q}$: Otherwise λ^{-1} is rational, which is never the case for an eigenvalue of a hyperbolic matrix in $\mathrm{SL}(2,\mathbb{Z})$. It follows that (N_{ij}) is the unique solution of (6.2) in integers. □

Next we calculate *incidence matrix* of $\mathscr{G}$, $T = (t_{PQ})_{\mathfrak{P}\times\mathfrak{P}}$, where

$$t_{PQ} = \begin{cases} 1 & P \to Q, \\ 0 & \text{otherwise} \end{cases} \qquad\qquad (P, Q \in \mathfrak{P}).$$

Lemma 6.5 (Adler & Weiss)**.** $t_{Q_{ij}Q_{k\ell}} = 1 \Leftrightarrow Q_{ij} \subset Q_k$. *Thus,* $\mathrm{rank}(T) = 2$ *and every* $\mathfrak{P}$*–element in* Q_i *connects to* N_{ik} $\mathfrak{P}$*–elements in* Q_k*.*

Proof. Suppose $Q_{ij} \in \mathfrak{P}$. Since $\mathfrak{P}$ refines $\{Q_1, Q_2\}$, $Q_{ij} \subset Q_k$ for $k = 1$ or 2. By construction, $\psi_0(Q_k) = \bigcup Q_{k\ell}$, so if $Q_{ij} \to Q$ then $\mathrm{int}(Q) \subset \bigcup Q_{k\ell}$, which means that $\mathrm{int}(Q)$ intersects $\mathrm{int}(Q_{k\ell})$ for some ℓ. Since $\{\mathrm{int}(Q_{k\ell})\}$ are pairwise disjoint, $Q = Q_{k\ell}$, which proves the $(\Rightarrow)$ direction.

The $(\Leftarrow)$ direction is also true, otherwise $\psi_0(\mathrm{int}(Q_k))$ intersects $\partial^u Q_{kj}$. This is false, because $\partial^u Q_{k\ell} \subset \psi_0(\partial^u Q_k) \subset \psi_0(\mathrm{int}(Q_k))^c$. So $t_{Q_{ij}Q_{k\ell}} = 1 \Leftrightarrow Q_{ij} \subset Q_k$.

We see that the incidence matrix T has two types of rows: those of $\mathfrak{P}$–elements $P \subset Q_1$, and those of $\mathfrak{P}$–elements $P \subset Q_2$. These rows are different, because

- if $P \in \mathfrak{P}, P \subset Q_1$, then $\#\{Q \in \mathfrak{P} : Q \subset Q_k, t_{PQ} = 1\} = N_{1k}$,
- if $P \in \mathfrak{P}, P \subset Q_2$, then $\#\{Q \in \mathfrak{P} : Q \subset Q_k, t_{PQ} = 1\} = N_{2k}$,
- $\binom{N_{11}}{N_{12}} \neq \binom{N_{21}}{N_{22}}$, otherwise by Lemma 6.4 $\ell^u(Q_1) = \ell^u(Q_2)$ and λ is rational.

Since different rows of zeroes and ones are linearly independent, $\mathrm{rank}(T) = 2$. □

Next we determine the *transition matrix* of the Markov chain $\{X_k\}_{k\in\mathbb{Z}}$: the matrix $(p_{PQ})_{P,Q\in\mathfrak{P}}$ s.t. $p_{PQ} = \mathbb{P}(X_1 = Q | X_0 = P)$.

Lemma 6.6 (Adler & Weiss)**.** $(p_{PQ})_{P,Q\in\mathfrak{P}} = \lambda M^{-1} T M$ *where* M *is the diagonal matrix with diagonal entries* $M_{PP} = \ell^u(P)$*.*

Proof. By the Adler–Weiss Theorem, $m_0 = \mathbb{P} \circ \pi_0^{-1}$, where m_0 is the normalized area measure on St_0 and $\mathbb{P} := \widehat{m}_0$ is the joint distribution measure given by

$$\mathbb{P}(E) := \mathbb{P}[(X_k)_{k\in\mathbb{Z}} \in E], \quad (E \subset \Sigma(\mathscr{G}) \text{ Borel}).$$

Therefore, if $P = Q_{ij}, Q = Q_{k\ell}$, then $p_{P,Q} = m_0[P \cap \psi_0^{-1}(Q)]/m_0(P)$.

P, Q, and $P \cap \psi_0^{-1}(Q)$ are parallelograms with sides in the stable and unstable directions. Let $\ell^u(\cdot), \ell^s(\cdot)$ denote the lengths of these sides, then $\ell^s(P) = \lambda\ell^s(Q_i)$, $\ell^s[P \cap \psi_0^{-1}(Q)] = \ell^s(P) = \lambda\ell^s(Q_i)$, and $\ell^u[P \cap \psi_0^{-1}(Q)] = \lambda\ell^u(Q)$. Denoting the angle between the stable and unstable directions by β, we see that $p_{PQ} = t_{PQ}\frac{\lambda^2\ell^s(Q_i)\ell^u(Q)\sin\beta}{\lambda\ell^s(Q_i)\ell^u(P)\sin\beta} = \lambda\ell^u(P)^{-1}t_{PQ}\ell^u(Q)$. □

Proof of (2.4) and calculation of g. Let ξ denote the associate $\mathbb{Z}$–coordinate of ψ. The definition of g is based on (2.4), which says that the Frobenius function F_ψ of ξ is either (a) $\mathfrak{P} \vee \psi_0^{-1}(\mathfrak{P})$–measurable or (b) $\mathfrak{P} \vee \psi_0(\mathfrak{P})$–measurable. In case (a), $g(P, Q)$ is the value of F_ψ on $\mathrm{int}(P) \cap \psi_0^{-1}[\mathrm{int}(Q)]$. In case (b), $g(P, Q)$ is the value of F_ψ on $\psi_0^{-1}[\text{int}(P)] \cap \text{int}(Q)$. In this section we prove (2.4), and give an explicit formula for g.

Recall from §2 that ψ or ψ^{-1} has a fundamental polygon of the form $R = \theta_0(R_0)$, where $\theta_0 : \text{St}_0 \to \text{St}_0$ is a toral automorphism which fixes the punctures of St_0^* and R_0 is one of the shapes in figure 2.

We will limit ourselves to the case when ψ has such a fundamental polygon. The case of ψ^{-1} can be handled by the identity $F_{\psi^{-1}} = -F_\psi \circ \psi_0$.

Suppose W^u is a u–fibre in R. The $\mathbb{Z}$*–displacement of* W^u is defined by

$$\phi(W^u) := \xi(\text{endpoint of } \widetilde{W}^u) - \xi(\text{beginning point of } \widetilde{W}^u)$$

for some (any) lift $\widetilde{W}^u$ of W^u to St_0. $\phi(W^u)$ is independent of the lift, and can be easily determined from the endpoint of W^u as follows. Looking at figure 2, divide the top side of the fundamental domain into three pieces: The top side of Q_1 ("left"), the part of the top side of Q_2 to the left of $H := (2, 1)$ ("middle"), and the part of the top side of Q_2 to the right of H ("right"). Then

- the $\mathbb{Z}$–displacement of u–fibres terminating at θ_0("left") is (-1);
- the $\mathbb{Z}$–displacement of u–fibres terminating at θ_0("middle") is (+1);
- the $\mathbb{Z}$–displacement of u–fibres terminating at θ_0("right") is (-1).

The unique $\mathfrak{P}$–element which has a u–fibre terminating at $H = (2, 1)$ is called *critical*. The non-critical elements of $\mathfrak{P}$ have the virtue that all their u–fibres have the same $\mathbb{Z}$–displacement. Let

$$\phi(Q_{ij}) := \text{the value of the } \mathbb{Z}\text{–displacement of } u\text{–fibres in } Q_{ij} \quad (Q_{ij} \text{ non-critical}).$$

Lemma 6.7. *The critical element is* Q_{2N_2}.

Proof. Call the critical element $Q_{k\ell}$.

H lies on the top s–side of Q_2, $\psi_0(H) = H$, and $\psi_0(\partial^s Q_2) \subset \partial^s Q_2$, therefore $H \in \text{int}[\partial^s \psi_0(Q_2)]$. It follows that $k = 2$ and $1 \le \ell \le N_2$. Let W^u be the u–fibre in $Q_{2\ell}$ whose closure contains H. If $\ell < N_2$ then $H = \psi_0^{-1}(H) \in \psi_0^{-1}[W^u] \subset \text{int}(Q_2) \cup$bottom s–side of Q_2. This is false, so $\ell = N_2$. □

Let $\widetilde{q}_0$ denote one of the singularities in the middle of the horizontal side of one of the horizontal rectangles. Since, by assumption, ψ fixes $\widetilde{q}_0$ and has positive eigenvalues, there is a constant τ s.t. $\psi[L_i(\widetilde{q}_0, \underline{w})] = L_{i+\tau}(\widetilde{q}_0, \underline{w})$ ($\tau = \tau_\psi(\widetilde{q}_0, \underline{w})$). A continuity argument shows that $\psi[L_i(\widetilde{q}_0, -\underline{w})] = L_{i+\tau}(\widetilde{q}_0, -\underline{w})$

Lemma 6.8. *If* ψ *has a fundamental domain of the form* $\theta(R_0)$ *with* R_0 *as in figure 2, then* F_ψ *is* $\mathfrak{P} \vee \psi_0^{-1}(\mathfrak{P})$*–measurable, and* $g(Q_{ij}, Q_{k\ell}) = \tau + \sum_{s=1}^{\ell-1} \phi(Q_{ks})$.

(The last expression makes sense because Q_{ks} is non-critical when $s \le \ell - 1$.)

Proof. Let $P := Q_{ij}$, $Q := Q_{k\ell}$, and suppose $p \in \text{int}(P) \cap \psi_0^{-1}[\text{int}(Q)]$. By the definition of the Frobenius function, $F_\psi(p) = \xi(\psi(\widetilde{p})) - \xi(\widetilde{p})$ for some (any) $\widetilde{p} \in \pi^{-1}(p)$. We choose the $\widetilde{p}$ s.t. $\xi(\widetilde{p}) = 0$, then $F_\psi(p) = \xi(\psi(\widetilde{p}))$.

To calculate this we construct a path γ in St_0 from the fixed point $q_0 = (1, 0)$ to p and analyze the lift of $\psi_0[\gamma]$ to the infinite staircase. Let q denote the intersection of $W^u(p)$ and $W^s(q_0)$ (the u and s fibres of p and q_0). The path γ we use is the concatenation of $[q_0, q] \subset W^s(p_0)$ and $[q, p] \subset W^u(p)$.

The curve γ begins with a piece of a ray emanating from q_0 in direction $\underline{w}$. Let $\widetilde{\gamma}$ denote its unique lift to St which begins with a segment in $L_0(\widetilde{q}_0, \underline{w})$. Since γ

does not cross ∂R, all points in $\widetilde{\gamma}$, in particular its end point, have $\mathbb{Z}$–coordinate equal to zero. It follows that $\widetilde{\gamma}$ ends at $\widetilde{p}$.

Let $\widetilde{\zeta} := \psi[\widetilde{\gamma}]$. This curve ends at $\psi(\widetilde{p})$, so $\xi[\text{end of } \widetilde{\zeta}] = \xi[\psi(\widetilde{p})]$.

As for its beginning, since $\psi[L_0(\widetilde{q}_0, \underline{w})] \subset L_\tau(\widetilde{q}_0, \underline{w})$, $\widetilde{\zeta} := \psi[\widetilde{\gamma}]$ begins with a segment in $L_\tau(\widetilde{q}_0, \underline{w})$. It follows that $\xi[\text{beginning of } \widetilde{\zeta}] = \tau$.

The curve $\widetilde{\zeta}$ projects to $\psi_0[\gamma]$. To calculate $\psi_0[\gamma]$, we first recall that $Q_{ij} \to Q_{k\ell}$, and therefore $P \subset Q_k = \bigcup_{s=1}^{N_k} \psi_0^{-1}(Q_{ks})$. $\{Q_{ks}\}_{s=1}^{N_k}$ are ordered so that $[q,p] = \bigcup_{s=1}^{\ell-1}[q_s, q_{s+1}] \cup [q_\ell, p]$, where $[q_s, q_{s+1}] = W^u(p) \cap \psi_0^{-1}(Q_{ks})$ and $[q_\ell, p] \subsetneq \psi_0^{-1}(Q_{k\ell})$. So $\psi_0[\gamma]$ is a concatenation of

- $[q_0, \psi(q)]$ (a subsegment of $W^s(q_0)$), followed by
- a u–fibre in Q_{k1} ($\psi_0[q_1, q_2]$), followed by
- a u–fibre in Q_{k2} ($\psi_0[q_2, q_3]$), etc

  ⋮ and so one until we reach
- a u–fibre in $Q_{k,\ell-1}$ ($\psi_0[q_{\ell-1}, q_\ell]$), followed by
- the beginning of a u–fibre in $Q_{k\ell}$, which terminates at $\psi_0(p)$.

It follows that $\xi[\text{end of } \widetilde{\zeta}] - \xi[\text{beginning of } \widetilde{\zeta}] = \sum_{s=1}^{\ell-1} \phi(Q_{ks})$. Substituting the values of ξ at the endpoint of $\widetilde{\zeta}$, we find that $F_\psi(p) = \tau + \sum_{s=1}^{\ell-1} \phi(Q_{ks})$. ☐

Proof of Theorem 6.1. That the twists always belong to $\frac{1}{2}\mathbb{Z}$ was proved in Lemma 2.11, so we focus on the value of σ^2. There is no loss of generality in assuming that ψ has zero drift.

Let g be the function found in Lemma 6.8, and define a family of $\mathfrak{P} \times \mathfrak{P}$ matrices $\Phi(\theta)$ by

$$\Phi_{P,Q}(\theta) := p_{P,Q} \exp[\theta g(P,Q)] \quad (P, Q \in \mathfrak{P}).$$

These are a positive matrices, and the mixing of $\sigma : \Sigma(\mathscr{G}) \to \Sigma(\mathscr{G})$ (Adler–Weiss Theorem) guarantees that they are primitive. By the Perron–Frobenius Theorem, $\Phi(\theta)$ has a simple positive eigenvalue $\lambda(\theta)$ such that $\lambda(\theta)$ is larger than the modulus of all other eigenvalues. When $\theta = 0$, Φ is stochastic, and $\lambda(0) = 1$.

Since $\Phi(\theta)$ depends analytically on θ, $\lambda(\theta)$ is analytic on some interval $(-\varepsilon, \varepsilon)$. It is known that

$$\lambda(0) = 1, \ \left.\frac{d}{d\theta}\right|_{\theta=0} \ln \lambda(\theta) = \mathbb{E}[g(X_0, X_1)] \text{ and } \left.\frac{d^2}{d\theta^2}\right|_{\theta=0} \ln \lambda(\theta) = \sigma^2. \tag{6.3}$$

See Doeblin [**D**], Nagaev [**Ngv**], or chapter 4 in [**PP**].

We use this formula to show that $\sigma^2 \in \mathbb{Q}[\alpha]$, where α is the angle normalized by ψ. Let A denote the derivative of ψ and let λ denote the eigenvalue of A in $(0,1)$. Since ψ renormalizes α, $\alpha = \frac{1}{2} + \frac{1}{2}\tan\theta (\text{mod } 1)$ where $A\binom{1}{\tan\theta} = \lambda\binom{1}{\tan\theta}$. Since A is a matrix of integers, $\lambda \in \mathbb{Q}[\tan\theta] = \mathbb{Q}[\alpha]$. We'll show that $\sigma^2 \in \mathbb{Q}[\lambda]$, and deduce that $\sigma^2 \in \mathbb{Q}[\alpha]$.

We need the following claim. Let $\mathscr{A}$ denote the collection of functions of the form $\varphi(\theta) = \sum_{k=-n}^{n} a_k e^{k\theta}$ with arbitrary $n \in \mathbb{N}$ and $a_k \in \mathbb{N} \cup \{0\}$, and set $\mu(\theta) := \lambda(\theta)/\lambda$.

CLAIM. *There are $\beta_{ij}(\theta) \in \mathscr{A}$ s.t. $\mu(\theta)$ is the largest eigenvalue of $\begin{pmatrix} \beta_{11}(\theta) & \beta_{12}(\theta) \\ \beta_{21}(\theta) & \beta_{22}(\theta) \end{pmatrix}$, for all $\theta \in (-\varepsilon, \varepsilon)$.*

Proof of the claim. Let $\Psi(\theta)$ denote the $\mathfrak{P} \times \mathfrak{P}$ matrix $(t_{PQ} \exp[\theta g(P,Q)])_{P,Q \in \mathfrak{P}}$. By lemma 6.6, $\Psi = \lambda^{-1} M \Phi M^{-1}$, so $\mu(\theta)$ is the leading eigenvalue of $\Psi(\theta)$.

As our formulas for $t_{Q_{ij},Q_{k\ell}}$ and $g(Q_{ij},Q_{k\ell})$ show, if $P,Q\in\mathfrak{P}$ are both included in the same Q_k, then the P–row of $\Psi(\theta)$ is equal to the Q–row of $\Psi(\theta)$, and if P,Q are not included in the same Q_k then the P–row and the Q–row are linearly independent. In particular, $\mathrm{rank}[\Psi(\theta)]=2$.

We think of $\Psi(\theta)$ as of the linear transformation $\underline{u}\mapsto\underline{u}\Psi(\theta)$ on $\mathbb{R}^{\mathfrak{P}}$. Let $V_\theta:=\mathrm{Im}[\Psi(\theta)]$. Then $\dim V_\theta=2$ and

$$V_\theta=\mathrm{Span}\{\underline{e}_P\Psi(\theta),\underline{e}_Q\Psi(\theta)\}$$

when $P,Q\in\mathfrak{P}$, $P\subset Q_1$, $Q\subset Q_2$, and $\underline{e}_P,\underline{e}_Q$ are the row vectors $(\underline{e}_P)_R=\delta_{PR}$, $(\underline{e}_Q)_R=\delta_{QR}$ where δ_{PQ} is the Kronecker symbol.

$\Psi(\theta)$ preserves V_θ, and since V_θ contains all the (left) eigenvectors of $\Psi(\theta)$, $\mu(\theta)$ is the leading eigenvalue of $\Psi(\theta)|_{V_\theta}:V_\theta\to V_\theta$. We represent $\Psi(\theta)|_{V_\theta}:V_\theta\to V_\theta$ in the basis $\{\underline{e}_P\Psi(\theta),\underline{e}_Q\Psi(\theta)\}$. For every $S\in\mathfrak{P}$,

$$\begin{aligned}((\underline{e}_P\Psi)\Psi)_S=(\underline{e}_P\Psi^2)_S=(\Psi^2)_{PS}&=\sum_{R\in\mathfrak{P},R\subset Q_1}\Psi_{P,R}\Psi_{R,S}+\sum_{R\in\mathfrak{P},R\subset Q_2}\Psi_{P,R}\Psi_{R,S}\\
&=\sum_{R\in\mathfrak{P},R\subset Q_1}\Psi_{P,R}\Psi_{P,S}+\sum_{R\in\mathfrak{P},R\subset Q_2}\Psi_{P,R}\Psi_{Q,S}\\
&=\Big(\sum_{R\in\mathfrak{P},R\subset Q_1}\Psi_{P,R}\Big)(\underline{e}_P\Psi)_S+\Big(\sum_{R\in\mathfrak{P},R\subset Q_2}\Psi_{P,R}\Big)(\underline{e}_Q\Psi)_S\\
&=\Big(\sum_{R\in\mathfrak{P},R\subset Q_1}t_{PR}e^{\theta g(P,R)}\Big)(\underline{e}_P\Psi)_S+\Big(\sum_{R\in\mathfrak{P},R\subset Q_2}t_{PR}e^{\theta g(P,R)}\Big)(\underline{e}_Q\Psi)_S.\end{aligned}$$

The terms in the brackets belong to $\mathscr{A}$. A similar formula holds for $(\underline{e}_Q\Psi)\Psi$. So $\Psi(\theta):V_\theta\to V_\theta$ is represented by a 2×2–matrix with entries in $\mathscr{A}$, and $\mu(\theta)$ is the leading eigenvalue of that matrix, as claimed.

Call the matrix in the claim B_θ, and let $f_\theta(t)=t^2-a(\theta)t-b(\theta)$, be the characteristic polynomial of B_θ, then $a(\theta)=\mathrm{tr}(B_\theta)\in\mathscr{A}$ and $b(\theta)=-\det(B_\theta)\in\mathscr{A}-\mathscr{A}$. It follows that $a^{(k)}(0),b^{(k)}(0)\in\mathbb{Z}$ for all $k\geq0$.

The eigenvalues of B_θ are zeroes of f_θ, therefore $f_\theta(\mu(\theta))=0$. We differentiate this identity twice with respect to θ and then substitute $\theta=0$, noting that $\mu'(0)=\lambda'(0)=\mathbb{E}(g)=$ drift of $\psi=0$. Rearranging terms, we obtain

$$\mu''(0)=\frac{a''(0)\mu(0)+b''(0)}{2\mu(0)-a(0)}.$$

Similarly, $(\ln\lambda)''(0)=(\ln\mu)''(0)=\frac{\mu''(0)}{\mu(0)}$, so $\sigma^2=(\ln\lambda)''(0)=\frac{a''(0)\mu(0)+b''(0)}{2\mu(0)^2-a(0)\mu(0)}$. Since $\mu(0)^2-a(0)\mu(0)-b(0)=0$, we obtain

$$\sigma^2=\frac{a''(0)\mu(0)+b''(0)}{a(0)\mu(0)+2b(0)}\in\mathbb{Q}[\mu(0)].$$

It remains to recall that $\lambda(0)=1$, therefore $\mu(0)=1/\lambda$, so $\mathbb{Q}[\mu(0)]\subset\mathbb{Q}[\alpha]$. □

Proof of Theorem 6.2. We now specialize to the case of $\alpha=\sqrt{2}$.

Let ϕ denote the homogeneous automorphism ϕ with derivative $\left(\begin{smallmatrix}11&-42\\-6&23\end{smallmatrix}\right)$, and which fixes $L_0[\widetilde{p}_0,\underline{w}]$ ($\widetilde{p}_0=$singularity in the bottom left/right corner of horizontal rectangle zero, $\underline{w}=$contracted eigenvector). We will find the function g which drives the random walk of ϕ, and then use (6.1) to calculate σ^2 and the drift of ϕ. There are four steps:

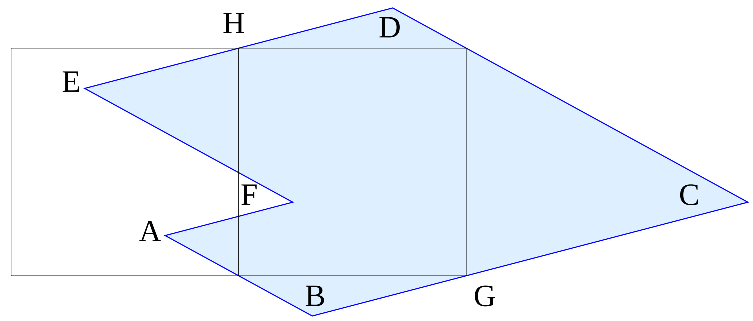

FIGURE 4. Fundamental domain for the renormalizing automorphism of $\sqrt{2}$

(1) Finding the fundamental polygon of ϕ
(2) Calculating the Markov partition and the transition matrix $(t_{PQ})_{\mathfrak{P}\times\mathfrak{P}}$
(3) Calculating g and $\Psi(\theta)$
(4) Finding a closed form for the leading eigenvalue $\mu(\theta)$ of $\Psi(\theta)$, and using the identity $(\ln\mu)''(0) = \sigma^2$.

All the calculations can be done in closed form, but $\mathfrak{P}$ is too large to do this reliably by hand ($\Psi(\theta)$ is a 58×58 matrix). We will supply alternative formulas for t_{PQ} and $g(P,Q)$ which can be easily implemented on a computer (with absolute precision).

Step 1. The fundamental polygon. The derivative matrix is $\left(\begin{smallmatrix} 11 & -42 \\ -6 & 23 \end{smallmatrix}\right)$. The eigenvalues are $\lambda = 17-12\sqrt{2}$, $\lambda^{-1} = 17+12\sqrt{2}$, and the eigenvectors are $\underline{v} = \binom{1-2\sqrt{2}}{1}$ and $\underline{w} = \binom{1+2\sqrt{2}}{1}$.[1]

This places us outside of the cases covered in figure 2. Instead of looking for a reduction to one these cases by means of Lemma 2.7, we find a fundamental polygon directly. It is given in figure 4. The vertices are $A(\frac{4+\sqrt{2}}{8}, \frac{1}{4\sqrt{2}})$, $B(\frac{12-\sqrt{2}}{8}, -\frac{1}{4\sqrt{2}})$, $C(2+\frac{7\sqrt{2}}{8}, \frac{4-\sqrt{2}}{8})$, $D(\frac{12+\sqrt{2}}{8}, \frac{8+\sqrt{2}}{8})$, $E(\frac{4-\sqrt{2}}{8}, \frac{8-\sqrt{2}}{8})$, $F(\frac{7\sqrt{2}}{8}, \frac{4-\sqrt{2}}{8})$.

Step 2. The Markov partition and the incidence matrix

$\mathfrak{P} = \{Q_{11},\ldots,Q_{1N_1}; Q_{21},\ldots,Q_{2N_2}\}$. We can find N_1 and N_2 using Lemma 6.4. The first step is find $\ell^u(Q_1) = |AB|$ and $\ell^u(Q_2) = |CD|$. A direct calculation with the coordinates found in step 1 leads to nested roots, but this can be avoided by first expressing the coordinates of A, B, C, D in the form "puncture + $t\underline{v}$." This leads to the presentation

$$\ell^u(Q_1) = \frac{\sqrt{2}}{4}\|\underline{v}\| \ , \ \ell^u(Q_2) = \frac{2+\sqrt{2}}{4}\|\underline{v}\|,$$

which makes the equations in Lemma 6.4 easy to solve.

The solution is $N_{11} = 5, N_{12} = 12$, $N_{21} = 12, N_{22} = 29$. Thus $N_1 = N_{11}+N_{12} = 17$ and $N_2 = N_{21}+N_{22} = 41$. As a result, $\mathfrak{P}$ contains 58 elements, of which $N_{11}+N_{21} = 17$ are in Q_1 and $N_{12}+N_{22} = 41$ are in Q_2. Every $\mathfrak{P}$–element in Q_1 connects to 5 elements in Q_1 and 12 elements in Q_2; and every $\mathfrak{P}$-element in Q_2 connects to 12 elements in Q_1 and 29 elements in Q_2.

[1]Here we deviate from our convention to choose the eigenvectors in the form $\binom{1}{*}$.

So far so good. But to find the incidence matrix we also needs to know which of the Q_{ij} fall in Q_1 and which fall in Q_2. A calculation by hand or "by inspection" is possible in principle, but not very reliable due to the size of the problem. We look for a method for calculating the position of Q_{ij} using a computer.

Let $\ell^s(Q_i)$ denote the lengths of the stable sides of Q_i, then side BC of our fundamental polygon has length $\ell^s(R) := \ell^s(Q_1) + \ell^s(Q_2)$, and this side contains $p_0 = (2, 0)$ (figure 2).

We parameterize this side by $L^s := [0, 1]$, representing a point by its normalized distance from the left endpoint. The puncture p_0, for example, is parameterized by $\widehat{p}_0 := \frac{\sqrt{2}}{4}$ (nested roots can be avoided as above); the s–sides of Q_1 and Q_2 are parameterized by $L^s_1 := [0, \frac{\ell^s(Q_1)}{\ell^s(R)}) = [0, 1 - \frac{\sqrt{2}}{2})$ and $L^s_2 := [\frac{\ell^s(Q_1)}{\ell^s(R)}, 1) = [1 - \frac{\sqrt{2}}{2}, 1)$.

The key observation is that with this parametrization, a u–fibre which begins at $\tau \in [0, 1]$, ends at $\tau - \frac{\ell^s(Q_1)}{\ell^s(R)} \bmod 1 = \tau + \frac{\sqrt{2}}{2} \bmod 1$. We can use this to keep track of the position of Q_{ij}, by following the image of a u–fibre in the interior of Q_i.

Suppose first that $i = 1$. Q_1 contains the u-fibre W^u_1 which starts at $\frac{\ell^s(Q_1)}{2\ell^s(R)}$. Since ψ_0 contracts L^s towards p_0 by a factor λ, it maps $\frac{\ell^s(Q_1)}{2\ell^s(R)}$ to $\widehat{q}_1 := \widehat{p}_0 - \lambda\left(\widehat{p}_0 - \frac{\ell^s(Q_1)}{2\ell^s(R)}\right) = \frac{2-57\sqrt{2}}{4} \bmod 1$, so $\psi_0(W^u_1)$ can be broken to u–fibres starting at

$$\begin{aligned} \tau_{1j} &:= \widehat{q}_1 - (j-1)\frac{\ell^s(Q_1)}{\ell^s(R)} \bmod 1 \\ &= \frac{2 - 57\sqrt{2}}{4} + (j-1)\frac{\sqrt{2}}{2} \bmod 1 \quad (j = 1, \dots, 17). \end{aligned} \tag{6.4}$$

Q_{1j} is the parallelogram which contains the u–fibre which starts at τ_{1j}. Similarly, Q_2 contains the u–fibre W^u_2 which starts at the fixed point p_0, so Q_{2j} is the parallelogram which contains the u–fibre which starts at

$$\begin{aligned} \tau_{2j} &:= \widehat{p}_0 - (j-1)\frac{\ell^s(Q_1)}{\ell^s(R)} \quad \bmod 1 \\ &= \frac{\sqrt{2}}{4} + (j-1)\frac{\sqrt{2}}{2} \quad (j = 1, \dots, 41). \end{aligned} \tag{6.5}$$

It follows that $Q_{ij} \subset Q_k$ iff $L^s_k \ni \tau_{ij}$, and therefore, by Lemma 6.5 we have the following explicit formula for the incidence matrix:

$$t_{Q_{ij}Q_{k\ell}} = 1_{L^s_k}(\tau_{ij}). \tag{6.6}$$

This can be calculated easily on a computer, *provided the precision of the calculation is smaller than the distance between τ_{ij} and the endpoints of L^s_k.*

We estimate the precision we need. The endpoints of L^s_k are $a \in \{0, 1, 1 - \frac{\sqrt{2}}{2}\}$. Since $\operatorname{dist}(\tau_{ij}, a + \mathbb{Z}) \geq \min\{\frac{1}{4}\operatorname{dist}(4\tau_{ij}, 4a + \mathbb{Z}), \frac{1}{4}\}$, we have the (generous) lower bound $\operatorname{dist}(\tau_{ij}, \{0, 1 - \frac{\sqrt{2}}{2}, 1\} + \mathbb{Z}) \geq \min\{\operatorname{dist}(k\sqrt{2}, \mathbb{Z}) : k = 1, \dots, 100\}$. The last quantity is bounded below by $2 \cdot 10^{-4}$, as can be seen from the sixth principal convergent of $\sqrt{2}$, $\frac{239}{169}$. So the precision we need for the calculation is just 10^{-4}, which is easily available on a standard machine.

Step 3. Calculating $g(Q_{ij}, Q_{k\ell})$.

We use Lemma 6.8. Note that by choice of ϕ, $\tau = 0$, and the calculation of g boils down to finding the $\mathbb{Z}$–displacement of suitable u–fibres.

The $\mathbb{Z}$–displacement of a u–fibre can be determined from the location of its endpoint, see figure 4. This in turn can be determined from the location of the beginning point as follows. Suppose a u–fibre starts at $\tau \in L^s = [0,1]$.

- If $\tau \in [0, \frac{\ell^s(Q_1)}{\ell^s(R)}) = [0, 1-\frac{\sqrt{2}}{2})$, then the endpoint is in AF and the $\mathbb{Z}$–displacement is 0.
- If $\tau \in [\frac{\ell^s(Q_1)}{\ell^s(R)}, \frac{\ell^s(Q_1)}{\ell^s(R)} + \frac{\ell^s(Q_2)}{2\ell^s(R)}) = [1-\frac{\sqrt{2}}{2}, 1-\frac{\sqrt{2}}{4})$, then the endpoint is between E and $(1,1)$ and the $\mathbb{Z}$–displacement is (-1)
- If $\tau \in [\frac{\ell^s(Q_1)}{\ell^s(R)} + \frac{\ell^s(Q_2)}{2\ell^s(R)}, 1) = [1-\frac{\sqrt{2}}{4}, 1)$, then the endpoint is between $(1,1)$ and D and the $\mathbb{Z}$–displacement is $(+1)$

In summary, a u–fibre which starts at $\tau \in L^s$ has $\mathbb{Z}$–displacement

$$\gamma(\tau) = 1_{[1-\frac{\sqrt{2}}{4},1)}(\tau) - 1_{[1-\frac{\sqrt{2}}{2},1-\frac{\sqrt{2}}{4})}(\tau).$$

It follows that $g(Q_{ij}, Q_{k\ell}) = \sum_{m=1}^{\ell-1} \gamma(\tau_{im})$, and by step 2,

$$\Psi_{Q_{ij},Q_{k\ell}}(\theta) = 1_{L^s_k}(\tau_{ij}) \exp\left(\theta \sum_{m=1}^{\ell-1} [1_{[1-\frac{\sqrt{2}}{4},1)}(\tau_{im}) - 1_{[1-\frac{\sqrt{2}}{2},1-\frac{\sqrt{2}}{4})}(\tau_{im})]\right),$$

where τ_{ij}, t_{im} are given by (6.4) and (6.5), $L^s_1 = [0, 1-\frac{\sqrt{2}}{2})$, and $L^s_2 = [1-\frac{\sqrt{2}}{2}, 1)$.

As in the case of the incidence matrix, this can be calculated with complete precision on a standard computer.

Step 4. Calculation of σ^2. We implemented the formulas in the previous step on `Mathematica`, and found $\Psi(\theta)$.

As predicted by the general theory, $\text{rank}[\Psi(\theta)] = 2$, so the characteristic polynomial of $\Psi(\theta)$ takes the form $t^{|\mathfrak{P}|-2}[t^2 + b(\theta)t + c(\theta)]$. The largest eigenvalue can therefore be found in closed form. We did this using `Mathematica` and got

$$\mu(\theta) = \frac{1}{2}e^{-2\theta}\left(9 + 16e^{\theta} + 9e^{2\theta} + 3\sqrt{(1+e^{\theta})^2(9+14e^{\theta}+9e^{2\theta})}\right). \tag{6.7}$$

It follows that $\sigma^2 = (\log\mu)''(0) = \frac{3}{8}\sqrt{2}$. □

Proof of Theorem 6.3. The calculations in the previous proof were done for the renormalizing automorphism ϕ which fixes the rays $L_0(p_0, \underline{w})$. By Lemma 2.12, $\tau_\psi(p_0, \underline{w}) = -\delta(\phi)$. The drift of ϕ is $\mathbb{E}(g)$, and by (6.3) $\mathbb{E}(g) = (\log\mu)'(0)$. It follows that $\tau_\psi(p_0, \underline{w}) = -(\log\mu)'(0)$. By (6.7), $\tau_\psi(p_0, \underline{w}) = 1$.

Here is another proof that $\tau_\psi(p_0, \underline{w}) = 1$. The first step is to express the derivative of ψ through generators of $\Gamma(2)$:

$$\begin{pmatrix} 11 & -42 \\ -6 & 23 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 6 & 1 \end{pmatrix}\begin{pmatrix} 1 & -4 \\ 0 & 1 \end{pmatrix}.$$

Let ψ_i denote the unique homogeneous automorphism with zero drift and derivative $\left(\begin{smallmatrix} 1 & -4 \\ 0 & 1 \end{smallmatrix}\right)$ $(i=1)$, $\left(\begin{smallmatrix} 1 & 0 \\ 6 & 1 \end{smallmatrix}\right)$ $(i=2)$, $\left(\begin{smallmatrix} 1 & -2 \\ 0 & 1 \end{smallmatrix}\right)$ $(i=3)$, and $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ $(i=4)$. By the uniqueness of homogeneous automorphisms with zero drift and given derivative, $\psi = \psi_4 \circ \psi_3 \circ \psi_2 \circ \psi_1$.

The automorphisms ψ_i are known explicitly (see appendix). To describe them, note that St has two canonical cylinder decompositions of St, one into horizontal cylinders and the other into vertical cylinders (Figure 5).

(1) ψ_1 acts on every horizontal cylinder by $\binom{x}{y} \mapsto \binom{x-4y \mod 2}{y \mod 1}$, where (x,y) are measured from the bottom left corner of the corresponding horizontal rectangle. So $\psi_1[L_0(p,\underline{w})] = L_0(p', \binom{-3+2\sqrt{2}}{1})$, where p' :=bottom right corner of horizontal rectangle #0 (p' is congruent to p)
(2) ψ_2 acts on every vertical cylinder by $\binom{x}{y} \mapsto \binom{x \mod 1}{y+6x \mod 2}$, where (x,y) are measured from the bottom left corner of the corresponding vertical rectangle. So $\psi_2[L_0(p', \binom{-3+2\sqrt{2}}{1})] = L_0(p', -\binom{3-2\sqrt{2}}{17-12\sqrt{2}}) = L_0(p', -\binom{3+2\sqrt{2}}{1}) = L_1(q, -\binom{3+2\sqrt{2}}{1})$, where q is the singularity at middle of the top side of horizontal rectangle #1 (q is congruent to p and p'). We had to move one horizontal rectangle up, because $-\binom{3+2\sqrt{2}}{1}$ based at p' points outside of horizontal rectangle #0.
(3) ψ_3 acts every horizontal cylinder by $\binom{x}{y} \mapsto \binom{x-2y \mod 2}{y \mod 1}$, where (x,y) are measured from the bottom left corner of the corresponding horizontal rectangle. So $\psi_3[L_1(q, -\binom{3+2\sqrt{2}}{1})] = L_1(q, -\underline{w})$.
(4) $\psi_4 = D^{-1} \circ \phi^2$, where ϕ maps horizontal cylinders into vertical cylinders by rotating the horizontal rectangles 90° counterclockwise around the midpoint of the top side of the corresponding horizontal rectangle. Now
   (i) $\phi[L_1(q, -\underline{w})] = L_1(q, \binom{1}{-(1+2\sqrt{2})})$;
   (ii) $\phi[L_1(q, \binom{1}{-(1+2\sqrt{2})})] = L_1(q, \underline{w}) = L_2(q, \underline{w}) = L_2(D^2(p), \underline{w})$ (we moved up, because $\underline{w}$ based at q, points outside of rectangle #1);
   (iii) $D^{-1}[L_2(D^2(p), \underline{w})] = L_1(D(p), \underline{w})$.

   Consequently, $\psi_4[L_1(q, -\underline{w})] = L_1(D(p), \underline{w})$.

In summary, $\psi[L_0(p,\underline{w})] = L_1(D(p), \underline{w}) = D[L_0(p, \underline{w})]$. So $\tau_\psi(p, \underline{w}) = 1$. □

7. Characterization of generic points

In this section we leave the study of the deterministic random walk, and turn to a different problem: The description of the generic points of the cylinder map.

The cylinder map T_α is ergodic and conservative with respect to the infinite invariant measure $m_{\mathbb{T}\times\mathbb{Z}}$ for every α irrational ([**C**],[**CK**],[**Sch**],[**AK**]). By Hopf's ratio ergodic theorem, for every $F, G \in L^1(\mathbb{T}\times\mathbb{Z})$ s.t. $\int G dm_{\mathbb{T}\times\mathbb{Z}} > 0$,

$$\frac{\sum_{j=0}^{n-1}(F\circ T_\alpha^j)(x,k)}{\sum_{j=0}^{n-1}(G\circ T_\alpha^j)(x,k)} \xrightarrow[n\to\infty]{} \frac{\int F dm_{\mathbb{T}\times\mathbb{Z}}}{\int G dm_{\mathbb{T}\times\mathbb{Z}}} \tag{7.1}$$

for $m_{\mathbb{T}\times\mathbb{Z}}$-almost every $(x,k) \in \mathbb{T}\times\mathbb{Z}$.

But (7.1) does not hold for *every* $(x,k) \in \mathbb{T}\times\mathbb{Z}$. This is because T_α admits other ergodic conservative locally finite measures [**Nkd**]. If μ is one of the other measures, then the limit in (7.1) is $\int F d\mu / \int G d\mu$ μ–almost everywhere, and not $\int F dm_{\mathbb{T}\times\mathbb{Z}} / \int G dm_{\mathbb{T}\times\mathbb{Z}}$.

This raises the question what exactly is the domain of validity of (7.1). To state the problem in a meaningful way, we need the following definition.

Definition 7.1. *A point (x,k) is called* generic *(for T_α and $m_{\mathbb{T}\times\mathbb{Z}}$), if it satisfies* (7.1) *for every $F, G \in C_c(\mathbb{T}\times\mathbb{Z})$ s.t. $\int G dm_{\mathbb{T}\times\mathbb{Z}} > 0$.*

By the discussion above almost every point is generic, but some points are not generic. It was asked in [**Sa**] what are the generic points of T_α.

In this section we give the answer in the special case when α is a quadratic irrational. Let $\omega : \mathbb{T} \to \mathrm{St}$ be as in Theorem 3.1.

Theorem 7.2. *Suppose α is a quadratic irrational, with renormalizing automorphism ψ, then (x,k) is generic for T_α and $m_{\mathbb{T}\times\mathbb{Z}}$ iff* $\frac{1}{n}\xi[\psi^n(\omega(x))] \xrightarrow[n\to\infty]{} 0$.

Let $\binom{\sin\theta}{\cos\theta}$ denote the stable direction of ψ, and φ_θ the linear flow in direction θ on the infinite staircase. A point $\omega\in\mathrm{St}$ is called generic for φ_θ and the area measure m, if $\int_0^T F[\varphi_\theta^t(\omega)]dt/\int_0^T G[\varphi_\theta^t(\omega)]dt \xrightarrow[T\to\infty]{} \int Fdm/\int Ggm$ for every $F,G\in C_c(\mathrm{St})$ s.t. $\int Gdm>0$. We will obtain Theorem 7.2 from the following result.

Theorem 7.3. $\omega\in\mathrm{St}$ *is generic for φ_θ and m iff* $\frac{1}{k}\xi[\psi^k(\omega)] \xrightarrow[k\to\infty]{} 0$.

Theorem 7.2 follows from Theorem 7.3 in the same way Theorem 3.1 follows from Theorem 3.2.

Proof of Theorem 7.3. If the theorem holds for one choice of a $\mathbb{Z}$–coordinate, then it works for all choices of $\mathbb{Z}$–coordinates, therefore we are free to use the $\mathbb{Z}$–coordinate of our choice. We choose the $\mathbb{Z}$–coordinate associated to ψ.

Suppose $\xi[\psi^k(\omega_0)]/k \xrightarrow[k\to\infty]{} 0$, then ω_0 is generic because of Theorem 3.2 : the fluctuating exponential term cancels out upon division.

The remainder of the proof deals with the implication "genericity⇒zero drift." We use the strategy of [**SS**].

Fix a generic point $\omega_0\in\mathrm{St}$, let $A_T:=\{\varphi_\theta^t(\omega_0):0\le t\le T\}$, and define λ_T to be the normalized length measure on $A_T\cap[\xi=0]$. Since ω_0 is generic, λ_T converges weak star to the normalized Lebesgue measure on $[\xi=0]$.

Construction: Fix some $N\ge 1$ to be chosen later, and define for every $k\ge 0$

$$X_k^N := \xi\circ\psi^{(k+1)N}-\xi\circ\psi^{kN} = \left(\sum_{j=kN}^{(k+1)N-1} F_\psi\circ\psi_0^j\right)\circ\pi.$$

We think of X_k^N as of bounded random variables on $(A_T,\mathscr{B}(A_T),\lambda_T)$. The bound is $|X_k^N|\le N\max|F_\psi|$.

Let $\log^*:=\log_{\lambda^{-1}}$, where λ is the eigenvalue of the derivative of ψ in $(0,1)$. Since $\xi=0$ λ_T–a.e. and F_ψ is uniformly bounded,

$$\begin{aligned}\sum_{k=0}^{\frac{1}{N}[\log^* T]-2} X_k^N &= \sum_{k=0}^{\frac{1}{N}[\log^* T]-1} X_k^N + O(N) = \xi\circ\psi^{[\log^* T]}-\xi+O(N)\\ &= \xi\circ\psi^{[\log^* T]}+O(N) \text{ uniformly on } \mathrm{supp}(\lambda_T).\end{aligned}$$

The right hand side is nearly constant λ_T–a.s., because $\psi^{[\log^* T]}$ contracts the support of λ_T (a subset of A_T) to a set of diameter less than λ^{-1}, and the $\mathbb{Z}$–coordinates of points in such a set must be uniformly bounded away from one another. It follows that for λ_T–a.e. $\omega\in A_T$,

$$\sum_{k=0}^{\frac{1}{N}[\log^* T]-2} X_k^N(\omega) = \xi[\psi^{[\log^* T]}(\omega_0)]+O(N) \text{ uniformly in } T.$$

Taking expectations with respect to λ_T and dividing by $[\log^* T]$, we obtain

$$\frac{\xi[\psi^{[\log^* T]}(\omega_0)]}{[\log^* T]} = \mathbb{E}_{\lambda_T}\left(\frac{1}{[\log^* T]}\sum_{k=0}^{\frac{1}{N}[\log^* T]-2} X_k^N\right)+o(1), \text{ as } T\to\infty. \tag{7.2}$$

The expectation of $\frac{1}{[\log^* T]}\sum_{k=0}^{\frac{1}{N}[\log^* T]-2} X_k^N$ with respect to the *normalized Lebesgue's measure* on $[\xi = 0]$ is zero (because ψ is an automorphism with zero drift). We will use the genericity (in the form $\lambda_T \xrightarrow[T\to\infty]{w^*}$Normalized Lebesgue$\restriction_{[\xi=0]}$) to show that something close happens to the λ_T–expectation. More precisely:

Claim. *If ω_0 is generic then for every ε there exists N s.t. for all T large enough*

$$\max\left\{|\mathbb{E}_{\lambda_T}(\tfrac{1}{N}X_k^N)| : 0 \le k \le \tfrac{1}{N}[\log^* T] - 2\right\} < \varepsilon. \tag{7.3}$$

Together with (7.2), this implies that $\frac{1}{k}\xi_k(\omega_0) \xrightarrow[k\to\infty]{} 0$ and finishes the proof.

We begin the proof of (7.3). Fix ω_0 generic and $\varepsilon > 0$, and let C, δ_0, N_0 be some parameters that will be calibrated at the end of the proof. Let λ_T^C denote the length measure on $A_T \cap [|\xi| \le C]$, normalized so that $\lambda_T^C[\xi = 0] = 1$ (this is not a probability measure).

Step 1. Choosing $N > N_0$ and τ s.t. $\lambda_T^C\left[|\frac{1}{N}X_0^N| > \delta_0\right] < \delta_0$ for all $T > \tau$.

Proof. Fix $0 < \delta < \delta_0$. Since $\frac{1}{k}\xi \circ \psi^k \xrightarrow[k\to\infty]{} 0$ almost everywhere with respect to Lebesgue's measure, we can use Egoroff's theorem to find $N = N(\delta, C) > N_0$ s.t. $\Lambda_0 := \{\omega \in \mathrm{St} : |\xi(\omega)| \le C, |\frac{1}{N}X_0^N| > \delta_0\}$ satisfies $m(\Lambda_0) < \delta$, where m is the area measure on St.

Since ω_0 is generic, λ_T^C converges w^* to the Lebesgue measure on $[|\xi| \le C]$. The indicator functions of Λ_0 and $[\xi = 0]$ are discontinuous, but all discontinuities lie on the boundaries of the parallelograms of the Markov partition, and their images under ψ^{-N}. N is fixed, therefore the closure of the singular set has measure zero. Using a standard approximation argument, it is easy to show that $\lambda_T^C(\Lambda_0) \xrightarrow[T\to\infty]{} m(\Lambda_0) < \delta$. It follows that there exists $\tau = \tau(\delta, C)$ s.t. for all $T > \tau(\delta, C)$, $\lambda_T^C(\Lambda_0) < \delta$, proving step 1.

Step 1 allows us to bound $\mathbb{E}_{\lambda_T}(\frac{1}{N}X_0^N)$ as follows. Choose $\delta_0 < \varepsilon/(1+\max|F_\psi|)$, then for all $T > \tau$,

$$|\mathbb{E}_{\lambda_T}(\tfrac{1}{N}X_0^N)| \le \delta_0 + \max|\tfrac{1}{N}X_0^N| \cdot \lambda_T\left[|\tfrac{1}{N}X_0^N| \ge \delta_0\right] \le \delta_0(1 + \max|F_\psi|) < \varepsilon.$$

It is tempting to try to bound $\mathbb{E}_{\lambda_T}(\frac{1}{N}X_k^N)$ for $k \ne 0$ in the same way. Unfortunately the methods of step 1 can only be used for bounded k, whereas (7.3) calls for a uniform bound for $0 \le k \le [\log^* T]$, as $T \to \infty$.

We will take an indirect approach. Imagine we were able to construct self maps $\theta_k : A_T \to A_T$ $(0 \le k \le \frac{1}{N}[\log^* T] - 2)$ with the following properties:

- $X_k^N = X_0^N \circ \theta_k$+error, uniformly bounded by E_0,
- θ_k is Borel, one-to-one, and $\theta_k(A_T \cap [\xi = 0]) \subset A_T \cap [|\xi| \le C]$,
- $C_r^{-1} \le \frac{d\ell\circ\theta_k}{d\ell} \le C_r$ where $d\ell$ is the (Lebesgue) length measure and C_r is a global constant, independent of N, T, k, and ε.

Then it would follow that $\lambda_T \le C_r\lambda_T^C \circ \theta_k|_{[\xi=0]}$, and if $E_0/N_0 < \delta_0$ then

$$\begin{aligned}\lambda_T[|\tfrac{1}{N}X_k^N| > 2\delta_0] &\le C_r(\lambda_T^C \circ \theta_k)([|\tfrac{1}{N}X_k^N| > 2\delta_0] \cap [\xi = 0])\\ &\le C_r\lambda_T^C[|\tfrac{1}{N}X_k^N \circ \theta_k^{-1}| > \delta_0] \le C_r\lambda_T^C[|\tfrac{1}{N}X_0^N| > \delta_0] < C_r\delta_0.\end{aligned}$$

This, and the fact that $\sup|X_k^N|$ are uniformly bounded, is sufficient to bound $\mathbb{E}_{\lambda_T}(|\frac{1}{N}X_k^N|)$ uniformly and prove (7.3).

In reality we do not know how to construct such θ_k, because of edge effects at the endpoints of A_T. Luckily these edge effects can be controlled well enough to push this argument through with minor modifications. The details follow.

Step 2. Breaking A_T into the "interior" and "edge" s–fibres.

We use the Adler–Weiss coding of Theorem 2.8 and Lemma 3.3. Recall that an s–fibre is a set of the form $W^s(\underline{x}, k) :=$lift to $[\xi = k]$ of $W^s(\underline{x}) := \{\pi_0(\underline{y}) : y_0^\infty = x_0^\infty\}$, where $z_0^\infty := (z_0, z_1, \ldots)$. This is a stable linear segment, with length $h(x_0)$.

We define an n_0*–stable block* to be the lift to $[\xi = k]$ of a set of the form $\{\pi_0(\underline{y}) : y_{-n_0}^\infty = x_{-n_0}^\infty\}$. These are closed stable linear segments, and their length is $\lambda^{n_0} h(x_{-n_0}) \asymp \lambda^{n_0} \xrightarrow[n_0\to\infty]{} 0$. ($A_n \asymp B_n$ means $C^{-1} < A_n/B_n < C$ for all n large.) Different n_0–stable blocks are disjoint, or they meet at one or two endpoints. Every point belongs to at most two n_0–stable blocks.

$B_T := \psi^{[\log^* T]}(A_T)$ is a stable linear segment with length in $[1, \lambda^{-1}]$. The n_0–stable blocks which intersect the relative interior of B_T fall into two groups:

- two or less "edge" n_0–stable blocks which cover the endpoints of B_T;
- $(1 - 4\lambda^{n_0} \max h)/(\lambda^{n_0} \min h)$ or more "interior" n_0–stable blocks which lie completely inside B_T.

Since the number of interior n_0–blocks tends to infinity as $n_0 \to \infty$, it is possible to fix once and for all n_0 in such a way that there is at least one interior block.

This choice is independent of N_0, therefore it is possible to assume without loss of generality that $N_0 > n_0$.

If $[\log^* T] \gg n_0$, then the decomposition of $B_T = \psi^{[\log^* T]}(A_T)$ into interior and boundary n_0–blocks induces a decomposition of A_T into interior and edge s–fibres. Here and throughout:

- An *edge s–fibre* of A_T is an s–fibre in $\psi^{-[\log^* T]}$(edge n_0–block of B_T).
- An *interior s–fibre* of A_T is a stable fibre in $\psi^{-[\log^* T]}$(interior n_0–block of B_T).

Let $\mathscr{W}_{int}, \mathscr{W}_{bnd}$ be the collection of interior and edge stable fibres in A_T. The *interior* of A_T is $A_T(int) := \bigcup \mathscr{W}_{int}$, and the *boundary* of A_T is $A_T(bnd) := \bigcup \mathscr{W}_{bnd}$.

Step 3. Defining θ_k on $A_T(int)$ ($0 \le k \le \frac{1}{N}[\log^* T] - 2$).

Recall that $\mathscr{G}$ is the dynamical graph of the Markov partition $\mathfrak{P}$. An *admissible word* is a finite word $w_0 \cdots w_n \in \mathfrak{P}^n$ s.t. $w_0 \to \cdots \to w_n$ is a path on $\mathscr{G}$.

Since $\Sigma(\mathscr{G})$ is topologically mixing, there exists a constant M_{br} such that for every pair of $a, b \in \mathfrak{P}$ there is a path $\underline{w}_{ab}$ on $\mathscr{G}$ of length M_{br} s.t. $a\underline{w}_{ab}b$ is admissible. Fix for such a, b a word $\underline{w}_{ab}$, and call it the *bridge* from a to b. In what follows

$$(\underline{a}, \text{bridge}, \underline{b}) := (\underline{a}, \underline{w}_{a_{\text{last}} b_{\text{first}}}, \underline{b}).$$

We define $\theta_k : A_T(int) \to A_T$ as follows. Suppose $\omega \in A_T(int)$. Fix a stable fibre $W^s(\underline{x}, \eta) \subset A_T(int)$. All but countably many points in $W^s(\underline{x}, \eta)$ can be uniquely represented in the form $(\underline{y}, \eta)$ where

$$\underline{y} \in \Sigma(\mathscr{G}) \text{ , } y_0^\infty = x_0^\infty,\ \xi(\omega) = \eta, \text{ and } \pi(\omega) = \pi_0(\underline{y}).$$

Write $\underline{y} = (y_{-\infty}^{-1}|\underline{B}_0, \underline{B}_1, \ldots, \underline{B}_k, y_{kN}^{\infty})$, where $\underline{B}_i$ are words of length of N, and the zeroth coordinate is to the immediate right of $|$. We let $\theta_k(\underline{y}, \eta) := (\underline{z}, \eta')$ where

$$\underline{z} = \sigma^{4M_{br}}(y_{-\infty}^{-1}|\text{bridge}, \underline{B}_k, \text{bridge}, \underline{B}_1, \ldots, \underline{B}_{k-1}, \text{bridge}, \underline{B}_0, \text{bridge}, y_{(k+1)N}^{\infty})$$

$$\eta' = \eta + \sum_{j=0}^{[\log^* T]-1} F(\sigma^j \underline{y}) - F(\sigma^j \underline{z}), \text{ where } F := F_\psi \circ \pi_0.$$

What we have done here is to exchange block zero with block k, plug-in bridge words to ensure admissibility, and at the end apply the shift and modify η to ensure that we remain inside A_T (see below). Here are some properties of θ_k.

(i) $X_0^N \circ \theta_k = X_k^N$+*bounded error:* θ_k exchanges the zeroth block with the k–th block, and therefore $|X_0^N \circ \theta_k - X_k^N| \leq 6M_{br} \max |F_\psi|$. Here we are using (2.4).

(ii) $\theta_k[A_T(int)] \subset A_T(int)$ *for all* $0 \leq k \leq \frac{1}{N}[\log^* T] - 2$*:* We abuse notation and identify a point in St with its symbolic coding in $\Sigma(\mathscr{G}) \times \mathbb{Z}$. Suppose $(\underline{y}, \eta)$ belongs to an interior n_0–block and $0 \leq k \leq \frac{1}{N}[\log^* T] - 2$, then

$$(\sigma^{[\log^* T]}\underline{z})_{-N}^{\infty} = \sigma^{[\log^* T]}(\underline{y})_{-N}^{\infty}, \text{ because of the } \sigma^{4M_{br}} \text{ in the definition of } \underline{z}, \text{ and}$$

$$\begin{aligned}\psi^{[\log^* T]}(\underline{z}, \eta') &= \big(\sigma^{[\log^* T]}(\underline{z}), \eta' + \sum_{j=0}^{[\log^* T]-1} F(\sigma^j \underline{z})\big) \\ &= \big(\sigma^{[\log^* T]}(\underline{z}), \eta + \sum_{j=0}^{[\log^* T]-1} F(\sigma^j \underline{y})\big), \text{ by the definition of } \eta'.\end{aligned}$$

Since $N > n_0$, $\psi^{[\log^* T]}(\underline{z}, \eta')$ belongs to the same n_0–block which contains $\psi^{[\log^* T]}(\underline{y}, \eta)$. This is an interior n_0–block of $B_T = \psi^{[\log^* T]}(A_T)$. So $(\underline{z}, \eta') \in A_T$.

(iii) $\theta_k[\text{supp}(\lambda_T)] \subset \text{supp}(\lambda_T^{C'})$ *for* $C' := 100[M_{br} + 1] \max |F_\psi|$*:* It is enough to check that $|\eta' - \eta| = \left|\sum_{j=0}^{[\log^* T]-1} F(\sigma^j \underline{z}) - \sum_{j=0}^{[\log^* T]-1} F(\sigma^j \underline{y})\right| \leq C'$. To check this we recall that F is constant on symbolic 2–cylinders, therefore the difference between the two sums can only come from the following sources:

- the effect of the shift by $\sigma^{4M_{br}}$, bounded by $2 \cdot 4M_{br} \max |F_\psi|$
- the sum over the bridge words, bounded by $4M_{br} \max |F_\psi|$
- the value of $F(\sigma^j \underline{z})$ for the j at the end of $\underline{B}_k$, $\underline{B}_{k-1}$, and $\underline{B}_0$, with a total effect bounded by $3 \max |F_\psi|$

This gives the bound above with C much smaller than claimed.

(iv) θ_k *is one-to-one on* $A_T(int)$*:* To reconstruct $\underline{y}$ from $\underline{z}$ one just needs to erase the bridge words (whose position is always the same), and then exchange the k–th block with the zeroth block of what remains. Once $\underline{y}$ is known, η can be easily calculated from η' and $\underline{z}$.

(v) $d\ell \circ \theta_k / d\ell$ *is uniformly bounded away from* $0, \infty$*, where* ℓ *is the length measure on* A_T*:* For every $\underline{a} \in \Sigma(\mathscr{G})$, the length of the stable linear segment

$$[a_{-n}^{\infty}] := \{\omega \in A_T(int) \cap [\xi = 0] : \pi(\omega) = \pi_0(\underline{y}), y_j = a_j \quad (j \geq -n)\}$$

is $\lambda^n h(a_{-n})$, because $\psi^{-n}[a_{-n}^{\infty}] = W^s(\sigma^{-n}\underline{a})$, $\ell[W^s(\sigma^{-n}\underline{a})] = h(a_{-n})$, and ψ^{-n} expands linear stable segments by a factor of λ^{-n}. With this formula at hand, it is easy to see that $\frac{d\ell \circ \theta_k}{d\ell} \in [\lambda^{4M_{br}}(\frac{\min h}{\max h}), \lambda^{4M_{br}}(\frac{\max h}{\min h})]$.

Step 4. Defining θ_k on $A(bnd)$.

Fix once and for all an *interior* stable fibre $W^s(\underline{x},\eta)$ of A_T. We will define θ_k on an *edge* stable fibre by first mapping it into $W^s(\underline{x},\eta)$, and then applying $\theta_k|_{A(int)}$ as defined in step 3.

The resulting transformation is $\theta_k : (\underline{y},\eta) \mapsto (\underline{z},\eta')$ where

$$\underline{z} = \sigma^{5M_{br}}(y_{-\infty}^{-1}|\text{bridge}, \underline{B}_k, \text{bridge}, \underline{B}_1, \ldots, \underline{B}_{k-1}, \text{bridge}, \underline{B}_0, \text{bridge}, \\ y_{kN}^{[\log^* T]-1}, \text{bridge}, x_{[\log^* T]}^{\infty})$$

$$\eta' = \eta + \sum_{j=0}^{[\log^* T]} F_\psi(\sigma^j \underline{y}) - F_\psi(\sigma^j \underline{z})$$

The following properties can be verified as in the previous step:

(i) $X_0^N \circ \theta_k = X_k^N$+bounded error. The error is bounded by $8M_{br}\max|F_\psi|$.
(ii) $\theta_k[A(bnd)] \subset A_T(int)$ and $\theta_k[A(bnd)] \cap [\xi = 0]) \subset A_T(bnd) \cap [|\xi| \leq C']$ with C' as above.
(iii) θ_k is one-to-one on each boundary stable fibre, and its Radon-Nikodym derivative takes values in $[\lambda^{5M_{br}}(\frac{\min h}{\max h}), \lambda^{5M_{br}}(\frac{\max h}{\min h})]$.

Step 5. Proof of (7.3).

We estimate $\mathbb{E}_{\lambda_T}(\frac{1}{N}X_k^N)$ for $0 \leq k \leq \frac{1}{N}[\log_{\lambda^{-1}} T] - 2$. Since $|X_k^N| \leq N \max|F_\psi|$,

$$\left|\mathbb{E}_{\lambda_T}(\tfrac{1}{N}X_k^N)\right| \leq 2\delta_0 + \max|F_\psi| \cdot \lambda_T(\Lambda_k), \tag{7.4}$$

where $\Lambda_k := \{\omega \in S : \xi(\omega) = 0,\ |\frac{1}{N}X_k^N| > 2\delta_0\}$. We will bound $\lambda_T(\Lambda_k)$.

We now choose the constants C, N_0, δ_0: $C := C' = 100(M_{br}+1)\max|F_\psi|$, $\delta_0 := \varepsilon/(2 + 3\lambda^{-5M_{br}}(\frac{\max h}{\min h})\max|F_\psi|)$, and N_0 so large that $\frac{10M_{br}\max|F_\psi|}{N_0} < \delta_0$.

By construction, $|X_k^N - X_0^N \circ \theta_k| \leq 8M_{br}\max|F_\psi|$, so if $|\frac{1}{N}X_k^N(\omega)| > 2\delta_0$ then $|\frac{1}{N}X_0^N(\theta_k(\omega))| > 2\delta_0 - \frac{10M_{br}\max|F_\psi|}{N_0} > \delta_0$. We see that

$$\theta_k(\Lambda_k) \subset \{\omega \in A_T : |\xi(\omega)| \leq C,\ |\tfrac{1}{N}X_0^N| > \delta_0\}.$$

Applying θ_k^{-1} to both sides and recalling that θ_k is piecewise invertible, at worst three-to-one (edge effects), and $\frac{d\ell\circ\theta_k}{d\ell} \geq \lambda^{5M_{br}}(\frac{\min h}{\max h})$ we find that

$$\lambda_T(\Lambda_k) \leq 3\lambda^{-5M_{br}}\left(\tfrac{\max h}{\min h}\right)\lambda_T^C\left[|\tfrac{1}{N}X_0^N| > \delta_0\right].$$

By step 1, $\lambda_T(\Lambda_k) \leq 3\lambda^{-5M_{br}}(\frac{\max h}{\min h})\delta_0$. Substituting this at (7.4) we find that $\left|\mathbb{E}_{\lambda_T}(\frac{1}{N}X_k^N)\right| \leq \delta_0(2 + 3\lambda^{-5M_{br}}(\frac{\max h}{\min h})\max|F_\psi|) < \varepsilon$ as required. □

Appendix A. Proofs of Proposition 2.3, Lemma 2.7, and Lemma 2.9

Classification of homogeneous automorphisms. We prove Proposition 2.3:

(1) If $A \in \mathrm{SL}(2,\mathbb{Z})$, $A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \bmod 2$ and $\delta_0 \in \mathbb{Z}$, then there is a unique homogeneous automorphism with derivative A and drift δ_0.
(2) If $A \in \mathrm{SL}(2,\mathbb{Z})$, $A = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \bmod 2$ and $\delta_0 \in \frac{1}{2} + \mathbb{Z}$, then there is a unique homogeneous automorphism with derivative A and drift δ_0.
(3) No other homogeneous automorphisms exist.

Step 1. Existence of homogeneous automorphisms as in parts 1 and 2.

Proof. Let $\Gamma(2) := \{A \in \mathrm{SL}(2,\mathbb{Z}) : A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \mod 2\}$, $\Gamma := \Gamma(2) \cup \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\Gamma(2)$. $\Gamma(2)$ is generated by $\left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right)$, and $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. We will construct

- A homogeneous automorphism with derivative $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and drift $+1$;
- A homogeneous automorphism with derivative $\left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$ and drift 0;
- A homogeneous automorphism with derivative $\left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right)$ and drift 0;
- A homogeneous automorphism with derivative $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and drift 0;
- A homogeneous automorphism with derivative $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and drift $\frac{1}{2}$.

The automorphisms in part (1) can be constructed from the first three automorphisms. The automorphisms in part (2) require an additional composition with the fourth automorphism.

A homogeneous automorphism with derivative $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ *and drift* 1*:* D.

A homogeneous automorphism with derivative $\left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$ *and drift* 0*:* Divide St into horizontal cylinders, as indicated by the dashed lines in figure 5(a). Act on each cylinder by the map $(x, y) \mapsto (x + 2y \mod 2, y)$ up to identifications, where (x, y) are measured relative to the bottom left corner. These maps equal the identity on the boundary of the cylinder (the horizontal sides of the rectangle), therefore they glue to an automorphism ψ. The derivative of ψ is $\left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$, ψ commutes with D, and ψ fixes the singularities of St. It has zero drift, because the Frobenius function with respect to the horizontal rectangles vanishes.

A homogeneous automorphism with derivative $\left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right)$ *and drift* 0*:* The same construction, but using the decomposition of St into vertical cylinders (figure 5(a)).

A homogeneous automorphism with derivative $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ *and drift* $\frac{1}{2}$*:* Decompose St into hotizontal rectangles. Rotate every rectangle 90 degrees counterclockwise around the midpoint of its top side, turning it into a vertical rectangle. These maps glue continuously to an automorphism of St (figure 5(b)). Using the $\mathbb{Z}$–coordinate defined by the horizontal rectangles, one sees that the average drift is $\frac{1}{2}$.

A homogeneous automorphism with derivative $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ *and drift* 0*:* Suppose ψ is the automorphism with drift $\frac{1}{2}$ and derivative $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ constructed above, then $\varphi := D^{-1} \circ \psi^2$ has zero drift and derivative $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.

Step 2. A homogeneous automorphism with derivative $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ is equal to D^k for some $k \in \mathbb{Z}$. Two homogeneous automorphisms with the same derivative and drift are equal.

Proof. Let ψ be a homogeneous automorphism with derivative $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Fix a horizontal rectangle R and let p_0 denote the singularity at its lower right corner. Given $p \in \mathrm{int}(R)$, let $\gamma_p \subset R$ denote the linear segment from p_0 to p. Since ψ fixes the D–orbit of p_0, $\psi[\gamma_p]$ is a linear segment from p_0 or $D(p_0)$ to $\psi(p)$. Since the derivative of ψ is $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $\psi[\gamma_p]$ has the same slope, length, and direction as γ_p.

There are infinitely many such segments, one for every horizontal rectangle $D^k(R)$, $k \in \mathbb{Z}$. By reasons of continuity there is some fixed $k \in \mathbb{Z}$ such that $\psi[\gamma_p] \subset D^k(R)$ for all $p \in \mathrm{int}(R)$. It follows that $\psi = D^k$ on R. Since ψ commutes with D, $\psi = D^k$ on St. This proves the first part of step 2.

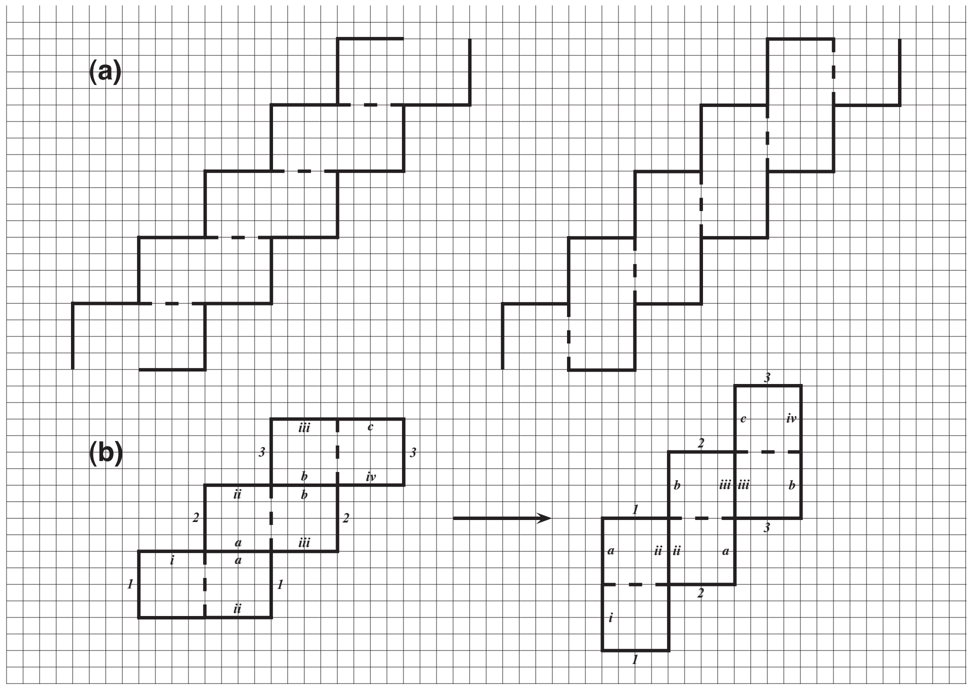

FIGURE 5. (a) decomposition of St into horizontal and vertical cylinders; (b) a homogeneous automorphism with drift $\frac{1}{2}$

For the second part, suppose ψ_1, ψ_2 have the same derivative and drift, then $\psi_1 \circ \psi_2^{-1}$ has derivative $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and drift zero (Lemma 2.2). So $\psi_1 \circ \psi_2^{-1} = D^k$ with $k = \text{drift} = 0$.

Step 3. For every homogeneous automorphism, either the derivative is $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \bmod 2$ and the drift is in $\mathbb{Z}$, or the derivative is $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \bmod 2$ and the drift is in $\frac{1}{2} + \mathbb{Z}$.

Proof. Suppose ψ is a homogeneous automorphism, and let $\psi_0 : \mathrm{St}_0 \to \mathrm{St}_0$ be the projection of ψ to St_0.

$\mathrm{St}_0 = \mathbb{R}^2/G$ where G is generated by the translations by $\binom{1}{1}$ and $\binom{1}{-1}$, see figure 1(c). The change of coordinates $\Theta\binom{x}{y} = \frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$ gives the identification $\mathrm{St}_0 \simeq \mathbb{R}^2/\sqrt{2}\mathbb{Z}^2$. In these coordinates, the punctures are $\sqrt{2}\mathbb{Z}^2$ and $\frac{1}{\sqrt{2}}\binom{1}{1} + \sqrt{2}\mathbb{Z}^2$, and ψ_0 is represented by $\widehat{\psi}_0 := \Theta \circ \psi_0 \circ \Theta$.

Let B denote the derivative of $\widehat{\psi}_0$. The linear segments in $\mathbb{R}^2$ from $\binom{0}{0}$ to $\sqrt{2}\binom{1}{0}, \sqrt{2}\binom{0}{1}$ project to closed curves γ_1, γ_2 on $\mathbb{R}^2/\sqrt{2}\mathbb{Z}^2$. $\widehat{\psi}_0$ fixes $\sqrt{2}\mathbb{Z}^2$ (a singularity), therefore if we apply $\widehat{\psi}_0$ to γ_1, γ_2, and lift the result to $\mathbb{R}^2$ at $\binom{0}{0}$, then we get linear segments from $\binom{0}{0}$ to $B\binom{\sqrt{2}}{0}$ and $B\binom{0}{\sqrt{2}}$. These segments project to the closed curves $\widehat{\psi}_0[\gamma_1], \widehat{\psi}_0[\gamma_2]$ on $\mathbb{R}^2/\sqrt{2}\mathbb{Z}^2$, so necessarily $B\binom{\sqrt{2}}{0}, B\binom{0}{\sqrt{2}} = \binom{0}{0} \bmod \sqrt{2}\mathbb{Z}^2$. It follows that $B \in \mathrm{GL}(2, \mathbb{Z})$.

Actually, $B \in \mathrm{SL}(2,\mathbb{Z})$: $|\det B| = 1$ because $\widehat{\psi}_0$ is an orientation preserving self–bijection of a surface of finite area, and $\det B > 0$ because $\widehat{\psi}_0$ is orientation preserving.

Next we use the fact that $\widehat{\psi}_0$ fixes $\frac{1}{\sqrt{2}}\binom{1}{1} + \sqrt{2}\mathbb{Z}^2$. Using the linear segment from $\binom{0}{0}$ to $\frac{1}{\sqrt{2}}\binom{1}{1}$ as above, we see that $B\binom{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} \in \binom{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} + \sqrt{2}\mathbb{Z}^2$. Multiplying by $\sqrt{2}$ and considering the result modulo 2 we see that the rows of B have entries of different parity. So $B \,(\mathrm{mod}\, 2) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, or $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right) (\mathrm{mod}\, 2)$. Since $\det B$ is odd, $B = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \mathrm{mod}\, 2$.

Returning to $\psi_0 = \Theta \circ \widehat{\psi}_0 \circ \Theta$, we see by direct calculation that the derivative of ψ also has entries with different parity at every row. As before, this means that the derivative of ψ is $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \mathrm{mod}\, 2$.

Suppose the derivative of ψ is $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \mathrm{mod}\, 2$. We saw in step one that there exists a homogeneous automorphism ϕ with the same derivative and with drift zero. By step two, $\psi \circ \phi^{-1} = D^k$ for some $k \in \mathbb{Z}$. It follows that $\delta(\psi) = k \in \mathbb{Z}$.

If the derivative of ψ is $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \mathrm{mod}\, 2$, then there is a homogeneous automorphism ϕ with the same derivative and with drift $\frac{1}{2}$. By step two, $\psi \circ \phi^{-1} = D^k$ for some $k \in \mathbb{Z}$, and $\delta(\psi) = k + \frac{1}{2}$. □

Proof of Lemma 2.7. Suppose $\xi, \eta \in \mathbb{R} \setminus \mathbb{Q}$ and $\xi \neq \eta$. We are asked to produce a matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(2)$ such that $s_1 := \frac{a\xi+b}{c\xi+d}$ and $s_2 := \frac{a\eta+b}{c\eta+d}$ satisfy one of the following: One of s_1, s_2 is in $(-1,0)$ and the other is in $(1,\infty)$ ("case 1"); Or one of s_1, s_2 is in $(0,1)$ and the other is in $(1,\infty)$ ("case 2").

Let $\mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$, then $\Gamma(2)$ acts on the upper half plane by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot z = \frac{az+b}{cz+d}$. It is well known that the hyperbolic polygon F with vertices $-1, 0, 1, \infty$ is a fundamental domain for this action [**Fo**]. So $\{g(F) : g \in \Gamma(2)\}$ is a tesselation of $\mathbb{H}$. Notice that the vertices of $g(F)$ belong to $\mathbb{Q} \cup \{\infty\}$.

Label the sides of F *on the inside* by $\{a, \overline{a}, b, \overline{b}\}$ as in figure 6. Notice that a is mapped to $\overline{a}$ by $\left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$, and b is mapped to $\overline{b}$ by $\left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right)$. Extend the labeling to $g(F)$ ($g \in \Gamma(2)$) in the natural way. Now every side in the tesselation has two labels $x, \overline{x}$, one internal and the other external (which is which depends on the tile we use as reference).

Let γ denote the (open) upper half of the circle with diameter $[\xi, \eta]$ or $[\eta, \xi]$. We think of γ as of a geodesic in the upper half plane, from ξ to η. Let $\underline{x} = (\cdots, x_{-1}, x_0, x_1, \cdots)$ denote the ordered sequence of *internal* labels of the sides of the tiles γ *enters*. The position of the zeroth coordinate is not important. The following facts follow from the geometric structure of the tessellation:

(1) $\underline{x}$ is a doubly infinite (otherwise ξ or η is a vertex of $g(F)$ for some $g \in \Gamma(2)$, in contradiction to the irrationality of ξ, η).
(2) For all i, $x_i \neq \overline{x}_{i+1}$ and $\overline{x}_i \neq x_{i+1}$.
(3) $\underline{x}$ does not begin or terminate with a constant ray (otherwise ξ or η is a vertex of $g(F)$ for some $g \in \Gamma(2)$, in contradiction to their irrationality).

Suppose first that $\underline{x}$ contains the symbol $\overline{a}$. Then it must contain $\overline{a}b$ or $\overline{a}\overline{b}$ (otherwise it terminates with the constant sequence $\overline{a}\,\overline{a}\cdots$). If $\underline{x}$ does not contain $\overline{a}$, then it must contain b or $\overline{b}$ (otherwise it equals $\cdots aaa \cdots$). If $\underline{x}$ contains b but not $\overline{a}$, then $\underline{x}$ contains ba (otherwise it terminates with $bbb\cdots$). If $\underline{x}$ contains $\overline{b}$ but

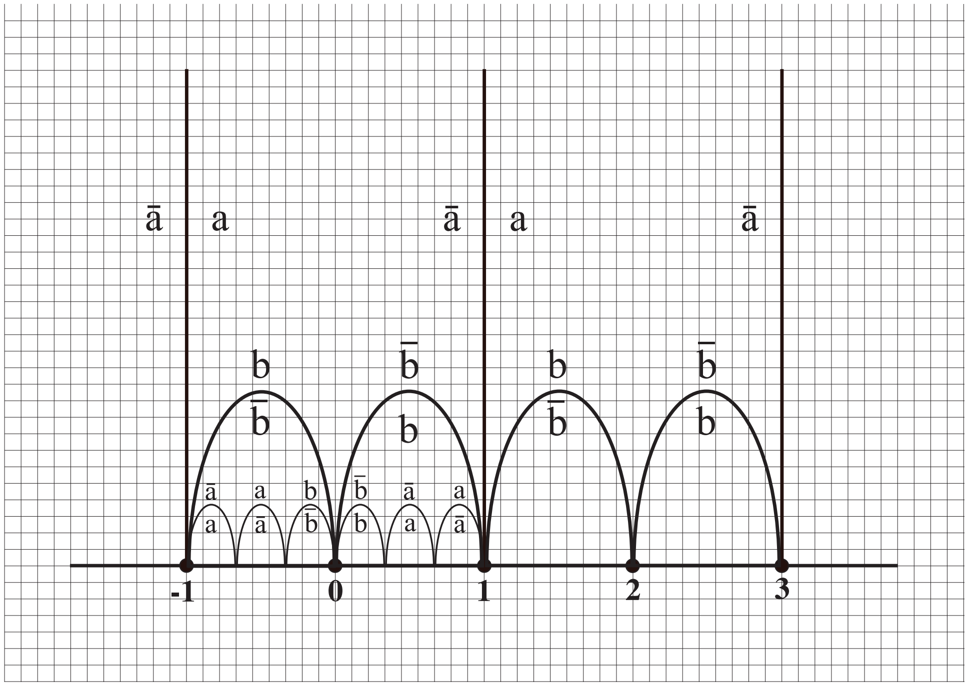

FIGURE 6. Fundamental domain of $\Gamma(2)$

not $\overline{a}$ then $\underline{x}$ contains $\overline{b}a$ (otherwise it terminates with $\overline{b}\,\overline{b}\,\overline{b}\cdots$). In summary, $\underline{x}$ must contain at least one of the words $\overline{a}b, \overline{b}a, \overline{a}\overline{b}, ba$.

Notice that if the cutting sequence of the geodesic from ξ to η is $(x_i)_{i\in\mathbb{Z}}$, then the cutting sequence of the geodesic from η to ξ is $(\overline{x}_{-i})_{i\in\mathbb{Z}}$. Therefore we may assume without loss of generality that $\underline{x}$ contains $\overline{a}b$ or $\overline{a}\overline{b}$, otherwise exchange $\xi \leftrightarrow \eta$.

Suppose first that $\underline{x}$ contains the word $\overline{a}b$, then γ enters some tile F^* through side $\overline{a}$ and leaves it through side $\overline{b}$ (entering an adjacent tile with side b). There is $P \in \Gamma(2)$ which maps F^* onto F. Since $P\cdot\gamma$ enters F through side $\overline{a}$, $P\cdot\xi \in (1,\infty)$. Since $P\cdot\gamma$ leaves F through side $\overline{b}$ (entering the adjacent tile through side b), $P\cdot\eta \in (0,1)$. This is case 1.

Next suppose $\underline{x}$ contains the word $\overline{a}\overline{b}$, then γ enters some tile F^* through side $\overline{a}$ and leaves it through side b (entering an adjacent tile with side $\overline{b}$). There is $P \in \Gamma(2)$ which maps F^* onto F. Since $P\cdot\gamma$ enters F through side $\overline{a}$, $P\cdot\xi \in (1,\infty)$. Since $P\cdot\gamma$ leaves F through side b, $P\cdot\eta \in (-1,0)$. This is case 2. □

Proof of Lemma 2.9 (Aperiodicity Lemma). Let $F := F_\psi \circ \pi_0$. We have to show that if $e^{itF} = \lambda h/h\circ\sigma$ for some $t \in \mathbb{R}$, $\lambda \in S^1$, and $h : \Sigma(\mathscr{G}) \to S^1$ continuous, then $t \in 2\pi\mathbb{Z}$, $\lambda = 1$ and $h = \text{const}$.

Proof. We first consider the special case when ψ fixes all the singularities of St.

The idea is to construct $\underline{x}, \underline{y} \in \Sigma(\mathscr{G})$ and $n \in \mathbb{N}$ s.t. $\sigma^n(\underline{x}) = \underline{x}, \sigma^n(\underline{y}) = \underline{y}$, and $F_n(\underline{x}) - F_n(\underline{y}) = \pm 1$ where $F_n = \sum_{k=0}^{n-1} F\circ\sigma^n$.

Given such points the lemma can be proved as follows: Suppose $e^{itF} = \lambda h/h\circ\sigma$, then $e^{itF_n} = \lambda^n h/h\circ\sigma^n$, whence $e^{itF_n(\underline{x})} = \lambda^n$ and $e^{itF_n(\underline{y})} = \lambda^n$. Dividing, we find that $e^{it} = 1$, whence $t \in 2\pi\mathbb{Z}$. But if $t \in 2\pi\mathbb{Z}$ then $\lambda h/h \circ \sigma = e^{itF} = 1$, whence h is a continuous eigenfunction of σ. Since $\sigma : \Sigma(\mathscr{G}) \to \Sigma(\mathscr{G})$ is topologically mixing, $\lambda = 1$ and $h = \text{const}$.

We will now construct such $\underline{x}, \underline{y}$. It is enough to do this in the case when ψ has a fundamental polygon of the form $R = \theta_0(R_0)$, where $\theta_0 : \mathrm{St}_0 \to \mathrm{St}_0$ is a toral automorphism which fixes the punctures of St_0^*, and R_0 is as in figure 2.

Case (a). *The slope of the unstable direction is bigger than one, and the slope of the stable direction belongs to* $(-1, 0)$.

Let $\partial^s Q_i, \partial^u Q_i$ denote the stable and unstable boundaries of Q_i. In case (a), $\partial^s Q_2 \ni (1, 0) =: p_0$ and $\partial^u Q_2 \ni (2, 0) =: q_0$. These are fixed points of ψ_0. Their lifts to St are fixed points of ψ (by assumption).

Recall that $\psi_0(Q_2)$ is the union of parallelograms $Q_{2,k}$, $k = 1, \ldots, N_2$, where the bottom stable side of $Q_{2,1}$ is part of the bottom stable side of Q_2, and the bottom stable side of $Q_{2,k+1}$ is the top stable side of $Q_{2,k}$, $k = 1, \ldots, N_2 - 1$.

In what follows we write $P \to P'$ if $P, P' \in \mathfrak{P}$ and $\mathrm{int}(P) \cap \psi_0^{-1}[\mathrm{int}(P')] \neq \varnothing$, and $g(P, P')$:=value of F_ψ on $\mathrm{int}(P) \cap \psi_0^{-1}[\mathrm{int}(P')]$.

Using the fixed point p_0 and the relation $\psi_0(\partial^s Q_2) \subset Q_2$, it is easy to see that $Q_{2,1} \subset Q_2$, $Q_{2,1} \to Q_{2,1}$, and $g(Q_{2,1}, Q_{2,1}) = 0$. So $\underline{x} = (\cdots, Q_{2,1}, Q_{2,1}, Q_{2,1}, \cdots)$ is a well defined point in $\Sigma(\mathscr{G})$, $\sigma^n(\underline{x}) = \underline{x}$, and $F_n(\underline{x}) = 0$ for all n. [Caution: for other k, $Q_{2,k}$ is not necessarily in Q_2.]

Using the fixed point q_0 we find $Q_{i,j} \subset Q_2$ which contains q_0 in its right u–boundary. We claim that $i = 2$ and $1 < j < N_2$:

- $q_0 = \psi_0^{-1}(q_0) \subset \psi_0^{-1}(\partial^u Q_{ij}) \subset \partial^u Q_i$. In case (a), this forces $i = 2$.
- The u–side of $Q_{2,j} \cap \psi_0^{-1}(Q_{2,j})$ which contains q_0 equals $\psi_0^{-1}[W^u(q_0)]$, where $W^u(q_0)$ is the u–fibre of q_0. Since ψ_0 is expanding on $W^u(q_0)$, the u–side of $Q_{2,j} \cap \psi_0^{-1}(Q_{2,j})$ does not meet the endpoints of $W^u(q_0)$. So $1 < j < N_2$.

Since $Q_{2,1} \subset Q_2$, $Q_{2,1} \to Q_{2,j}$. Since q_0 is a fixed point, $g(Q_{2,1}, Q_{2,j}) = 0$.

Since $Q_{2,j+1}$ follows $Q_{2,j}$, and $Q_{2,j}$ is the rightmost $\mathfrak{P}$–element in Q_2, figure 2 tells us that $Q_{2,1} \to Q_{2,j+1}$, $Q_{2,j+1} \subset Q_2$, and $g(Q_{2,1}, Q_{2,j+1}) = -1$.

Since $Q_{2,j+1} \subset Q_2$, and $Q_{2,1} \ni p_0$, $Q_{2,j+1} \to Q_{2,1}$ and $g(Q_{2,j+1}, Q_{2,1}) = 0$.

We now define $\underline{y} := (\cdots, Q_{2,1}, Q_{2,j+1}; Q_{2,1}, Q_{2,j+1}; \cdots)$, then $\underline{y} \in \Sigma(\mathscr{G})$, $\sigma^2(\underline{y}) = \underline{y}$ and $F_2(\underline{y}) = -1$. Using $\underline{x}, \underline{y}$ and $n = 2$, we get the aperiodicity of F in case 1.

Case (b). *The slope of the unstable direction is bigger than one, and the slope of the stable direction is in* $(0, 1)$.

Just like in case (a), the $\mathfrak{P}$–element in Q_2 which contains p_0 in its bottom s–side is $Q_{2,1}$, and $Q_{2,1} \to Q_{2,1}$ with $g(Q_{2,1}, Q_{2,1}) = 0$. So $\underline{x} = (\cdots, Q_{2,1}, Q_{2,1}, Q_{2,1}, \cdots)$ belongs to Σ, $\sigma(\underline{x}) = \underline{x}$, and $F(\underline{x}) = 0$.

To construct $\underline{y}$, we separate cases according to whether $Q_{2,2} \subset Q_1$ or $Q_{2,2} \subset Q_2$.

Suppose first that $Q_{2,2} \subset Q_2$. Looking at figure 2 and noting that $p_0 \in \partial^s Q_2$, we see that $Q_{2,1} \subset Q_2$, and that every $\mathfrak{P}$–element P in Q_2 satisfies $P \to Q_{2,1}$ and $g(P, Q_{2,1}) = 0$. In particular $Q_{2,2} \to Q_{2,1}$ and $g(Q_{2,2}, Q_{2,1}) = 0$. Since $Q_{2,1} \subset Q_2$, $Q_{2,1} \to Q_{2,2}$. Using the assumption that the slope of the unstable direction is bigger than one, it is not difficult to see that $g(Q_{2,1}, Q_{2,2}) = 1$. We now set

$$\underline{y} = (\cdots; Q_{2,1}, Q_{2,2}; Q_{2,1}, Q_{2,2}; \cdots).$$

This is a point in $\Sigma(\mathscr{G})$, $\sigma^2(\underline{y}) = \underline{y}$, and $F_2(\underline{y}) = 1$. Using $\underline{x}$ and $\underline{y}$ and $n = 2$, we get the aperiodicity of F, assuming that $Q_{2,2} \subset Q_2$.

Now suppose that $Q_{2,2} \subset Q_1$. The following observations follow from figure 2 and the fact that $p_0 \in \partial^s Q_2$:

- As before, $Q_{2,1} \to Q_{2,2}$, $Q_{2,2} \subset Q_1$ and $g(Q_{2,1}, Q_{2,2}) = 1$.
- $Q_{1,1} \subset Q_2$, and every $\mathfrak{P}$–element P in Q_1 satisfies $P \to Q_{1,1}$, $g(P, Q_{1,1}) = 0$. In particular, $Q_{2,2} \to Q_{1,1}$, and $g(Q_{2,2}, Q_{1,1}) = 0$.
- All $\mathfrak{P}$–elements P in Q_2 satisfy $P \to Q_{2,1}$ with $g(P, Q_{2,1}) = 0$. In particular, $Q_{1,1} \to Q_{2,1}$ and $g(Q_{1,1}, Q_{2,1}) = 0$.

We now let $\underline{y} := (\cdots ; Q_{2,1}, Q_{2,2}, Q_{1,1}; Q_{2,1}, Q_{2,2}, Q_{1,1}; \cdots)$. This is a point in $\Sigma(\mathscr{G})$ s.t. $\sigma^3(\underline{y}) = \underline{y}$ and $F_3(\underline{y}) = 1$. Using $\underline{x}, \underline{y}$ and $n = 3$, we see that F is aperiodic.

This proves the lemma in case ψ fixes the singularities of St. The general case can be reduced to this case as follows.

Suppose ψ is a homogeneous automorphism, and assume that $F := F_\psi \circ \pi_0$ satisfies $e^{itF} = \lambda h / h \circ \sigma$ for some $t \in \mathbb{R}$, $\lambda \in S^1$, and $h : \Sigma(\mathscr{G}) \to S^1$ continuous.

Let $G := F + F \circ \sigma$ and $g := h(h \circ \sigma)$, then $e^{itG} = \lambda^2 g / g \circ \sigma$. Observe that $G = F_{\psi^2} \circ \pi_0$, and that ψ^2 fixes the singularities of St (this holds for any homogeneous automorphism, by virtue of the fact that it preserves the D–orbits of the singularities of St). By the first part of the proof, $t \in 2\pi\mathbb{Z}$. It follows that $\lambda h / h \circ \sigma = e^{itF} = 1$, whence h is a continuous eigenfunction of σ. Since σ is topologically mixing, $h = \text{const}$ and $\lambda = 1$. $\square$

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References

[**A1**] J. Aaronson: *On the ergodic theory of non-integrable functions and infinite measure spaces.* Israel J. Math. **27** (1977), no. 2, 163–173.

[**A2**] J. Aaronson: *On the pointwise ergodic behaviour of transformations preserving infinite measures.* Israel J. Math. **32** (1979), no. 1, 67–82.

[**ADF**] J. Aaronson; M. Denker; A.M. Fisher: *Second order ergodic theorems for ergodic transformations of infinite measure spaces.* Proc. Amer. Math. Soc. **114** (1992), no. 1, 115–127.

[**AK**] J. Aaronson; M. Keane: *The visits to zero of some deterministic random walks.* Proc. London Math. Soc. (3) **44** (1982), no. 3, 535–553.

[**ANSS**] J. Aaronson; H. Nakada; O. Sarig; R. Solomyak: *Invariant measures and asymptotics for some skew products.* Israel J. Math. **128** (2002), 93–134.

[**AS**] J. Aaronson and O. Sarig: *Exponential chi squared distributions in infinite ergodic theory.* To appear in Erg. Th. Dyn. Syst. doi:10.1017/etds.2012.160

[**AW**] R.L. Adler; B. Weiss: *Similarity of automorphisms of the torus.* Memoirs of the American Mathematical Society, No. **98**, American Mathematical Society, Providence, R.I. 1970 ii+43 pp.

[**At**] G. Atkinson: *Recurrence of co-cycles and random walks.* J. London Math. Soc. (2) **13** (1976), no. 3, 486–488.

[**BL1**] M. Babillot; F. Ledrappier: *Geodesic paths and horocycle flow on abelian covers.* Lie groups and ergodic theory (Mumbai, 1996), 1–32, Tata Inst. Fund. Res. Stud. Math., **14**, Tata Inst. Fund. Res., Bombay, 1998.

[**BL2**] M. Babillot; F. Ledrappier: *Lalley's theorem on periodic orbits of hyperbolic flows.* Ergodic Theory Dynam. Systems **18** (1998), no. 1, 17–39.

[**BS**] L. Barreira; J. Schmeling: *Sets of "non-typical" points have full topological entropy and full Hausdorff dimension.* Israel J. Math. **116** (2000), 29–70.

[**B1**] J. Beck: *Randomness of the square root of 2 and the giant leap, part 1*, Periodica Math. Hungarica **60** (2010), 137–242.

[**B2**] J. Beck: *Randomness of the square root of 2 and the giant leap, Part 2.* Periodica Math. Hungarica **62** (2011), 127–246.

[**Bor**] A. N. Borodin: *On the distribution of random walk local time.* Ann. Inst. H. Poincaré Probab. Statist. **23** (1987), no. 1, 63–89.

[**BR**] M. Boshernitzan; D. Ralston: *Continued fractions and heavy sequences.* Proc. Amer. Math. Soc. **137** (2009), no. 10, 3177–3185.

[**BD**] R. Burton; M. Denker: *On the central limit theorem for dynamical systems.* Trans. Amer. Math. Soc. **302** (1987), no. 2, 715–726.

[**C**] J.-P. Conze: *Equirépartition et ergodicité de transformations cylindriques.* (French) Séminaire de Probabilités, I (Univ. Rennes, Rennes, 1976), Exp. No. 2, 21 pp. Dépt. Math. Informat., Univ. Rennes, Rennes, 1976.

[**CK**] J.-P. Conze; M. Keane: *Ergodicité d'un flot cylindrique.* (French) Séminaire de Probabilités, I (Univ. Rennes, Rennes, 1976), Exp. No. 5, 7 pp. Dépt. Math. Informat., Univ. Rennes, Rennes, 1976.

[**D**] W. Doeblin: *Sur deux problèmes de M. Kolmogoroff concernant les chaînes dénombrables.* (French) Bull. Soc. Math. France **66** (1938), 210–220.

[**E**] G. K. Eagleson: *Some simple conditions for limit theorems to be mixing.* (Russian) Teor. Verojatnost. i Primenen. **21** (1976), no. 3, 653–660. *Engl. Transl. in* Th. Probab. Appl. **21** (1977), 637–642.

[**Fi1**] A. M. Fisher: *Convex invariant means and a pathwise central limit theorem*, Adv. Math. **63** (1987), 213–246.

[**Fi2**] A. M. Fisher: *Integer Cantor sets and an order-two ergodic theorem.* Ergodic Theory Dynam. Systems **13** (1993), no. 1, 45–64.

[**Fo**] L.R. Ford: Automorphic functions. *Second edition, AMS Chelsea Publishing, American Mathematical Society, Providence, RI, 1957, xii + 333pp.*

[**GH**] Y. Guivarc'h; J. Hardy: *Théorèmes limites pour une classe de chaînes de Markov et applications aux difféomorphismes d'Anosov.* Ann. Inst. H. Poincaré Probab. Statist. **24** (1988), no. 1, 73–98.

[**Her**] M.-R. Herman: *Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations.* (French) Inst. Hautes Études Sci. Publ. Math. No. **49** (1979), 5–233.

[**HHW**] W.P. Hooper, P. Hubert and B. Weiss: *Dynamics on the infinite staircase surface*, Dis. Cont. Dyn. Sys. (series A) **33** (2013), 4341–4347.

[**HW**] W.P. Hooper and B. Weiss: *Generalized staircases: recurrence and symmetry.* Ann. Inst. Fourier (Grenoble) **62** (2012), no. 4, 1581–1600.

[**Hu**] F. Huveneers *Subdiffusive behavior generated by irrational rotations,* Erg Th. Dyn. Sys. **29** (2009) 1217–1233

[**Kel**] G. Keller: *Generalized bounded variation and applications to piecewise monotonic transformations*, Z. Wahrsch. Verw. Gebiete **69** (1985), no. 3, 461–478.

[**Kes**] H. Kesten: *Uniform distribution mod 1.* Ann. of Math. (2) **71** 1960 445–471.

[**Kh**] A. Ya. Khinchin: Continued fractions. With a preface by B. V. Gnedenko. *Translated from the third (1961) Russian edition. Reprint of the 1964 translation. Dover Publications, Inc., Mineola, NY, 1997. xii+95 pp.*

[**KN**] L. Kuipers and H. Niederreiter: Uniform distribution of sequences. *Dover Publ. Inc., Mineola, New York (2006).*

[**LS1**] F. Ledrappier; O. Sarig: *Unique ergodicity for non-uniquely ergodic horocycle flows.* Discrete Contin. Dyn. Syst. **16** (2006), no. 2, 411–433.

[**LS2**] F. Ledrappier; O. Sarig: *Fluctuations of ergodic sums for horocycle flows on $\mathbb{Z}^d$-covers of finite volume surfaces.* Disc. Contin. Dyn. Syst. **22** (2008), no. 1-2, 247–325.

[**Ngv**] S. V. Nagaev: *More exact limit theorems for homogeneous Markov chains.* (Russian) Teor. Verojatnost. i Primenen. **6** (1961) 67–86.

[**Leo**] V.P. Leonov: *On the dispersion of time means of a stationary stochastic process.* (Russian. English summary) Teor. Verojatnost. i Primenen. 6 (1961) 93–101.

[**Nkd**] H. Nakada: *Piecewise linear homeomorphisms of type III and the ergodicity of cylinder flows.* Keio Math. Sem. Rep. No. **7** (1982), 29–40.

[**PP**] W. Parry; M. Pollicott: Zeta functions and the periodic orbit structure of hyperbolic dynamics. *Astérisque* **187–188** (1990), *Soc. Math. de France*, 268pp.

[Pe] Y. Peres: *A combinatorial application of the maximal ergodic theorem,* Bull. London Math. Soc. **20** (1988), 248–252.

[Pet] K. Petersen: Ergodic theory. *Cambridge Studies in Advanced Mathematics, 2. Cambridge University Press, Cambridge, 1983. xii+329 pp. ISBN: 0-521-23632-0*

[PS] W. Philipp; W. Stout: Almost sure invariance principles for partial sums of weakly dependent random variables. *Mem. Amer. Math. Soc.* 2 **161** iv+140pp.

[Ra] D. Ralston: $\frac{1}{2}$-*Heavy Sequences Driven By Rotation.* Preprint arXiv:1106.0577.

[RS] L. Roçadas; J. Schoißengeier: *On the local discrepancy of* $(n\alpha)$*-sequences.* J. Number Theory **131** (2011), no. 8, 1492–1497.

[Ru] D. Ruelle: Thermodynamic formalism. The mathematical structures of classical equilibrium statistical mechanics. With a foreword by Giovanni Gallavotti and Gian-Carlo Rota. *Encyclopedia of Mathematics and its Applications*, **5** Addison-Wesley Publishing Co., Reading, Mass., 1978. xix+183 pp.

[Sa] O.M. Sarig: *Unique ergodicity for infinite measures.* Proceedings of the International Congress of Mathematicians. Volume III, 1777–1803, Hindustan Book Agency, New Delhi, 2010.

[SS] O. Sarig; B. Schapira: *The generic points for the horocycle flow on a class of hyperbolic surfaces with infinite genus.* Int. Math. Res. Not. 2008, Art. ID rnn 086, 37 pp.

[Sch] K. Schmidt: *A cylinder flow arising from irregularity of distribution.* Compositio Math. **36** (1978), no. 3, 225–232.

[V] D. Volný: *Invariance principles and Gaussian approximation for strictly stationary processes.* Trans. Amer. Math. Soc. **351** (1999), no. 8, 3351–3371.

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