Random walks in one dimensional environment

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CHAPTER 1

Introduction

We discuss several models of random walks in random environment on $\mathbb{Z}$. Both the case of fixed environment and environment changing with time will be considered. Many one dimensional models can be analyzed completely and they exhibit a wide range of different asymptotic behaviours. We will present the resultss, methods and discuss several open problems.

CHAPTER 2

Homogeneous random walk.

We review several properties of classical random walk on $\mathbb{Z}^1$. The goals are to provide comparison with models studied later and to introduce a number of methods which will be useful in the sequel. Let Δ_k be $\mathbb{Z}$-valued iid random variables taking finitely many values. Let $X_n = \sum_{k=1}^n \Delta_k$. Denote $p_j = \mathbb{P}(\Delta = j)$, $\mu = \mathbb{E}(\Delta)$, $\sigma^2 = \mathrm{Var}(\Delta)$.

1. Preliminaries.

We recall two results which extend familiar Central Limit Theorem for random walks.

Let $W_N(t)$ be a continuous time process such that

$$W_N\left(\frac{n}{N}\right) = \frac{S_n - n\mu}{\sqrt{N}}$$

and $W_N(t)$ is linear between those points.

THEOREM 2.1. *(Weak Invariance Principle) As $N \to +\infty$, $W_N(t)$ converges in distribution to the Brownian Motion with zero mean and variance $\sigma^2 t$.*

THEOREM 2.2. *(Local Limit Theorem) Assume that the walk is irreducible and aperiodic that is for any site i there exists N such that $\mathbb{P}(X_n = i)$ for all $n \geq N$. Then*

$$\sup_j \left| \sqrt{n}\mathbb{P}(X_n = j) - \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(j-n\mu)^2}{2\sigma^2 n}} \right| = 0.$$

2. Large deviations.

2.1. Large deviations for walker's position. Local Limit Theorem gives a good estimate of $\mathbb{P}(X_n = j)$ only if j is close to $n\mu$. For j far from $n\mu$ we need a different approach. To fix our ideas we suppose that $j > n\mu$, $j/n \sim a$ where a is a number satifying $\mu < a < R$ and R is the maximum right jump of the walker. We want to study $\mathbb{P}(X_n \geq an)$. We will see that this probability is of the same order as $\mathbb{P}(X_n = \lceil an \rceil)$

where $\lceil x \rceil$ denotes the largest number greater than or equal to x. Let $\gamma(s)$ be the logarithm of the moment generated function

$$e^{\gamma(s)} = \sum_j p_j e^{sj}.$$

It has the following properties

LEMMA 2.3. *(a)* $\gamma(0) = 0$, $\lim_{s \to +\infty} \frac{\gamma(s)}{s} = R$.
(b) $\gamma'(0) = 0$, $\lim_{s \to +\infty} \gamma'(s) = R$.
(c) γ *is convex.*

PROOF. (a) Follows from the definition of γ.

To prove (b) and (c) consider a random variable $\Delta^{(s)}$ such that $p(\Delta^{(s)} = j) = \frac{p_j e^{sj}}{e^{\gamma(s)}}$. Now (b) and (c) follow from identities

$$\gamma'(s) = \mathbb{E}(\Delta^{(s)}), \quad \gamma''(s) = \mathrm{Var}(\Delta^{(s)}).$$

□

By parts (b) and (c) of Lemma 2.3(b) there exists unique $s(a)$ such that $\gamma'(s(a)) = a$.

THEOREM 2.4.

$$(a) \quad \mathbb{P}(X_n \geq an) \sim \frac{1}{\sqrt{2\pi\gamma''(s(a))n}} e^{\gamma(s(a))n - s(a)\lceil an \rceil} \frac{1}{1 - e^{-s(a)}}.$$

$$(b) \quad \mathbb{P}(X_n = \lceil an \rceil) \sim \frac{1}{\sqrt{2\pi\gamma''(s(a))n}} e^{\gamma(s(a))n - s(a)\lceil an \rceil}.$$

PROOF. Let $X_n^{(s)} = \sum_{k=1}^n \Delta_k^{(s)}$. Then

$$\mathbb{P}(X_n^{(s)} = j) = \frac{e^{s(a)j}}{e^{\gamma(s(a))n}} \mathbb{P}(X_n = j),$$

that is

$$\mathbb{P}(X_n = j) = \frac{e^{\gamma(s(a))n}}{e^{s(a)j}} \mathbb{P}(X_n^{(s)} = j).$$

In particular if $j = \lceil an \rceil + m$ then by the Local Limit Theorem
(2.1)

$$\mathbb{P}(X_n = j) \sim \frac{1}{\sqrt{2\pi\gamma''(s)n}} e^{-m^2/(2\gamma''(s(a))n)} e^{n\gamma(s(a)) - s(a)(\lceil na \rceil} e^{-\gamma(s(a))m}$$

uniformly for $m \leq \sqrt{n}$.

$$\mathbb{P}(X_n = j) \leq e^{n\gamma(s(a)) - s(a)(\lceil na \rceil} e^{-\gamma(s(a))m} \tag{2.2}$$

Summation over $m \geq 0$ completes the proof (the main contribution comes from (2.1)). □

Part (a) of Theorem 2.4 can be rewritten as

$$\mathbb{P}(X_n \geq an) \sim \frac{1}{\sqrt{2\pi\gamma''(s(a))n}} e^{\gamma(s(a))n - s(a)an} \frac{e^{s(a)[an]}}{1 - e^{-s(a)}}$$

where $[an]$ denotes $an - \lceil an \rceil$. Accordingly

$$\frac{\ln \mathbb{P}(X_n \geq an)}{n} \sim I(a) := \gamma(s(a)) - as(a).$$

Note that, as expected, $I(a) < 0$ for $a > \mu$. Indeed

$I(\mu) = \gamma(a) - \mu \times 0 = 0$ and $I'(a) = \gamma'(s(a))s'(s) - s(a) - as'(a) = -s(a) < 0.$

Exercise 2.1. Compute $I(a)$ for the simple random walk.

Exercise 2.2. Let m_n be a sequence such that $\frac{m_n}{n} \to a$ as $n \to \infty$. Let $Y_n(t)$ denote the process $\frac{X_{nt} - \frac{m_n t}{n}}{\sqrt{n}}$ conditioned on $X_n = m_n$. Show that as $n \to \infty$ $Y_n(t)$ converges to a Brownian bridge–a Gaussian process with zero mean and covariance

$$\mathbb{E}(Y(t), Y(s)) = \gamma''(a) \min(s,t)(1 - \max(s,t)).$$

2.2. Large deviation for the maximum of the walk. Suppose that $\mu < 0$. We shall use Theorem 2.4 to estimate the probability that $M = \sup_{n \geq 0} X_n$ takes large values.

First we consider the case of simple(=nearest neighbor) random walk which moves to the right with probability p and to the left with probability q. Then $Z_n = \left(\frac{q}{p}\right)^{X_n}$ is a martingale. Let τ be the first time X_n reaches either a or $-b$. Let $\mathbf{p} = \mathbb{P}(X_\tau = a)$. By Optional Stopping Theorem

$$\mathbf{p}\left(\frac{q}{p}\right)^a + (1 - \mathbf{p})\left(\frac{q}{p}\right)^{-b} = 1$$

and so we obtain Gambler's Ruin Formula

$$\mathbf{p} = \frac{1 - (q/p)^{-b}}{(q/p)^a - (q/p)^{-b}}.$$

Letting $b \to \infty$ we obtain

$$\mathbb{P}(M \geq a) = \left(\frac{p}{q}\right)^a$$

so that M has geometric distribution.

Next we consider arbitrary random walk. In this case $\mathbb{P}(M = a)$ may have complicated form for small a but we will be able to compute its asymptotics for large a. Lemma 2.3 tells us that $\gamma(s) < 0$ for small positive s and $\gamma(s) > 0$ for large positive s. By convexity there is unique s_0 such that $\gamma(s_0) = 0$. Let $\alpha_0 = \gamma'(s_0)$.

THEOREM 2.5. *There is a constant c such that*

$$\mathbb{P}(M \geq k) \sim ce^{-ks(\alpha_0)} \text{ as } k \to \infty.$$

PROOF. Define

$$f(\alpha) = \frac{\gamma(s(\alpha))}{\alpha} - s(\alpha).$$

Note that

$$f'(\alpha) = \frac{\gamma'(s(\alpha))s'(\alpha)}{\alpha} - \frac{\gamma(s(\alpha)}{\alpha^2} - s'(\alpha) = -\frac{\gamma(s(\alpha)}{\alpha^2}$$

so that α_0 is a critical point of f. By Lemma 2.3(b) $f'(\alpha) < 0$ for α near R and $f(\alpha) = \frac{I(\alpha)}{\alpha} \to -\infty$ as $\alpha \to 0$. Thus $\alpha_0 = \arg\max f(\alpha)$. Note that $f(\alpha_0) = -s(\alpha_0)$.

Let $\rho_j = \mathbb{P}(\max_{n>0} X_n > -j)$. Considering the last time the walker is above level k we get

$$\mathbb{P}(M \geq k) = \sum_{n=0}^{\infty} \sum_{j=0}^{R} \mathbb{P}(X_n = j + k)\rho_j.$$

According to Theorem 2.4

$$\mathbb{P}(X_n = j + k) \sim \frac{1}{\sqrt{2\pi\gamma''(k/n)n}} e^{n\gamma(s(k/n)) - s(k/n)k} \frac{\rho_j}{e^{s(k/n)j}}.$$

The index of the exponent in the above expression equals to $kf(k/n)$. Take n_0 such that k/n_0 is closest to α_0. Then writting $n = n_0 + \Delta n$ we have

$$kf\left(\frac{k}{n}\right) \sim kf(\alpha_0) - \frac{f''(\alpha_0)\alpha_0^3}{2} \frac{(\Delta n)^2}{n}$$

so that

$$\begin{aligned}\mathbb{P}(X_n = j + k) &\sim \frac{\rho_j}{e^{s(\alpha_0)j}} e^{kf(\alpha_0)} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\gamma''(\alpha_0)}} e^{-\frac{f''(\alpha_0)\alpha_0^3 h^2}{2}} dh \\ &= \frac{\rho_j}{e^{s(\alpha_0)j}} e^{kf(\alpha_0)} \frac{1}{\sqrt{\gamma''(\alpha_0) f''(\alpha_0)\alpha_0^3}}.\end{aligned}$$

Summation over j proves the result. □

REMARK. The proof of Theorems 2.4 and 2.5 also shows that conditioned on $X_n \geq \lceil an \rceil$, $X_n - \lceil an \rceil$, has asymptotically geometric distribution and conditioned on $M \geq k$, $M - k$ also has asymptotically geometric distribution.

The argument presented above is valid in much more general setting than sums of finite range lattice random variables. For example it can give the following result.

EXERCISE 2.3. Let $X_n = \sum_{j=1}^n \Delta_j$ where Δ_j are iid, have finite range and are **non-lattice** in the sense that there exist no β such that $\Delta \in \beta\mathbb{Z}$ with probability 1. Show that

$$\mathbb{P}(X \geq an) \sim \frac{1}{\sqrt{2\pi\gamma''(s(a))n}s(a)} e^{(\gamma(s(a))-as(a))n}$$

and

$$\mathbb{P}(M \geq t) \sim ce^{-ts(\alpha_0)}.$$

Show also that conditioned on $X_n \geq an$, $X - an$ has asymptotically geometric distribution and conditioned on $M \geq t$, $M - t$ has asymptotically geometric distribution.

Hint. Follow the argument presented above using the Local Limit Theorem for non-lattice random variables instead of the Local Limit Theorem for lattice random variables.

On the other hand there is a proof of Theorem 2.5 not using Theorem 2.4. Namely, since $\gamma(s(\alpha_0)) = 0$ it follows that $e^{s(\alpha_0)X_n}$ is a martingale. Let $\tau_+(a,b)$ be the first time when the walker reaches either $\{-b, -b+1 \cdots -b+M\}$ or $\{a, a+1, \dots a+R\}$ where M denotes the maximal negative jump of the walk. Likewise let $\tau_-(a,b)$ be the first time when the walker reaches either $\{-b, -b+1 \cdots -b+M\}$ or $\{a-R, a-R+1, \dots a\}$. By applying Optional Stopping Theorem to τ_- and letting $b \to \infty$ we see that

$$\mathbb{P}(M \geq a) \leq c_1 e^{-s(\alpha_0)a}. \tag{2.3}$$

By applying Optional Stopping Theorem to τ_+ and letting $b \to \infty$ we see that

$$\mathbb{P}(M \geq a) \geq c_2 e^{-s(\alpha_0)a}. \tag{2.4}$$

On the other hand $\mathbf{q}_a = \mathbb{P}(M \geq a)$ satisfies a linear recurrence relation

$$q_a = \sum_j p_j q_{a-j}.$$

It follows that q_a can be represented as a finite sum $\sum_r \mathcal{P}_r(a)e^{\lambda_r a}$ where $\mathcal{P}_r$ are polynomials. We claim that there is only one root of the characteristic equation with absolute value $e^{-s(\alpha_0)}$. Indeed let $\lambda = e^{-[s(\alpha_0)+i\kappa]}$ satisfy $1 = \sum_j p_j \lambda^j$ for some

$$0 \leq \kappa < 2\pi \tag{2.5}$$

Subtracting

$$1 = \sum_j p_j e^{s(\alpha_0)j+ij\kappa} \text{ from } 1 = \sum_j p_j e^{s(\alpha_0)j}$$

we get

$$\sum_j p_j e^{s(\alpha_0)j}(1-e^{ij\kappa}).$$

Note that the RHS has nonnegative real part. It follows that

$$j\kappa \equiv 0 \mod (2\pi)$$

for each $j \in \text{supp}\Delta$. Since the walk is irreducible we have can reach one at some time with positive probability. Thus $1 = \sum_{l=1}^t j_l$ where $j_l \in \text{supp}(\Delta)$. Accordingly $\kappa = 0 \mod (2\pi)$ and hence $\kappa = 0$ due to (2.5). Now (2.3) and (2.4) imply that $q_a \sim ce^{-ks(\alpha_0)}$.

3. Visits to zero before given time.

A useful tool in understanding one dimensional random walk is its local time. In this subsection we assume that $\mu = 0$ so that the random walk is recurrent. By recurrence the walker visits each site infinitely many times. We are interested in statistics of such visits. Let $L_m(N)$ be the number of visits to site m before time N.

THEOREM 2.6. *As* $N \to \infty$ $\frac{\sigma L_0(N)}{\sqrt{N}} \Rightarrow |\mathbf{N}|$ *where* $\mathbf{N}$ *is the standard Gaussian random variable.*

PROOF.

$$L_0(N) = \sum_{n=1}^N 1_{X_n=0}.$$

Therefore

$$\mathbb{E}(L_0^k(N)) = k! \sum_{0<j_1<j_2\cdots<j_k} \mathbb{P}(X_{j_1}=0, X_{j_2}=0\dots X_{j_k}=0) + m_N$$

there m_N stands for the contribution of terms where not all j_is are different. By the computations presented below $m_N = O(N^{(k-1)/2})$. On the other hand by the Local Limit Theorem the main term is asymptotic to

$$\left(\frac{1}{\sqrt{2\pi}\sigma}\right)^k \sum_{0<j_1<j_2\cdots<j_k} \prod_{s=0}^k \frac{1}{\sqrt{j_s - j_{s-1}}}$$

where $j_0 = 0$. Therefore

$$\mathbb{E}\left(\left(\frac{L_0(N)}{\sqrt{N}}\right)^k\right) = \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^k \left[\sum \frac{1}{N^k} \prod_{s=1}^k \frac{1}{\sqrt{(j_s/N) - (j_{s-1}/N)}}\right].$$

The second term here is asymptotic to

$$I_k = \int_{0<t_1<t_2\cdots<t_k<1} \prod_s \frac{1}{\sqrt{t_s - t_{s-1}}} dt_s.$$

Note that $I_1 = 2$ and

$$I_{k+1} = \int_0^1 \frac{1}{\sqrt{t_1}} \int_{0<u_2\cdots<u_{k+1}<1-t_1} \prod_s \frac{1}{\sqrt{u_s - u_{s-1}}} du_s = \int_0^1 (1-t_1)^{k/2} t_1^{-1/2} I_k dt_1.$$

Thus

$$\frac{I_{k+1}}{I_k} = \frac{\Gamma(\frac{k}{2}+1)\Gamma(\frac{1}{2})}{\Gamma(\frac{k+1}{2}+1)} = \sqrt{\pi}\left[\frac{1}{\Gamma(\frac{k+1}{2}+1)} : \frac{1}{\Gamma(\frac{k}{2}+1)}\right].$$

The last equation implies that $I_k = C\frac{\sqrt{\pi}^k}{\Gamma(\frac{k}{2}+1)}$. Plugging $I_1 = 2$ we get $C = 1$. Therefore

$$\lim_{N\to\infty} \mathbb{E}\left(\left(\frac{\sigma L_0(N)}{\sqrt{N}}\right)^k\right) = \frac{k!}{\Gamma(\frac{k}{2}+1)\sqrt{2}^k}.$$

From the doubling identity

$$\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) = 2^{1-2z}\sqrt{\pi}\Gamma(2z)$$

we get

$$\frac{k!}{\Gamma(\frac{k}{2}+1)} = \frac{\Gamma(\frac{k+1}{2})2^k}{\sqrt{\pi}}$$

so that

$$\lim_{N\to\infty} \mathbb{E}\left(\left(\frac{\sigma L_0^N}{\sqrt{N}}\right)^k\right) = \frac{\Gamma(\frac{k+1}{2})\left(\sqrt{2}\right)^k}{\sqrt{\pi}} = \mathbb{E}(|\mathbf{N}|^k).$$

Since the family $\{\frac{\sigma L_0}{\sqrt{N}}\}$ is uniformly integrable, it is tight. Let Y be the limit point. Then $\mathbb{E}(Y^k) = \mathbb{E}(|\mathbf{N}|^k)$. It follows that Y and $\mathbf{N}$ has the same distribution and hence $\frac{L_0}{\sqrt{N}} \Rightarrow |\mathbf{N}|$. $\square$

EXERCISE 2.4. Show that there exists unique distribution such that $\mathbb{E}(Y^k) = \frac{\Gamma(\frac{k+1}{2})\left(\sqrt{2}\right)^k}{\sqrt{\pi}}$.

Hint. Use the fact that the characteristic function of Y is analytic around the origin.

EXERCISE 2.5. Prove that Theorem 2.6 remains valid for reducible walks.

Hint. Consider $Y_N = X_{rN}$ where r is such that the random walk can reach zero only at times which are multiples of r.

EXERCISE 2.6. Show that conditioned on $X_n = 0$ $\frac{L_0(N)}{N}$ has a limiting distribution.

Theorem 2.6 allows to study the distribution of $L_m(N)$ as well. Let τ_m be the first time the walker visits m.

LEMMA 2.7. *(Reflection Principle) As $m \to \infty$*

$$\mathbb{P}(\tau_m > tm^2) \sim 2\mathbb{P}(X_{tm^2} > m) = \frac{2}{\sqrt{2\pi\tau}} \int_1^\infty e^{-u^2/(2\tau)} du = \sqrt{\frac{2}{\pi}} \int_{1/\sqrt{t}}^\infty e^{-z^2/2} dz.$$

PROOF. Since the random walk is recurrent for each $\varepsilon > 0$ there exists l such that

$\mathbb{P}$(there exists site $0 < j \leq R$ which is not visited by X_n up to time l) $\leq \varepsilon$.

Next

$$\mathbb{P}(X_{tm^2} > m)$$

$$= \sum_{k=1}^{tm^2} \mathbb{P}(\tau_m = k, X_{tm^2} > m) + \sum_{tm^2}^{tm^2+l} \mathbb{P}(\tau_m = k, X_{tm^2} > m) + \mathbb{P}(\tau_m > tm^2 + l, X_{tm^2} > m).$$

The last term is less than ε by the foregoing discussion, the second term is less than

$$\sum_{tm^2}^{tm^2+l} \mathbb{P}(X_k = m) = O(l/m)$$

by Local Limit Theorem while the first term equals

$$\sum_{k=1}^{tm^2} \mathbb{P}(\tau_m = k)\mathbb{P}(X_{tm^2-k} > m) \sim \frac{1}{2} \sum_{k=1}^{tm^2} \mathbb{P}(\tau_m = k) = \frac{1}{2}\mathbb{P}(\tau_m \leq m^2 t).$$

□

Lemma 2.7 says that as $m \to \infty$ $\tau_m/m^2 \Rightarrow T$-the random variable with density

$$\frac{1}{\sqrt{2\pi t^3}} e^{-1/(2t)}.$$

COROLLARY 2.8. *Suppose that $N \to \infty$, $m/\sqrt{N} \to x$ then*

$$\frac{L_m(N)}{\sqrt{N}} \Rightarrow \sqrt{\max(0, 1 - x^2 T)}|\mathbf{N}|.$$

PROOF. By Lemma 2.7 $\frac{\tau_m}{N} = \frac{\tau_m}{m^2}\frac{m^2}{N} \Rightarrow x^2 T$. On the other hand by Theorem 2.6 conditioned on $\tau_m = k$ we have $L_m(N) \approx \sqrt{N-k}|\mathbf{N}|$. □

We say that a random variable Y has a Mittag Leffler distribution if

$$\mathbb{E}(Y^k) = \frac{c^k k!}{\Gamma(\alpha k + 1)}.$$

As we saw above for $\alpha = 1$ we get a one-side Gaussian distribution.

EXERCISE 2.7. [**7**] Let X_n be a Markov process and B be a set such that there is a number $Q > 0$ such that uniformly for $x \in B$ we have $P_n(x, B)n^\alpha \to Q$. Show that $\frac{L_B(N)}{N^\alpha}$ converges as $N \to \infty$ to a Mittag Leffler distribution.

4. Ray-Knight Theorems.

Using moment asymptotics one can also obtain the joint distribution of local times at several points. However the local times at different sites are strongly correlated due to the constraint $\sum_m L_m(N) = N$. To remove this constraint we will randomize the time. In fact, we will kill two birds with one stone by prescribing the position of the walker at the final moment which will remove an unpleasant need to consider several case depending on where the walker ends up. Thus in this section we deal with the local times considered not at time N but at the time when the walker has visited certain site certain number of times. To simplify the analysis we treat in this section simple symmetric random walk postponing the extension to general mean zero walks till later.

LEMMA 2.9. *As* $n \to \infty$ $\frac{L_0(n)}{2n}$ *converges to exponential random variable with mean 1.*

PROOF. We claim that $P(L_0(\tau_n) = 0) = \frac{1}{2n}$. Indeed for this to happen the first step should be to the left which happens with probability $\frac{1}{2}$. Next since X_k is a martingale

$$\mathbb{P}(X \text{ visits } 0 \text{ before } n | X_1 = 1) = \frac{1}{2n}.$$

Consequently $L_0(\tau_n)$ has geometric distribution with parameter $\frac{1}{2n}$. Now the result follows easily. □

Next let $\tau_{n,k}$ be the time of k-th visit to site n.

LEMMA 2.10. *As* $n \to \infty$, $\frac{m}{n} \to x$

$$\frac{L_m(\tau_{0,n})}{n} \Rightarrow \sum_{j=1}^{N} \xi_j$$

where ξ_j *are iid variable having Exp*$(\frac{1}{2x})$ *distribution and* N *has Pois*$(\frac{1}{2x})$ *distribution* ($\sum_{j=1}^{0} \xi_j = 0$ *by definition).*

PROOF. Consider the random walk only at times it visits either 0 or m. Let $\bar{\sigma}_1, \bar{\sigma}_2 \dots \bar{\sigma}_k \dots$ be the lengths of consecutive stays at 0 and $\hat{\sigma}_1, \hat{\sigma}_2 \dots \hat{\sigma}_k \dots$ be the lengths of consecutive stays at m. Then $\{\bar{\sigma}_j\}$, $\{\hat{\sigma}_j\}$ are mutually independent and by Lemma 2.9 $\frac{\bar{\sigma}_j}{n} \to$Esp$(\frac{1}{2x})$,

$\frac{\hat{\sigma}_j}{n}$ $\to$Esp$(\frac{1}{2x})$. Accordingly the number of transitions from 0 to m converges to Pois$(\frac{1}{2x})$ and the result follows. □

The same reasoning also gives

LEMMA 2.11. *As $n \to \infty$, $\frac{n-m}{n} \to x$, $\frac{k}{n} \to t$*

$$\frac{L_m(\tau_{n,k})}{n} \Rightarrow \sum_{j=1}^{N+1} \xi_j$$

where ξ_j are iid variable having Exp$(\frac{1}{2x})$ distribution and N has Pois$(\frac{t}{2x})$ distribution.

PROOF. Consider the random walk only at times it visits either m or n and note that the walk comes to m first. □

Next we turn to joint distribution of the number of visits to several sites. As before we deal with two stopping times τ_n and $\tau_{0,n}$. In the first case let $m_1 > m_2 \cdots > m_k$ be the sites such that $\frac{n-m_j}{n} \to t_j$. As before we consider the walker only at times he visits either one of m_js or n. Let Z_s be the corresponding discrete time Markov chain with state space $\{0, 1, 2 \dots k\}$. If the walker is at site m_j then the next time he visits m_{j-1} with probability $\frac{1}{2(m_{j-1}-m_j)}$, m_{j+1} with probability $\frac{1}{2(m_j-m_{j+1})}$, and m_j with probability $1 - \frac{1}{2(m_{j-1}-m_j)} - \frac{1}{2(m_j-m_{j+1})}$. Accordingly the time the walker spends at m_j has geometric distribution with parameter $\frac{1}{2(m_{j-1}-m_j)} + \frac{1}{2(m_j-m_{j+1})}$. Afterwards the walker goes to m_{j+1} with probability

$$\frac{1}{m_j - m_{j+1}} : \left(\frac{1}{m_j - m_{j+1}} + \frac{1}{m_{j-1} - m_j}\right) = \frac{m_{j-1} - m_j}{m_{j-1} - m_{j+1}}.$$

Therefore the scaling assumptions we made imply

LEMMA 2.12. *As $n \to \infty$ Z_{sn} converges to a continuous time Markov chain $\mathcal{Z}_s$ with generator G where*

$$G_{j,j-1} = \frac{1}{2(t_j - t_{j-1})}, \quad G_{j,j+1} = \frac{1}{2(t_{j+1} - t_j)}, \quad G_{j,j} = -\frac{1}{2(t_j - t_{j-1})} - \frac{1}{2(t_{j+1} - t_j)}$$

and $G_{j,i} = 0$ otherwise.

In other words $\mathcal{Z}_s$ stays at site j for a time having Exp$\left(\frac{1}{2(t_j-t_{j-1})} - \frac{1}{2(t_{j+1}-t_j)}\right)$ distribution. Whereafter it goes to $j-1$ with probability $\frac{t_{j+1}-t_j}{t_{j+1}-t_{j-1}}$ or to $j+1$ with probability $\frac{t_j-t_{j-1}}{t_{j+1}-t_{j-1}}$.

Corollary 2.13.

$$\left(\frac{L_{m_1}(\tau_n)}{n}, \frac{L_{m_2}(\tau_n)}{n} \dots \frac{L_{m_k}(\tau_n)}{n}\right) \Rightarrow (T_1, T_2 \dots T_k)$$

where T_j *is the time* $\mathcal{Z}_s$ *spends at site* j *before reaching site* 0 *given that* $\mathcal{Z}_0 = k$.

Similar arguments apply to the stopping time $\tau_{0,n}$. We just state the answer.

Corollary 2.14. *Suppose* $0 < m_1 < m_2 \dots < m_k$ *be the sites such that* $\frac{m_j}{n} \to t_j$. *Then*

$$\left(\frac{L_{m_1}(\tau_{0,n})}{n}, \frac{L_{m_2}(\tau_{0,n})}{n} \dots \frac{L_{m_k}(\tau_{0,n})}{n}\right) \Rightarrow (\tilde{T}_1, \tilde{T}_2 \dots \tilde{T}_k)$$

where $\tilde{T}_j$ *is the time* $\mathcal{Z}_s$ *spends at site* j *before spending time 1 at site* 0 *given that* $\mathcal{Z}_0 = 0$.

Corollaries 2.13 and 2.14 may create an impression that the joint distributions of the local times look quite complicated. However this is not the case due to the Markov properties of the processes $j \to T_j$ and $j \to \tilde{T}_j$. Indeed by the time Z_s spends time T_j at site j the number of left and right jumps form independent Poisson processes. Therefore if we know T_j the knowledge of $T_{j-1}, \dots T_1$ does not give us any additional information for the purpose of computing the distribution of T_{j+1}. To compute the transition probabilities of our Markov processes we use Lemmas 2.11 and 2.12. Thus

$$T_{j+1} = \sum_{s=1}^{N+1} \xi_s, \quad \tilde{T}_{j+1} = \sum_{s=1}^{N} \xi_s$$

where $\xi_s \sim$Exp$\left(\frac{1}{2(t_{j+1}-t_j)}\right)$, and $N \sim$Pois$\left(\frac{T_j}{2(t_{j+1}-t_j)}\right)$ (respectively Pois$\left(\frac{\tilde{T}_j}{2(t_{j+1}-t_j)}\right)$). To compute the transition densities we introduce modified Bessel functions

$$I_\nu(x) = \sum_{m=0}^{\infty} \frac{1}{k!\Gamma(\nu+k+1)} \left(\frac{x}{2}\right)^{2k+\nu}.$$

Exercise 2.8. Show that $I_\nu(x)$ satisfy the modified Bessel equation

$$x^2 I'' + xI' - (x^2 + \nu^2) I = 0$$

and that Bessel functions $J_\nu(x) = i^\alpha I_\nu(-ix)$ satisfy the Bessel equation

$$x^2 J'' + xJ' - (x^2 - \nu^2) J = 0$$

Denote $\lambda = \frac{1}{2(t_{j+1}-t_j)}$. Then

$$p(T_{j+1}, T_j) = \sum_{m=0}^{\infty} e^{-T_j\lambda} \frac{(\lambda T_j)^m}{m!} \frac{(\lambda T_{j+1})^m}{m!} \lambda e^{-T_{j+1}\lambda}$$

$$= e^{-\lambda(T_j+T_{j+1})} \lambda I_0(2\lambda\sqrt{T_j T_j + 1}). \tag{2.6}$$

Note that since $T_0 = 0$ the series for $p(T_1, 0)$ reduces to only one term $m = 0$ so that $p(T_1, 0) = \lambda e^{-\lambda T_1}$ in accordance with Lemma 2.10.

Next

$$\mathbb{P}(\tilde{T}_{j+1} = 0 | \tilde{T}_j) = e^{-\lambda \tilde{T}_j} \tag{2.7}$$

and for $\tilde{T}_{j+1} \neq 0$

$$p(\tilde{T}_{j+1}, \tilde{T}_j) = \sum_{m=0}^{\infty} e^{-\tilde{T}_j\lambda} \frac{(\lambda T_j)^{m+1}}{(m+1)!} \frac{(\lambda \tilde{T}_{j+1})^m}{m!} \lambda e^{-\tilde{T}_{j+1}\lambda}$$

$$= e^{-\lambda(\tilde{T}_j+\tilde{T}_{j+1})} \tilde{T}_j^{-1} I_1(2\lambda\sqrt{T_j T_j + 1}). \tag{2.8}$$

We summarize our discussion as follows

Theorem 2.15. *(Ray-Knight Theorems, 1st formulation)* [**21, 30**] *$j \to T_j$ and $j \to \tilde{T}_j$ are Markov processes with transition probabilities given by* (2.6) *and* (2.7)–(2.8) *respectively.*

To complete the picture we consider the values of $L_m(\cdot)$ for all m simultaneously. That is let $\mathbb{L}^{(n)}(t) = \frac{L_{n(1-t)}(\tau_n)}{n}$ if nt is integer with linear interpolation in between. Likewise let $\widetilde{\mathbb{L}}^{(n)}(t) = \frac{L_{nt}(\tau_{0,n})}{n}$ if nt is integer with linear interpolation in between. We shall check that the families $\{\mathbb{L}^{(n)}(t)\}$ and $\{\widetilde{\mathbb{L}}^{(n)}(t)\}$ are tight in the space of continuous functions. Thus they have limit points $T(t)$ and $\tilde{T}(t)$ respectively. Note that the finite dimensional distributions of T and $\tilde{T}$ are given by Theorem 2.15. Since continuous processes are uniquely determined by their finite dimensional distributions we conclude that $\mathbb{L}^{(n)}(t) \Rightarrow T(t)$ and $\widetilde{\mathbb{L}}^{(n)}(t) \Rightarrow \tilde{T}(t)$.

Let us check the tightness of $\{\widetilde{\mathbb{L}}^{(n)}(t)\}$, $\{\mathbb{L}^{(n)}(t)\}$ is similar. Let ρ_m^+ denote the number of passages of the edge $[m, m+1]$ in positive direction and and ρ_m^- denote the number of passages of the edge $[m, m+1]$ in negative direction. Observe that $L_m(n) = \rho_m^+ + \rho_m^-$ and conditionally on $L_m(n) = l$ $\rho_m^\pm \sim$Bin$(l, \frac{1}{2})$. It follows from the Law of Large numbers that $\frac{\rho_m^+(n)}{L_m(n)} \to \frac{1}{2}$ so before dealing with $\widetilde{\mathbb{L}}^{(n)}(t)$ we first establish that

$$\widetilde{\mathfrak{R}}^{(n)}(t) := \frac{\rho_{nt}^+(\tau_{0,n})}{n} \text{ is tight.} \tag{2.9}$$

To study this process it is useful to consider an alternative description of the simpler symmetric random walk. Namely instead of deciding each time there the walker will go next we decide for each site m and each $k \in \mathbb{N}$ where the walker will go after the kth visit to m. Once the decision is made for each site the trajectory of the walker is determined uniquely. Let $\xi_{j,m}$ be the number of left steps from the site m between $j-1$st and jth right step. Then $\xi_{j,m}$ are iid random variables having geometric distribution with parameter $\frac{1}{2}$. Note that $\rho^-_{m+1} = \rho^+_m$ since the walker ends up at 0. Therefore

$$\rho^+_{m+1} = \sum_{j=0}^{\rho^+_m} \xi_{j,m+1}. \tag{2.10}$$

In other wordsρ^+_m is the branching process with geometric distribution of offsprings started from $\mathrm{Bin}(n, \frac{1}{2})$.

Let $\mathcal{F}_m$ denote the σ-algebra generated by $\rho^+_1, \rho^+_2 \dots \rho^+_m$. Then (2.10) implies that $(\rho^+_m, \mathcal{F}_m)$ is a martingale.

By Kolmogorov's theorem to show the tightness of $\widetilde{\mathfrak{R}}^{(n)}(t)$ it suffices to check that

$$\mathbb{E}(|\widetilde{\mathfrak{R}}^{(n)}(t_2) - \widetilde{\mathfrak{R}}^{(n)}(t_2)|^4) \leq C|t_2 - t_1|^{1+\delta}$$

for some $\delta > 0$. Let $m_j = Nt_j$. We need to estimate $\mathbb{E}(|\rho^+_{m_2} - \rho^+_{m_1}|^4)$. Let γ_m denote the matringale difference $\gamma_m = \rho^+_{m+1} - \rho^+_m$. By the maximal inequality for martingles

$$\mathbb{E}(\rho^+_{m_2} - \rho^+_{m_1}|^4) \leq C\mathbb{E}\left(\left(\sum_{m=m_1+1}^{m_2} \gamma_m^2\right)^2\right) = \sum_{m',m''} \mathbb{E}(\gamma^2_{m'}\gamma^2_{m''}) + \sum_m \mathbb{E}(\gamma^4_m).$$

Using the identity $\gamma_m = \sum_j (\xi_{j,m} - 1)$ we get $\mathbb{E}(\gamma_m^4) \leq C\mathbb{E}((\rho^+_m)^2)$. Next conditioning on $\rho^+_{m-1} = k$ we get

$$\mathbb{E}_k((\rho^+_m)^2) = (\mathbb{E}_k(\rho^+_m))^2 + \mathrm{Var}_k(\rho^+_m) = k^2 + 2k.$$

Thus

$$\mathbb{E}((\rho^+_m)^2) = \mathbb{E}((\rho^+_{m-1})^2) + 2\mathbb{E}(\rho^+_{m-1}) = \mathbb{E}((\rho^+_m)^2) + 2\mathbb{E}(\rho^+_0) = \mathbb{E}((\rho^+_m)^2) + n.$$

Hence

$$\mathbb{E}((\rho^+_m)^2) = \mathbb{E}((\rho^+_0)^2) + mn.$$

It follows that there exists $D > 0$ such that for $m \leq \mathbf{T}n$ we have$\mathbb{E}((\rho^+_m)^2) \leq Dn^2$. Also by Cauchy-Schwartz inequality $\mathbb{E}(\gamma^2_{m'}\gamma^2_{m''}) \leq Dn^2$. Therefore

$$\begin{aligned} \mathbb{E}(\rho^+_{m_2} - \rho^+_{m_1}|^4) &\leq C[(m_2 - m_1)n^2 + (m_2 - m_1)^2 n^2] \\ &\leq Cn^4(t_2 - t_1)^2. \end{aligned} \tag{2.11}$$

Thus
$$\mathbb{E}(|\widetilde{\mathfrak{R}}^{(n)}(t_2) - \widetilde{\mathfrak{R}}^{(n)}(t_2)|^4) \le C|t_2 - t_1|^2$$
and $\{\widetilde{\mathfrak{R}}^{(n)}(t)\}$ is tight as claimed.

EXERCISE 2.9. Compute $\mathbb{E}(\gamma_{m'}^2 \gamma_{m''}^2)$ exactly.

Likewise the tightness of $\{\widetilde{\mathbb{L}}^{(n)}(t)\}$ follows from the estimate
$$\mathbb{E}((L_{m_2}(\tau_{0,n}) - L_{m_1}(\tau_{0,n}))^4) \le Cn^2(m_2 - m_1)^2. \tag{2.12}$$
(2.12) is a consequence of (2.11) since $L_m(\tau_{0,n}) = \rho_m^+ + \rho_{m-1}^+$. The tightness of $\{\mathbb{L}^{(n)}(t)\}$ follows a similar argument except that we move in the opposite direction. Namely let $\mathcal{G}_m = \sigma(\rho_{n-1}^+, \rho_{n-2}^+ \dots \rho_{n-m}^+)$. Then we argue the same way as for $\widetilde{\mathbb{L}}^{(n)}$ using the fact that $(\rho_{n-m} - m, \mathcal{G}_m)$ is a martingale.

EXERCISE 2.10. Complete the proof of the tightness of $\{\widetilde{\mathbb{L}}^{(n)}(t)\}$.

Recall that a Markov process $T(t)$ with continuous paths is called a diffusion process with drift $a(T)$ and diffusion coefficient $\sigma^2(T)$ is
$$\lim_{h\to 0} \frac{\mathbb{E}(T(t+h) - T(t)|T(t) = T)}{h} = a(T), \quad \lim_{h\to 0} \frac{\mathbb{E}((T(t+h) - T(t))^2|T(t) = T)}{h} = \sigma^2(T).$$
In our case we have $\tilde{T}(t+h) = \sum_{s=1}^N \xi_j$ where $\xi_j \sim \text{Exp}(\frac{1}{2h})$, $N \sim \text{Pois}(\frac{1}{2h})$. Conditioning on $N = n$ we get
$$\mathbb{E}_n(\tilde{T}(t+t)) = 2nh,$$
$$\mathbb{E}_n(\tilde{T}^2(t+t)) = (\mathbb{E}_n(\tilde{T}(t+h))^2 + \text{Var}_n(\tilde{T}(t+h)) = 4h^2n^2 + 4h^2 n.$$
Consequently conditioned on $\tilde{T}(t) = T$ we get
$$\mathbb{E}_T(\tilde{T}(t+h)) = 2h\mathbb{E}_T(N) = T,$$
$$\mathbb{E}_T(\tilde{T}^2(t+h)) = 4h^2\mathbb{E}_T(N) + 4h^2\mathbb{E}_T(N^2) = 2hT + 4h^2((\mathbb{E}_T(N))^2 + \text{Var}_T(N)) = T^2 + 4hT.$$
Accordingly $\mathbb{E}_T((\tilde{T}(t+h) - T)^2) = \mathbb{E}_T(\tilde{T}^2(t+h)) - T^2 = 4hT$. Therefore the drift and diffusion coefficients of $\tilde{T}$ are
$$\tilde{a}(T) = 0, \quad \tilde{\sigma}^2(T) = 4T. \tag{2.13}$$
For T similar computations give
$$\tilde{a}(T) = 2, \quad \tilde{\sigma}^2(T) = 4T. \tag{2.14}$$

EXERCISE 2.11. Prove (2.14).

EXERCISE 2.12. Prove (2.13) and (2.14) using the relation of the local time with the branching processes.

Therefore we can restate Theorem 2.15 as follows

THEOREM 2.16. *(Ray-Knight Theorems, 2d formulation)*

(a) $\tilde{T}$ *is a diffusion process with zero drift and diffusion coefficient* $4T$.

(b) $\tilde{T}$ *is a diffusion process with drift* 2 *and diffusion coefficient* $4T$.

EXERCISE 2.13. Find the limiting distribution of $\overline{\mathbb{L}}^{(n)}(t) = \frac{L_{[nt]}(\tau_{m,n})}{\sqrt{n}}$ under the assumption that $n \to \infty, \frac{m}{n} \to x$.

EXERCISE 2.14. Extend Theorems 2.15 and 2.16 to weakly asymmetric random walk moving to the left with probability $\frac{1}{2} + \frac{c}{\sqrt{n}}$ and to the right with probability $\frac{1}{2} - \frac{c}{\sqrt{n}}$.

REMARK. Theorem 2.16 is stronger than Theorem 2.15 in that it handles all sites simultaneously. It is possible to give relatively short proof of Theorem 2.16 using deep results from the theory of diffusion processes. We prefer to give a direct combinatorial proof since it gives some insight on properties of T and $\tilde{T}$.

REMARK. The diffusion processes with constant drift and linear diffusion coefficient are called square Bessel processes since their transition densities are expressible in terms of modified Bessel functions.

5. Arcsine law.

Consider a simpler symmetric random walk. Let T be the first positive time when $X_T = 0$.

LEMMA 2.17.

$$\mathbb{P}(T > 2n) = \mathbb{P}(X_{2n} = 0).$$

PROOF. Denote $\rho_{n,k} = \mathbb{P}(X_{2n} = 2k, T > 2n)$. For above event to occur we need that $X_1 = 1$ which happens with probability $\frac{1}{2}$. Note that $\rho_{n,k} = \mathbb{P}(X_{2n} = 2k) - \mathbb{P}(X_{2n} = 2k, T \leq 2n)$. To compute the second term let $\bar{X}$ denote the reflected trajectory: $\bar{X}_n = X_n$ for $X \leq T$ and $\bar{X}_n = -X_n$ for $X \geq T$. Then the original and reflected trajectory has the same probabilities so

$$\mathbb{P}(X_{2n} = 2k, T < 2n | X_1 = 1) = \mathbb{P}(X_{2n} = -2k, | X_1 = 1) = P(X_{2n-1} = 2k+1).$$

Hence

$$\rho_{n,k} = \frac{1}{2} \left[\mathbb{P}(X_{2n-1} = 2k - 1) - \mathbb{P}(X_{2n-1} = 2k + 1) \right].$$

Summation over $k > 0$ gives

$$\mathbb{P}(T > 2n, X_{2n} > 0) = \frac{1}{2} \mathbb{P}(X_{2n-1} = 1).$$

On the other hand

$$\mathbb{P}(X_{2n}=0)=\frac{1}{2}P(X_{2n-1}=1)+\frac{1}{2}P(X_{2n-1}=-1)=P(X_{2n-1}=1)$$

so that

$$\mathbb{P}(T>2n, X_{2n}>0)=\mathbb{P}(T>2n, X_{2n}<0)=\frac{1}{2}\mathbb{P}(X_{2n}=0).$$

□

Let Z_n denote the length of the last zero-free period for the walk, that if $X_{2n-Z_n}=0, X_m\neq 0$ for $m>2n-Z_n$.

LEMMA 2.18. *(a)* $\mathbb{P}(Z_n)=2k)=\mathbb{P}(X_{2k}=0)\mathbb{P}(X_{2(n-k)}=0)$.
(b) $\lim_{n\to\infty}P(\frac{Z_n}{n}\leq a)=\frac{2}{\pi}\arcsin\sqrt{a}$.

PROOF. (a) For $Z_n=2k$ two event must happen. First, $X_{2n-2k}=0$ and secondly the walk should avoid 0 for the last $2k$ steps. The probability of the first event is $\mathbb{P}(X_{2(n-k)}=0)$ while the probability of the second event is $\mathbb{P}(T>2k)=\mathbb{P}(X_{2k}=0)$.

(b) From (a) and Local Limit Theorem we get

$$\mathbb{P}(X_{2k}=0)\mathbb{P}(X_{2(n-k)}=0)\sim\frac{1}{\pi}\frac{1}{\sqrt{k(n-k)}}$$

(Recall that the simple symmetric random walk is not aperiodic so to apply the Local Limit Theorem we use the fact that $\mathbb{P}(X_{2k}=0)=\mathbb{P}(Y_k=0)$ where Y is the random walk with increments $\frac{\Delta_1+\Delta_2}{2}$. In particular $\mathrm{Var}(Y_k)=\frac{k}{2}$ so $\mathbb{P}(Y_k=0)\sim\frac{1}{2\pi\frac{1}{2}k}=\frac{1}{\pi k}$.) Thus

$$\mathbb{P}(Z_n\leq na)\sim\sum_{k=1}^{an}\frac{1}{\pi n}\frac{1}{\sqrt{\frac{k}{n})(1-\frac{k}{n})}}\to\int_0^a\frac{1}{\pi}\frac{dz}{\sqrt{z(1-z)}}=\frac{2}{\pi}\arcsin\sqrt{a}.$$

□

Let U_n be the time spent by the random walk above 0. More precisely, let $\bar{W}_n(t)$ be the process such that $W_n(\frac{m}{n})=\frac{X_m}{\sqrt{n}}$ with linear interpolation in between. Then $U_n=n\mathrm{mes}(t:\bar{W}_n>0)$.

LEMMA 2.19. *U_n and Z_n have the same distribution.*

PROOF. We use induction on n. Note that

$$=\mathbb{P}(Z_n=2n)=\mathbb{P}(T>2n)=\mathbb{P}(X_{2n}=0)=\mathbb{P}(Z_n=0).$$

Also $P(U_n=2n)=\mathbb{P}(X_m\geq 0 \text{ for } 0\leq m\leq 2n)$ Next we claim that

$$\mathbb{P}(X_m\geq 0 \text{ for } 0\leq m\leq 2n)=2\mathbb{P}(X_m>0 \text{ for } m=1\dots 2n)=\mathbb{P}(X_{2n}=0).$$

To this end consider the simple symmetric walk $Y_n = X_{n+1} - X_1$. Then

$$\mathbb{P}(X_m > 0 \text{ for } m = 1 \dots 2n) = \frac{1}{2}\mathbb{P}(Y_m \geq 0 \text{ for } m = 1, 2n-1) = \frac{1}{2}\mathbb{P}(Y_m \geq 0 \text{ for } m = 0, 2n).$$

It follows that $\mathbb{P}(U_n = 2n) = \mathbb{P}(X_{2n} = 0)$. By symmetry $\mathbb{P}(U_n = 0) = \mathbb{P}(X_{2n} = 0)$. Note that we have proved the statement for $n = 1$. For $n > 1$ we still have to show that $\mathbb{P}(U_n = 2k) = \mathbb{P}(Z_n = 2k)$ if $0 < k < n$. In this case the walk should return to 0 before the time $2n$. Let $2r$ be the time of the first return. Considering the cases where the walker stay above and below x-axis separately we obtain

$$\mathbb{P}(U_n = 2k) = \frac{1}{2}\sum_{r=1}^{k}\mathbb{P}(T = 2r)\mathbb{P}(U_{n-r} = 2(k-r)) + \frac{1}{2}\sum_{r=1}^{n-k}\mathbb{P}(T = 2r)\mathbb{P}(U_{n-r} = 2k).$$

By induction the first sum is

$$\sum_{r=1}^{k}\mathbb{P}(T = 2r)\mathbb{P}(Z_{n-r} = 2(k-r)) = \sum_{r=1}^{k}\mathbb{P}(T = 2r)\mathbb{P}(X_{2(k-r)} = 0)\mathbb{P}(X_{2(n-k)} = 0)$$

$$= \mathbb{P}(X_{2(n-k)} = 0)\mathbb{P}(X_{2k} = 0) = \mathbb{P}(Z_n = 2k)$$

and the second sum is

$$\sum_{r=1}^{n-k}\mathbb{P}(T = 2r)\mathbb{P}(U_{n-r} = 2k) = \sum_{r=1}^{n-k}\mathbb{P}(T = 2r)\mathbb{P}(X_{n-r-k} = 0)\mathbb{P}(X_{2k} = 0)$$

$$\mathbb{P}(X_{2k} = 0)\mathbb{P}(X_{2(n-k)} = 0) = \mathbb{P}(Z_n = 2k).$$

Adding the two terms above we obtain the result. □

EXERCISE 2.15. Show that the conditioned on $X_{2n} = 0$, U_n has uniform distribution on the set of even integers from 0 to $2n$.

Hint. Using the reflection principle show that $\mathbb{P}(U_n = 0) = \mathbb{P}(U_n = 2n) = \frac{1}{n+1}$. The proceed by induction as in the proof of Lemma 2.19.

COROLLARY 2.20.

$$\lim_{n\to\infty} P\left(\frac{U_n}{n} \leq a\right) = \frac{2}{\pi}\arcsin\sqrt{a}.$$

6. Extension to finite range random walks.

THEOREM 2.21. *Corollary 2.20 is valid for arbitrary finite range random walk with zero mean.*

PROOF. Let $W_n(t)$ denote the process on $[0,1]$ such that $W_n(\frac{m}{n}) = \frac{X_m}{\sqrt{n}}$ and let $\bar{W}_n(t)$ denote the corresponding process for a simple symmetric random walk $\bar{X}_n$. Let $W(t)$ be the standard Brownian Motion with zero mean and variance t. Let U_n and $\bar{U}_n$ be the time spent by the walker above 0. We have $U_n = n\Phi(W_n) = \Phi(\frac{W_n}{\sigma})$ where

$$\Phi(W) = \text{mes}(t : W(t) > 0).$$

We claim that

$$\lim_{n\to\infty} \mathbb{P}\left(\Phi\left(\frac{W_n}{\sigma}\right) \leq a\right) = \mathbb{P}(\Phi(W(t)) \leq a) \tag{2.15}$$

Since by Weak Invariance Principle $\frac{W_n(t)}{\sigma} \Rightarrow W(t)$ we need to check that

$$\mathbb{P}(W(t) \text{ is a point of continuity of } \Phi) = 1. \tag{2.16}$$

Observe that if $|V(t) - W(t)| < \varepsilon$ then

$$\text{mes}(t : W(t) > \varepsilon) \leq \Phi(V) \leq \text{mes}(t : W(t) > -\varepsilon).$$

Therefore it suffices to show that with probability 1

$$\text{mes}(t : |W(t)| < \varepsilon) \to 0 \text{ as } \varepsilon \to 0.$$

However by Fubini Theorem

$$\mathbb{E}(\text{mes}(t : |W(t)| < \varepsilon) = \int_0^1 \int_{-\varepsilon}^{\varepsilon} \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx dt \to 0 \text{ as } \varepsilon \to 0.$$

Hence (2.16) follows. Thus

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{U_n}{n} \leq a\right) = \lim_{n\to\infty} \mathbb{P}(\Phi(W_n\sigma) \leq a) = \mathbb{P}(\Phi(W(t)) \leq a)$$

$$= \lim_{n\to\infty} \mathbb{P}(\Phi(2\bar{W}_n) \leq a) = \lim_{n\to\infty} \mathbb{P}(\Phi(\bar{W}_n) \leq a) = \frac{2}{\pi} \arcsin\sqrt{a}.$$

□

EXERCISE 2.16. Show that W is a point of continuity of Φ iff

$$\text{mes}(t : W(t) = 0) = 0.$$

EXERCISE 2.17. Generalize the results of Section 4 to finite range random walks. That is, prove that that if X is a finite range random walk then $\frac{\sigma L_{t\sigma n}(\tau_0)}{2n}$ converge to the diffusion process with drift 2 and variance $4T$ and $\frac{\sigma L_{t\sigma n}(\tau_{0,n})}{2n}$ converge to the diffusion process with drift 0 and variance $4T$.

Hint. By considering the walk only at the moments it visits m and $m+1$ show that typically $L_{m+1}(\tau) = L_m(\tau) + O(\sqrt{L_m(\tau)})$. Conclude that for small δ $L_m(\tau) \sim \frac{1}{\delta n}\sum_{k=m}^{m+\delta n} L_k(\tau)$. Then compare L_m to $\frac{1}{\delta}\int_0^1 1_{[m/\sigma n, m/\sigma n+\delta]}(W(t))dt$.

7. Favorite sites of transient walk.

Here we consider the walks with $\mu \neq 0$. To fix the notation we assume that $\mu > 0$. In this case each site is visited finitely many times. In fact the number of visits forms asymptotically stationary sequence. Let L_n denote the total number of visits of site n.

LEMMA 2.22. *Given a finite sequence $l_0, l_1 \dots l_m$ there is a limit*

$$\lim_{n\to\infty} \mathbb{P}(L_n = l_0, L_{n+1} = l_1 \dots L_{n+m} = l_m).$$

PROOF. We shall show that there exists $C > 0, \theta < 1$ such that

$$|\mathbb{P}(L_n = l_0, L_{n+1} = l_1 \dots L_{n+m} = l_m) - \mathbb{P}(L_{n+1} = l_0, L_{n+2} = l_1 \dots L_{n+m+1} = l_m)| \leq C\theta^n.$$

Indeed keeping track of the initial point we have

$$\mathbb{P}_0(L_{n+1} = l_0, L_{n+2} = l_1 \dots L_{n+m+1} = l_m) = \mathbb{P}_{-1}(L_n = l_0, L_{n+1} = l_1 \dots L_{n+m} = l_m)$$

so we need to show that

$$|\mathbb{P}_{-1}(L_n = l_0, L_{n+1} = l_1 \dots L_{n+m} = l_m) - \mathbb{P}_{-1}(L_n = l_0, L_{n+1} = l_1 \dots L_{n+m} = l_m)| \leq C\theta^n.$$

To this end we will construct a coupling between the walk X_n started from 0 and the walk $\bar{X}_n$ started from -1 such that there exists random variables k_1, k_2 such that $X_{n+k_1} = \bar{X}_{n+k_2}$ and

$$\mathbb{P}(\max(\max_{j\in[0,k_1]} X_j, \max_{j\in[0,k_2]} \bar{X}_j) \geq n) \leq C\theta^n.$$

Assume first that X is aperiodic. Then there exists k, γ such that $\mathbb{P}(X'_k = X''_k) > \gamma$ where X' and X'' are two independent walks which start within distance R apart (R is the longest right jump of the walk). Then if at time k we have $X_k = \bar{X}_k$ then we can couple them requiring $\Delta_j = \bar{\Delta}_j$ for $j > k$. Otherwise we wait till X and $\bar{X}$ enter $\{x \geq kR\}$ and try to couple them again. Thus before the walkers enter $\{x \geq n\}$ we can make n/Const attempts to couple them. The probability that all attempts fail is $\gamma^{n/\mathrm{Const}}$. This completes the proof in the aperiodic case. If X is periodic we choose j_0 such that $\bar{X}_{j_0}$ is in the same periodicity class as 0 and then couple X_n and $\bar{X}_{j_0+n}$ using the above algorithm. □

In particular let $c = \lim_{n\to\infty} \mathbb{P}(L_n \neq 0)$. Let ρ denote the probability that the walk will return to its initial position.

LEMMA 2.23. $\frac{c}{1-\rho} = \frac{1}{\mu}$.

PROOF. Conditioned on $L_n \neq 0$, L_n has Geom$(1-\rho)$ distribution so $\mathbb{E}(L_n) = \frac{\mathbb{P}(L_n \neq 0)}{1-\rho}$. Accordingly

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mathbb{E}(L_n) = \frac{c}{1-\rho}.$$

Note that the sum in the LHS equals to the expected number of visits to the interval $[1, N]$. On the other hand by the Large Deviation estimate

$$\mathbb{P}\left(\exists k \in \left[\varepsilon N, \left(\frac{1}{\mu} - \varepsilon\right) N\right] : X_k \notin [1, N]\right) \leq C e^{-I_\varepsilon N}$$

and

$$\mathbb{P}(X_k \in [1, N]) \leq C e^{-I_\varepsilon k} \text{ if } k > \left(\frac{1}{\mu} + \varepsilon\right) N.$$

Therefore

$$\frac{1}{N} \sum_{n=1}^{N} \mathbb{E}(L_n) = \frac{1}{\mu}$$

and the result follows. □

Next since in this case we do not have any non-trivial scaling behavior for L_n we consider the places where it exhibits unlikely behavior. That is we consider favorite sites of the walk. Note that

$$\mathbb{P}(L_n > k) \to \frac{1-\rho}{\mu} \rho^k.$$

THEOREM 2.24. *Let $k_N \to \infty$ so that along a subsequence we have $\frac{1-\rho}{\mu} \rho^{k_N} N \to \lambda$ Then along this subsequnce*

$$\text{Card}(1 \leq n < N : L_n > k_N) \Rightarrow \text{Pois}(\lambda).$$

The proof will rely on an auxiliary estimate saying that the points where L_n takes high values are well separated.

LEMMA 2.25. *There is $\theta < 1$ such that $\mathbb{P}(L_n > k, L_{n-m} > k) \leq C\rho^k \theta^k$.*

PROOF OF THEOREM 2.24. According to Lemma 2.25 there exists $\beta > 0$ such that

$$\mathbb{P}(L_n > k_N, L_{n-m} > k_N) \leq N^{-(1+\beta)}. \tag{2.17}$$

Take $\gamma < \beta$ Divide $[0, N]$ into long segments I_s of length N^γ separated by short segments J_s of length $N^{\gamma/2}$. More precisely $I_s = [x_s, x_s + N^\gamma - 1]$ where x_s is the first site in $(s-1)(N^\gamma + N^{\gamma/2})$ visited by the walk. Note that

$$\mathbb{P}(\exists s, n \in J_s : L_n > k_N) \leq C \frac{N^{1-\gamma/2}}{N} = C N^{-\gamma/2} \to 0$$

where the numerator counts the total number of points in the short segments. Let $\xi_s = \sum_{n\in I_s} 1_{L_n>k_N}$. Thus ξ_s is the total number of favorite points in I_s. Let $\tilde{\xi}_s = \sum_{n\in I_s} \tilde{L}_n$ where we only count visits to I_s before the first time the walker leaves I_s and visits some $I_{s'}$ for $s'\neq s$. Note that if $\tilde{\xi}_s \neq \xi_s$ then the walker should backtrack by $N^{\gamma/2}$. Now by Theorem 2.5 given n

$$\mathbb{P}(X \text{ visits }]-\infty, n-N^{\gamma/2}] \text{ after visiting } [n,\infty[) \leq \theta^{N^{\gamma/2}}.$$

Hence

$$\mathbb{P}(\tilde{L}_n = L_n \text{ for all } n\in[1,N]) \geq 1 - N\theta^{N^{\gamma/2}}. \tag{2.18}$$

In particular (2.17) is valid with L_n replaced by $\tilde{L}_n$. Thus

$$\mathbb{P}(\tilde{\xi}_s \geq 2) \leq \sum_{n,n-m\in I_s} \mathbb{P}(\tilde{L}_n > k_N, \tilde{L}_{n-m} > k_N) \leq N^{2\gamma-1-\beta}$$

while by Bonferroni inequality

$$\mathbb{P}(\tilde{\xi}_s = 1) = \sum_{n\in I_s} \mathbb{P}(\tilde{L}_n > k_N) + O\left(\sum_{n,n-m\in I_s} \mathbb{P}(\tilde{L}_n > k_N, \tilde{L}_{n-m} > k_N)\right)$$
$$= \lambda N^{\gamma-1} + O(N^{2\gamma-1-\beta}).$$

Hence

$$\mathbb{E}(e^{it\tilde{\xi}_s}) = (1 + \frac{\lambda}{N^{1-\gamma}}(e^{it}-1) + O(N^{2\gamma-1-\beta}).$$

Since $\tilde{\xi}_s$ are independent $\sum_s \tilde{\xi}_s$ has characteristic function

$$\exp(\lambda(e^{it}-1))(1+o(1))$$

so $\sum_s \tilde{\xi}_s$ is asymptotically Poisson. By (2.18) $\sum_s \xi_s$ is also asymptotically Poisson. □

Proof of Lemma 2.25. Consider our walk only at the moments when it visits either n or $n-m$. The transition probability takes form

$$\begin{pmatrix} p_1 & q_1 & r_1 \\ q_2 & p_2 & r_2 \\ 0 & 0 & 1 \end{pmatrix}$$

where the entries depend on m and we added a terminal state to account for the fact that the walk visits each site only finite number of times. Let u be the probability that the walk started from $n-m$ never visit n. By one step analysis $u = q_1 + p_1 u$ so that $u = \frac{q_1}{1-p_1}$. Accordingly if the walk starts at n and is conditioned to return to n then it goes to $n-m$ with probability

$$\frac{q_1q_2}{1-p_1} : \left(\frac{q_1q_2}{1-p_1}+p_2\right)=\frac{q_1q_2}{q_1q_2+(1-p_1)p_2}.$$

Once the walker moves to $n-m$ he can stay where k times with probability

$$\frac{q_2p_2^{k-1}}{\sum_{j=1}^{\infty} q_2p_2^{j-1}}=p_1^{k-1}(1-p_1)$$

so the expected number of returns to $n-m$ before coming back to n is $\frac{1}{1-p_1}$. Multiplying this by the probability of going to $n-m$ we get

$$\mathbb{E}(\text{Visits to } n{-}m \text{ before return to } n|X \text{ returns to } n)=\frac{q_1q_2}{q_1q_2+(1-p_1)p_2}\frac{1}{1-p_1}.$$

Thus conditioned on $L_n=k$ we have $L_{n-m}=\sum_{j=1}^{k-1}\xi_j+\eta_0+\eta_k$ where η_0 counts visits to $n-m$ before the first visit to n, η_k counts visits to $n-m$ after the last visit to n and ξ_j counts visits to $n-m$ between the j-th and $j+1$-st visit to n. Hence $\mathbb{E}(L_{n-m}|L_n=k)=v(m)k+O(1)$ where

$$v(m)=\frac{q_1q_2}{q_1q_2+(1-p_1)p_2}\frac{1}{1-p_1}. \tag{2.19}$$

Due to the Large Deviation estimate it suffices to show that $v(m)<1$ for all $m>0$. Notice that as $m\to\infty$ the transition matrix approaches

$$\begin{pmatrix} \rho & (1-\rho)c & (1-\rho)(1-c) \\ 0 & \rho & 1-\rho \\ 0 & 0 & 1 \end{pmatrix}$$

So the numerator of (2.19) tends to 0 while the denominator stays bounded. Hence $v(m)\to 0$ as $m\to 0$ and so there exists m_0 such that $v(m)<1$ if $m\geq m_0$. It remains to handle $m\in[1,m_0-1]$. Assume by contradiction that $v(m)\geq 1$ for some $m<m_0$. Pick $\varepsilon_1\ll\varepsilon_2\cdots\ll\varepsilon_{m_0}\ll 1$. We will show by induction that for $j\in 1\dots m_0$

$$\mathbb{P}(L_{n-jm}>(1-\varepsilon_j)k|L_n>k)\to 1 \text{ as } k\to\infty. \tag{2.20}$$

For $j=1$ the statement follows from the assumption $v(m)\geq 1$ and the Large Deviation estimate. Next if the result is known for some j then

$$\mathbb{P}(L_{n-(j+1)m}<(1-\varepsilon_{j+1})k|L_n>k)=$$

$$\mathbb{P}(L_{n-(j+1)m}<(1-\varepsilon_{j+1})k, L_{n-mj}>(1-\varepsilon_j)k|L_n>k)+o(1).$$

The first term here equals to

$$\frac{\mathbb{P}(L_{n-(j+1)m} < (1-\varepsilon_{j+1})k, L_{n-mj} > (1-\varepsilon_j)k, L_n > k)}{\mathbb{P}(L_n > k)} \leq$$

$$\frac{\mathbb{P}(L_{n-(j+1)m} < (1-\varepsilon_{j+1})k | L_{n-mj} > (1-\varepsilon_j)k)\mathbb{P}(L_{n-jm} > (1-\varepsilon_j)k)}{\mathbb{P}(L_n > k)} \leq$$

$$\rho^{\varepsilon_j k} e^{-I(\varepsilon_j, \varepsilon_{j-1})k}.$$

The last expression tends to 0 provided that $\varepsilon_j \ll \varepsilon_{j+1}$. This proves (2.20). However (2.20) with $j = m_0$ contradicts to the fact that

$$v(n - mm_0) < 1.$$

Thus $v(m) < 1$ for all m as claimed. □

EXERCISE 2.18. Let Y be the simple random walk going to the right with probability $p > q$ conditioned on returning to 0 a number of times. Show that before the last return to 0

$$\mathbb{P}(Y_{n+1} - Y_n = +1 | Y_n) = \begin{cases} q & \text{if } Y_n > 0 \\ p & \text{if } Y_n < 0 \\ \frac{1}{2} & \text{if } Y_n = 0 \end{cases}$$

and use this formula to compute $v(m)$ explicitly.

COROLLARY 2.26. *Under the assumptions of Theorem 2.24*

$$\text{Card}(n : L_n(N) > k_N) \Rightarrow \text{Pois}\,(\lambda\mu)\,.$$

PROOF. Using the Strong Law of Large Numbers

$$\mathbb{P}(L_n(N) = L_n \text{ for all } n \leq (\mu - \varepsilon)\, N) \to 1.$$

Accordingly by Theorem 2.24

$$\liminf \mathbb{P}(\text{Card}(n : L_n(N) > k_N) \geq m) \geq \sum_{k=m}^{\infty} (\mu - \varepsilon)^k \frac{\lambda^k}{k!} \exp\left[-(\mu - \varepsilon)\,\lambda\right].$$

Likewise

$$\mathbb{P}(L_n(N) = 0 \text{ for all } n \geq (\mu + \varepsilon)\, N) \to 1.$$

Accordingly by Theorem 2.24

$$\liminf \mathbb{P}(\text{Card}(n : L_n(N) > k_N) \geq m) \geq \sum_{k=m}^{\infty} \left(\frac{1}{\mu} + \varepsilon\right)^k \frac{\lambda^k}{k!} \exp\left[-(\mu + \varepsilon)\,\lambda\right].$$

Since ε is arbitrary we can let $\varepsilon \to 0$ and get the required result. □

EXERCISE 2.19. Prove that under assumption of Theorem 2.24

$\mathrm{Card}(n : L_n(N) = k_N+1), \mathrm{Card}(n : L_n(N) = k_N+2), \dots \mathrm{Card}(n : L_n(N) = k+l_N)$

are asymptotically independent Poisson random variables with parameters

$$\left(\lambda(1-\rho)\mu, \lambda(1-\rho)\rho\mu, \dots, \lambda(1-\rho)\rho^{l-1}\mu\right).$$

EXERCISE 2.20. Prove that if $M_N = \max_m L_m(N)$ then

$$\lim_{N\to\infty} \mathbb{P}(M_N - k_N > a) = \exp\left(-\lambda\rho^a\mu\right).$$

EXERCISE 2.21. Prove the following limit theorem for the favorite site of recurrent walk. Let $M_N = \max_m L_m(\tau_{0,N})$. Then for $x \geq 1$

$$\lim_{N\to\infty} \mathbb{P}\left(\frac{M_N}{N} \leq x\right) = \left(1 - \frac{1}{x}\right)^2.$$

Hint. Let $M_N^{\pm} \max_{\pm m>0} L_m(\tau_{0,N})$. Use the analysis of Section 4 to show that M_N^- and M_N^+ are asymptotically independent. Next use the fact that the process $\tilde{T}(t)$ in Theorem 2.16 is martingale to show that

$$\lim_{N\to\infty} \mathbb{P}\left(\frac{M_N^{\pm}}{N} \geq x\right) = \frac{1}{x}.$$

EXERCISE 2.22. Prove the Central Limit Theorem for $\mathrm{Card}(1 \leq n \leq N : L_n = 0)$

Hint. Use the small block-long block decomposition employed in the proof of Theorem 2.24. Show that the contribution of the small blocks is negligible and verify the condition of Lindenberg-Feller Central Limit Theorem for the sum of contributions of the long blocks.

EXERCISE 2.23. Use the result of Exercise 2.22 to prove the Central Limit Theorem for the number of cites visited by the walk utill time N.

CHAPTER 3

Random walks in Markov environment

1. The result.

We consider the following model. At each site $u \in \mathbb{Z}$ there is a Markov chain x_u^n with finite state space $\mathcal{A}$ and transition probability $p_{ab} > 0$. We denote by $p_{ab}(n)$ the n-step transition probability of the Markov chain and by π_b the stationary distribution of the chain. The chains at different sites are independent and each chain is started from the stationary distribution π. Let Λ be a finite subset of $\mathbb{Z}$. For each $a \in \mathcal{A}$ let $q_{a,j}$ be a probability distribution on Λ. Given the environment $\{x_u^n\}$ the transition probability of the walk depend on the state of the chain at the present position of the walker. That is $X_{n+1} - X_n = \Delta_n$ where

$$\mathbb{P}(\Delta_n = j | X_n) = q_{x^u_{X_n},j}.$$

A useful example to keep in mind is then the Markov chain has two states L and R with transition matrix $\begin{pmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{pmatrix}$ and if the walker is at state L then he goes left while state R forces him to go right. One can also consider a softer situation when the Δ_n can take both values -1 and $+1$ at each state but L makes -1 more likely while R makes $+1$ more likely. That is possible probability distributions are $q_L = (0.8, 0.2)$, $q_R = (0.2, 0.8)$ or $q_L = (0.8, 0.2)$, $q_R = (0.4, 0.6)$.

THEOREM 3.1. *([**10**]) There exist numbers* $\mathbf{v}, D$ *such that*

$$\frac{X_n}{n} \Rightarrow \mathbf{v} \text{ and } \frac{X_n - n\mathbf{v}}{\sqrt{n}} \Rightarrow \mathcal{N}(0, D).$$

2. Environment viewed by the walker.

To prove Theorem 3.1 we employ the method called *environment viewed by the walker.* There are two ways to understand this notion. More common is the broad way when we look at all the sites at time n shifted so that the walker is at the origin and study the distribution of this random sequence. For the problem at hand it is more convenient to take a more narrow or "nearsighted" view where the walker only cares about the site he is presently in. That is, let ω_n denote the pair

(x_n, Δ_n) where $x_n = x^u_{X_n}$. We consider the sequence $\omega_0, \omega_1 \dots \omega_n$ and we will show that it is asymptotically stationary. However the length of this sequence changes with n so it will be convenient to introduce a "fake past". We say that a sequence $\{\omega_k\}$ is admissible if $q_{x_k,\Delta_k} > 0$. Choose any admissible sequence $\bar{\xi} = \{\bar{\omega}_k\}_{k=-\infty}^{-1}$. Let ξ_n be the sequence $\dots \bar{\omega}_{-k}\omega_0\omega_1 \dots \omega_n$ reindexed so that the symbols range from $-\infty$ to 0. We would like to study the distribution of ξ_n. Let σ denote the shift, that is, $\sigma\eta$ is the sequence obtained from η by erasing the last symbol. Let Σ denote the space of all admissible functions. To measure the closeness we introduce the distance $d(\eta', \eta'') = \theta^{n_0}$ where $n_0 = \max\{n : \eta'_{-j} = \eta''_{-j} \text{ for } j \le n\}$ where $\theta < 1$. Then the environment viewed from by the walker becomes a Markov chain with transition probability

$$\mathbb{P}_n(\eta|\xi) = r_n(x_0, \xi) q_{x_0,\Delta_0}$$

where $\eta = \xi(x_0, \Delta_0)$,

$$r_n(x_0, \xi) = \begin{cases} \pi_{x_0} & \text{if } L > n \\ \pi_{\xi_{-L},x_0}(L), & \text{if } L \le n \end{cases}$$

and L is the last time the walker was at the present site. Note that by mixing of the Markov chain

$$|\mathbb{P}_n(\eta|\xi) - \mathbb{P}(\eta|\xi)| \le C\rho^n \tag{3.1}$$

where

$$r(x_0, \xi) = \begin{cases} \pi_{x_0} & \text{if } L = \infty \\ \pi_{\xi_{-L},x_0}(L), & \text{if } L < \infty \end{cases}$$

and ρ denotes the spectral gap of the Markov chain. Consider the generator

$$(\mathcal{L}_n f)(\xi) = \sum_{\sigma\eta=\xi} \mathbb{P}_n(\eta|\xi) f(\eta).$$

Let

$$(\mathcal{L} f)(\xi) = \sum_{\sigma\eta=\xi} \mathbb{P}(\eta|\xi) f(\eta).$$

Let C_θ denote the space of d_θ-Lipshitz functions.

LEMMA 3.2. *(a)* $||\mathcal{L}_n - \mathcal{L}||_\infty < \text{Const}\rho^n$.
(b) $\mathcal{L}_n 1 = \mathcal{L} 1 = 1$ *and* $||\mathcal{L}_n||_\infty = ||\mathcal{L}||_\infty = 1$.
(c) If $\theta > \rho$ *then* $\mathcal{L}_n$ *and* $\mathcal{L}$ *preserve* C_θ.

PROOF. (a) follows directly from (3.1).

We prove parts (b) and (c) for $\mathcal{L}_n$. The proofs for $\mathcal{L}$ are the same. First, we have

$$\mathcal{L}_n 1 = \sum_{\sigma\eta=\xi} \mathbb{P}_n(\eta|\xi) = 1. \tag{3.2}$$

Next

$$|(\mathcal{L}_n f)(\xi)| \leq ||f||_\infty \sum_{\sigma\eta=\xi} \mathbb{P}_n(\eta|\xi) = ||f||_\infty.$$

Thus $||\mathcal{L}_n|| \leq 1$ which together with (3.2) proves part (b). Finally suppose that first k symbols of ξ' and ξ'' coincide. Given ω_0 let $\eta' = \xi'\omega_0$, $\eta'' = \xi''\omega_0$. Note that first $k+1$ symbols of η' and η'' coincide. Hence

$$|\mathbb{P}_n(\eta'|\xi')f(\eta')-\mathbb{P}_n(\eta''|\xi')f(\eta'')| \leq \mathbb{P}_n(\eta'|\xi')|f(\eta')-f(\eta'')|+|f(\eta'')||\mathbb{P}_n(\eta'|\xi')-\mathbb{P}_n(\eta''|\xi'')|$$
$$\leq \mathbb{P}_n(\eta'|\xi')L(f)\theta^{k+1} + 2C||f||_\infty \rho^k$$

where $L(f)$ is the Lipshitz constant of f. Summing over all admissible ω_0 we obtain

$$|\mathcal{L}_n(f)(\xi') - \mathcal{L}_n(f)(\xi'')| \leq [\theta L(f) + \text{Const}||f||_\infty]\, \theta^k.$$

This proves part (c). □

The next theorem is proven in Section 4.

THEOREM 3.3. *(Ruelle-Perron-Frobenius Theorem) There are constants $C > 0$, $\zeta < 1$ and a linear functional $\mu(f)$ such that*

$$||\mathcal{L}^n f - \mu(f)1||_\theta \leq C\zeta^n ||f||_\theta.$$

In Section 3 we derive the following result from Theorem 3.3. For $f \in C_\theta$ let $S_n = \sum_{j=0}^{n-1} f(\xi_n)$.

THEOREM 3.4. *(a)* $\frac{S_n}{n} \Rightarrow \mu(f)$.
(b) There exists $D(f)$ such that $\frac{S_n - n\mu(f)}{\sqrt{n}} \Rightarrow \mathcal{N}(0, D)$.

Letting $f = \Delta_0$ in Theorem 3.4 we obtain Theorem 3.1.

3. The Central Limit Theorem for Gibbs measures.

Here we prove Theorem 3.4.

PROOF. Without the loss of generality we may assume that $\mu(f) = 0$. We have $\mathbb{E}(S_n) = \sum_{k=0}^{n-1} \mathbb{E}(f(\xi_k))$ where

$$\mathbb{E}(f(\xi_k)) = \mathcal{L}_0\mathcal{L}_1 \dots \mathcal{L}_{k-1} f(\bar{\omega})$$
$$= \mathcal{L}_0\mathcal{L}_1 \dots \mathcal{L}_{k/2}(\mathcal{L}^{k/2} f) + O\left(\theta^{k/2}\right). \tag{3.3}$$

By Lemma 3.2 the first term here bounded by $||\mathcal{L}^{k/2}f||_\infty$ which by Theorem 3.3 is less that $C\zeta^{k/2}$. It follows that $\mathbb{E}(S_n)$ is bounded. Next,

$$\mathbb{E}(S_n^2) = \sum_k \mathbb{E}(f^2(\xi_k)) + 2\sum_{j<k} \mathbb{E}(f(\xi_j)f(\xi_k)).$$

Arguing as for $\mathbb{E}(S_n)$ we see that the first term equals $n\mu(f^2) + O(1)$ while for the second term we have

$$\mathbb{E}(f(\xi_j)f(\xi_k)) = (\mathcal{L}_0\mathcal{L}_1 \dots \mathcal{L}_{j-1}(f(\mathcal{L}_j \dots \mathcal{L}_{k-1}f)).$$

Using the analysis of (3.3) we see that $\mathcal{L}_j \dots \mathcal{L}_{k-1}f$ decays exponentially in $k-j$. Therefore $\mathbb{E}(S_n^2) = O(n)$. Now (a) follows from Chebyshev inequality.

EXERCISE 3.1. Show that

$$\frac{\mathbb{E}(S_n)}{n} \to \mu(f^2) + 2\sum_{l=1}^{\infty} \mu(f\mathcal{L}^l f).$$

EXERCISE 3.2. Show that in fact $\frac{S_n}{n} \to \mu$ almost surely. Moreover the same statement holds if the assumption that f is Lipshitz is replaced by the assumption that f is merely continuous.

To prove part (b) let $f_n = f(\xi_n)$. Using the general theory we represent

$$f_n = Y_n + \beta_n - \beta_{n+1}$$

where

$$\beta_n = \sum_{j=0}^{\infty} \mathbb{E}(f_{n+j}|\mathcal{F}_{n-1}) = \sum_{j=0}^{\infty} \mathcal{L}_{n-1} \dots \mathcal{L}_{n+j-1}(f)(\xi_{n-1}).$$

Then $S_n = (\sum_{k=0}^{n-1} Y_k) + \beta_0 - \beta_n$ and so it suffices to prove the Central Limit Theorem for $\sum_{k=0}^{n-1} Y_k$. Since Y_n is a martingale difference sequence we need to show that $\frac{1}{n}\sum_{k=0}^{n-1} \mathbb{E}(Y_k^2|\mathcal{F}_{k-1})$ converges to 0 in probability. Parts (a) and (b) of Lemma 3.2 tell us that $\beta_n = \tilde{\beta}_n + O(\zeta^n)$ where

$$\tilde{\beta}_n = \sum_{j=1}^{\infty} (\mathcal{L}^j f)(\xi_{n-1})$$

so it suffices to analyze $\sum_k \mathbb{E}(\tilde{Y}_k^2|\mathcal{F}_{k-1})$ where $\tilde{Y}_k = f_k - \tilde{\beta}_k + \tilde{\beta}_{k+1}$. Let $\bar{C}$ denote the subspace of C_θ consisting of functions such that $\mu(f) = 0$. Since $\mu(f) = \lim_{j\to\infty} \mathbb{E}(\mathcal{L}^j f)$ the space $\bar{C}$ is $\mathcal{L}$ invariant and by Theorem 3.3 $1-\mathcal{L}$ is invertible on $\bar{C}$. Note that $\tilde{Y}_n = g(\xi_n)$ where

$$g(\xi) = f(\xi) - h(\xi) + h(\sigma\xi_n) \text{ and } h = \mathcal{L}(1-\mathcal{L})^{-1} f.$$

By the foregoing discussion g and hence g^2 belong to C_θ. We have to show the convergence of $\frac{1}{n}\sum_{k=0}^{n-1}(\mathcal{L}_{k-1}g^2)(\xi_k)$. Due to Lemma 3.2(a) the numerator differs by $O(1)$ from $\frac{1}{n}\sum_{k=0}^{n-1}(\mathcal{L}g^2)(\xi_k)$. However the convergence of this expression follows from already proven part (a) of Theorem 3.4. □

Remark. Theorem 3.1 remains valid for random walks in Markov environment in $\mathbb{Z}^d$ for any d (see [**10**]). The argument is the same, the only distinction is that the proof of Theorem 3.4 is slightly more complicated.

Exercise 3.3. Use Exercise 3.1 to show that the limiting variance in Theorem 3.4 equals to

$$D(f) = \mu(f^2) + 2\sum_{l=1}^{\infty}\mu(f\mathcal{L}^l f).$$

Exercise 3.4. Prove that Theorem 3.1 remains valid if the transition probabilities depend not only on the state of site where the walker is located but also on the states of a finitely many sites around him.

Exercise 3.5. Let $\alpha : \mathcal{A}\times\mathcal{A} \to (0, \frac{1}{2})$ be a symmetric function. Assign to each site a pair x_n^t, E_n^t where x_n^t are Markov chains as above and E_n^t are conserved quantities evolved by the rule

$$E_n^{t+1} = \alpha(x_n^t, x_{n+1}^t)E_{n+1}^t + \alpha(x_n^t, x_{n-1}^t)E_{n-1}^t - (1-\alpha(x_n^t, x_{n+1}^t) - \alpha(x_n^t, x_{n-1}^t))E_n^t.$$

Suppose that $E_n^0 = \delta_{n0}$ Let A be a smooth observable. Show that

$$\sum_n A\left(\frac{n}{N}\right) E_n^{N^2} \to \mathbb{E}(A(\mathbf{N}))$$

where $\mathbf{N}$ is the Gaussian random variable with zero mean and variance D.

Hint. Take $M \gg N^2$. Consider the system where initially we have M particles of mass $\frac{1}{M}$ each concentrated at the origin. The particles move independently so that if a particle is at site n at time t then it moves to $n+1$ with probability $\alpha(x_n^t, x_{n+1}^t)$, moves to $n-1$with probability $\alpha(x_n^t, x_{n-1}^t)$ and stays at n with probability $(1 - \alpha(x_n^t, x_{n+1}^t) - \alpha(x_n^t, x_{n-1}^t))E_n^t$. Use Exercise 3.4 to obtain the Central Limit Theorem for position of each particle. Then take $M \to \infty$ and show that the mass at site n at time t converges to E_n^t.

4. Proof of Ruelle-Perron-Frobenius Theorem.

4.1. Cone geometry. The main difficulty in proving Theorem 3.3 is that we do not know $\mu(f)$. That is for each c vector $c1$ is invariant by

$\mathcal{L}$ and we do not know which of those vectors is approached by $\mathcal{L}^n f$. To overcome this difficulty it is convenient to consider the induced action of $\mathcal{L}$ on the projective space (that is we identify two functions if one is a multiple of the other). So we need a little bit of projective geometry which we now develop. Let $\mathcal{B}$ be a Banach space and $\mathcal{K}$ be a closed convex cone in $\mathcal{B}$ (that is if $f, g \in \mathcal{K}$ then $\alpha f + \beta g \in \mathcal{K}$). We assume that $\mathcal{K} \cap -\mathcal{K} = \{0\}$. We shall write $f \geq g$ if $f - g \in \mathcal{K}$. Set

$$\alpha(f,g) = \sup\{\lambda : \lambda f \leq g\}, \quad \beta(f,g) = \inf\{\mu : g \leq \mu f\}.$$

We define *Hilbert metric* by the equation

$$\Theta(f,g) = \ln \frac{\beta(f,g)}{\alpha(f,g)}.$$

Note that replacing f by cf changes λ to $\frac{\lambda}{c}$ and μ to $\frac{\mu}{c}$ so that $\Theta(cf,g) = \Theta(f,g)$. Likewise $\Theta(f,cg) = \Theta(f,g)$. To visualize Θ we consider the plane passing through f and g. $\mathcal{K}$ cuts an angle in this plane. By appropriate choice of coordinates $\mathcal{K}$ becomes the cone of positive vectors (that is, vectors with positive coordinates). Now if f and g have x coordinate 1 then $f = (1, A)$, $g = (1, B)$ where $A < B$ then

$$\Theta(f,g) = \ln B - \ln A = \int_A^B \frac{dy}{y}. \tag{3.4}$$

In general we have

$$\Theta(f,g) = \left| \ln \frac{f_2}{f_1} - \ln \frac{g_2}{g_1} \right|.$$

EXERCISE 3.6. Show that Θ satisfies the triangle inequality.

LEMMA 3.5. *Given Δ there exists $\gamma < 1$ such that the following holds. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be Banach spaces containing cones $\mathcal{K}_1$ and $\mathcal{K}_2$ respectively. Let $T : \mathcal{B}_1 \to \mathcal{B}_2$ be a linear map such $TK_1 \subset K_2$ and moreover* $\operatorname{diam}(T\mathcal{K}_1) \leq \Delta$. *Then*

$$\frac{\Theta_2(Tf, Tg)}{\Theta_1(f,g)} \leq \gamma.$$

PROOF. Restricting our attention to the planes generated by f, g and Tf, Tg respectively and choosing appropriate coordinates we can reduce the problem to the case where $\mathcal{B}_1 = \mathcal{B}_2 = \mathbb{R}^2$, $\mathcal{K}_1 = \mathcal{K}_2$ is the cone of positive vectors and T is orientation preserving. In this case T is given by a matrix with positive elements $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The condition

$\Theta(T(1,0), T(0,1)) \le \Delta$ reads

$$ad < bce^{\Delta}. \tag{3.5}$$

Note that if $(X,Y) = T(x,y)$ then

$$\frac{Y}{X} = \frac{a + b(y/x)}{c + d(y/x)}$$

so the induced action on the projective space takes form

$$z = \frac{a + by}{c + dy}.$$

Thus

$$\left|\frac{dz}{dy}\right| = \frac{ad - bc}{(c + dy)^2}$$

so that

$$\frac{y}{z}\left|\frac{dz}{dy}\right| = \frac{(ad - bc)y}{(a + by)(c + dy)} = \frac{(ad - bc)y}{(ad + bc)y + ac + bdy^2} \le \frac{ad - bc}{ad + bc}.$$

By (3.5) there exists $\gamma < 1$ such that $\frac{dz}{z} : \frac{dy}{y} \le 1$. In view of (3.4) the result follows. □

LEMMA 3.6. *Let* $||\cdot||$ *be a norm on* $\mathcal{B}$ *such that* $-f \le g \le f$ *implies that* $||f|| \ge ||g||$. *Let* f *and* g *be two vectors such that* $||f|| = ||g|| = 1$. *Then*

$$||f - g|| \le e^{\Theta(f,g)} - 1.$$

PROOF. $\Theta(f,g) = \ln \frac{\beta}{\alpha}$ where $\alpha f \le g$, $g \le \beta f$. Since $-g \le 0 \le \alpha f \le g$ we have $\alpha \le 1$. Since $-\beta g \le 0 \le g \le \beta f$ we have $\beta < 1$. Since

$$g - f \le (\beta - 1)f \le (\beta - \alpha)f \text{ and } g - f \ge (\alpha - 1)f \ge -(\beta - \alpha)f$$

we have

$$||g - f|| \le \beta - \alpha \le \frac{\beta - \alpha}{\alpha} = \frac{\beta}{\alpha} - 1 = e^{\Theta(f,g)} - 1.$$

□

To apply the above results to the problem at hand consider the cone

$$\mathcal{K}_A = \{f : f \ge 0 \text{ and for each } \xi_1, \xi_2 \quad f(\xi_1) \le f(\xi_2)e^{Ad(\xi_1,\xi_2)}\}.$$

LEMMA 3.7.

$$\Theta(f,g) = \ln \sup_{\xi_1 \ne \xi_2, \eta_1 \ne \eta_2} \frac{\left[e^{Ad(\xi_1,\xi_2)}g(\xi_2) - g(\xi_1)\right]}{\left[e^{Ad(\xi_1,\xi_2)}f(\xi_2) - f(\xi_1)\right]} \frac{\left[e^{Ad(\eta_1,\eta_2)}f(\eta_2) - f(\eta_1)\right]}{\left[e^{Ad(\eta_1,\eta_2)}g(\eta_2) - g(\eta_1)\right]}.$$

PROOF. By definition $g \geq \lambda f$ iff for each ξ $g(\xi) \geq \lambda f(\xi)$ and for each pair ξ_1, ξ_2

$$g(\xi_1) - \lambda f(\xi_1) \leq e^{Ad(\xi_1,\xi_2)}(g(\xi_1) - \lambda f(\xi_1)). \tag{3.6}$$

Thus $\alpha(f,g) = \min(\alpha_1, \alpha_2)$ where

$$\alpha_1 = \inf_{\xi} \frac{g(\xi)}{f(\xi)}, \quad \alpha_2 = \inf_{\xi_1 \neq \xi_2} \frac{\left[e^{Ad(\xi_1,\xi_2)} g(\xi_2) - g(\xi_1)\right]}{\left[e^{Ad(\xi_1,\xi_2)} f(\xi_2) - f(\xi_1)\right]}.$$

However if for some ξ_2 we have $g(\xi_2) = \lambda f(\xi_2)$ then (3.6) can not hold. Therefore $\alpha = \alpha_2$. Likewise

$$\beta(f,g) = \sup_{\xi_1 \neq \xi_2} \frac{\left[e^{Ad(\xi_1,\xi_2)} g(\xi_2) - g(\xi_1)\right]}{\left[e^{Ad(\xi_1,\xi_2)} f(\xi_2) - f(\xi_1)\right]}.$$

□

LEMMA 3.8. *If $B < A$ then $\mathcal{K}_B$ has finite diameter in $\mathcal{K}_A$.*

PROOF.

$$\frac{\left[e^{Ad(\xi_1,\xi_2)} g(\xi_2) - g(\xi_1)\right]}{\left[e^{Ad(\xi_1,\xi_2)} f(\xi_2) - f(\xi_1)\right]} \frac{\left[e^{Ad(\eta_1,\eta_2)} f(\eta_2) - f(\eta_1)\right]}{\left[e^{Ad(\eta_1,\eta_2)} g(\eta_2) - g(\eta_1)\right]}$$

$$\leq \frac{g(\xi_2)\left[e^{Ad(\xi_1,\xi_2)} - e^{-Bd(\xi_1,\xi_2)}\right]}{f(\xi_2)\left[e^{Ad(\xi_1,\xi_2)} - e^{Bd(\xi_1,\xi_2)}\right]} \frac{f(\eta_2)\left[e^{Ad(\eta_1,\eta_2)} - e^{-Bd(\eta_1,\eta_2)}\right]}{g(\eta_2)\left[e^{Ad(\eta_1,\eta_2)} - e^{Bd(\eta_1,\eta_2)}\right]}$$

$$\leq \frac{\left[e^{Ad(\xi_1,\xi_2)} - e^{-Bd(\xi_1,\xi_2)}\right]}{\left[e^{Ad(\xi_1,\xi_2)} - e^{Bd(\xi_1,\xi_2)}\right]} \frac{\left[e^{Ad(\eta_1,\eta_2)} - e^{-Bd(\eta_1,\eta_2)}\right]}{\left[e^{Ad(\eta_1,\eta_2)} - e^{Bd(\eta_1,\eta_2)}\right]} e^{2Bd(\xi_2,\eta_2)}.$$

□

4.2. Contraction of the transfer operator.

LEMMA 3.9. *There is a constant C such that $\mathcal{L} : \mathcal{K}_A \to \mathcal{K}_{\theta A + C}$.*

PROOF. Since $\mathcal{L}$ preserves positivity we just need to check the Lipshitz bound. Let ξ_1 and ξ_2 be two sequences and $\eta_j = \xi_j \omega$. Let $\phi(\eta) = \ln \mathbb{P}(\eta | \sigma\eta)$. Note that $\phi \in C_\theta$. We have

$$e^{\phi(\eta_1)} f(\eta_1) \leq e^{\phi(\eta_2)} f(\eta_2) e^{(L(\phi)+A)\theta d(\xi_1,\xi_2)}.$$

Summing over all admissible ω we obtain the result. □

PROOF OF THEOREM 3.3. Choose a large A so that $A\theta + C < A$ and consider $f \in \mathcal{K}_A$ with $||f|| = 1$. By Lemmas 3.9 and 3.8 $\Theta(\mathcal{L}^n f, 1) \leq \gamma^n$ and by Lemma 3.6

$$\left\| \frac{\mathcal{L}^n f}{||\mathcal{L}^n f||} - 1 \right\|_\infty \leq C\gamma^n.$$

Combining this with Lemma 3.2(b) we get

$$||\mathcal{L}^n f - ||\mathcal{L}^n f||1||_\infty \leq C\gamma^n$$

and invoking again Lemma 3.2(b) we see that

$$||\mathcal{L}^{n+1} f - ||\mathcal{L}^n f||1||_\infty \leq C\gamma^n$$

In particular

$$|||\mathcal{L}^{n+1} f||_\infty - ||\mathcal{L}^n f||_\infty| \leq C\gamma^n.$$

Let $\mu(f) = \lim_{n\to\infty} ||\mathcal{L}^n f||_\infty$. Then

$$||\mathcal{L}^n f - \mu(f)1||_\infty \leq \bar{C}\gamma^n. \tag{3.7}$$

To prove the Theorem we need the same estimate for $||\cdot||_\theta$-norm.

LEMMA 3.10. *(a) There exists* Γ *such that*

$$L(\mathcal{L}f) \leq \theta L(f) + \Gamma ||f||_\infty.$$

(b) There exists $\bar{\Gamma}$ *such that*

$$L(\mathcal{L}^m f) \leq \theta^m L(f) + \Gamma ||f||_\infty.$$

PROOF. Let ξ_1 and ξ_2 be two sequences and $\eta_j = \xi_j \omega_0$. Then

$$\left|e^{\phi(\eta_2)} f(\eta_2) - e^{\phi(\eta_2)} f(\eta_2)\right| \leq \left|e^{\phi(\eta_2)} - e^{\phi(\eta_1)}\right| f(\eta_2) + e^{\phi(\eta_1)} |f(\eta_2) - f(\eta_1)|$$
$$\leq C||f||_\infty + e^{\phi(\eta_1)}\theta L(f).$$

Summation over ω_0 completes the proof of (a). Iterating part (a) we get

$$L(\mathcal{L}^m f) \leq \theta^m L(f) + \left[\Gamma + \theta\Gamma + \dots \theta^{m-1}\Gamma\right] ||f||_\infty.$$

□

Applying Lemma 3.10(b) with $m = n/2$ we obtain

$$L(\mathcal{L}^n f) = L(\mathcal{L}^n f - \mu(f)1) \leq \theta^{n/2} L(\mathcal{L}^{n/2} f - \mu(f)1) + \bar{\Gamma} ||\mathcal{L}^{n/2} f - \mu(f)1||_\infty.$$

Lemma 3.10(b) tells us that the first term is $O(\theta^{n/2})$ while the second term is $O(\gamma^{n/2})$ in view of (3.7). Thus for $f \in \mathcal{K}_A$ with $||f|| = 1$ we have

$$||\mathcal{L}^n f - \mu(f)1||_\theta \leq \tilde{C}\zeta^n.$$

Consequently for any $f \in \mathcal{K}_A$ we have

$$||\mathcal{L}^n f - \mu(f)1||_\theta \leq \tilde{C}\zeta^n ||f||_\theta.$$

To prove the theorem it remains to note that any $f \in C_\theta$ can be represented as $f = f_1 - f_2$ where $f_j \in \mathcal{K}_A$ (for example, one can take $f_1 = 10||f||_\theta$). □

EXERCISE 3.7. Let $(L_n f)(\xi) = \sum_{\sigma\eta=\xi} e^{\phi_n(\eta)} f(\eta)$ be an infinite sequence of Markov transfer operators (that is we assume that $L_n 1 = 1$ for all n), $n \in \mathbb{Z}$. Assume that the functions ϕ_n have uniformly bounded norms $||\phi_n||_\theta \leq C$. Show that for each n there exists a measure μ_n such that $\mathcal{L}_{m,n} = \mathcal{L}_m \mathcal{L}_{m+1} \dots \mathcal{L}_{n-1}$ then for all $f \in C_\theta$

$$||\mathcal{L}_{m,n} - \mu(f)1|| \leq C\theta^{n-m}||f||_\theta.$$

Prove that if ξ_n is a realization of corresponding Markov process than

$$\frac{\sum_n [f(\xi_n) - \mu_n(f)]}{n} \to 0$$

almost surely and that all limit points of

$$\frac{\sum_n [f(\xi_n) - \mu_n(f)]}{\sqrt{n}}$$

are Gaussian.

CHAPTER 4

Random walk conditioned to stay positive.

Let X be a zero mean random walk. Let τ_x^+ (τ_x^-) be the first time the walker visits $(-\infty, x]$ (respectively $[x, +\infty)$.

LEMMA 4.1. *The limit* $-c := \lim_{m\to+\infty} \mathbb{E}_m(\tau_0^-)$ *exists. Moreover*

$$\mathbb{E}_m(\tau_0^-) + c = O(\theta^m)$$

for some $\theta < 1$.

PROOF. The proof is similar to the proof of Lemma 2.22. □

EXERCISE 4.1. Prove Lemma 4.1.

LEMMA 4.2.

$$\mathbb{P}_m(\tau_N^+ < \tau_0^-) = \frac{m - \mathbb{E}_m(\tau_0^-)}{N}\left(1 + O\left(\frac{m}{N}\right)\right).$$

PROOF. Since X_n is a martingale the optional stopping theorem gives

$$m = \mathbb{E}(\tau_0^- 1_{\tau_0^- < \tau_N^+}) + \mathbb{E}(\tau_N^+ 1_{\tau_N^+ < \tau_0^-}). \tag{4.1}$$

Since the first term here is $O(1)$ and the second term is

$$\mathbb{P}_m(\tau_N^+ < \tau_0^-)(N + O(1))$$

we see that

$$\mathbb{P}_m(\tau_N^+ < \tau_0^-) = \frac{m + O(1)}{N}.$$

Thus the first term in (4.1) equals to $\mathbb{E}_m(\tau_0^-) + O(\frac{m}{N})$ and the second term is $N\mathbb{P}_m(\tau_N^+ < \tau_0^-) + O(\frac{m}{N})$. Dividing (4.1) by N we obtain the result. □

Let X_n^N be our walk conditioned on $\tau_N^+ < \tau_0^-$. Thus if $X_n^N = m$ then $X_{n+1}^N = m + j$ with probability

$$\frac{p_j \mathbb{P}_{m+j}(\tau_N^+ < \tau_0^-)}{\sum_k p_k \mathbb{P}_{m+k}(\tau_N^+ < \tau_0^-)}.$$

(4.2) As $N \to \infty$, X_n^N converges to a Markov process Y_n

such that

$$\mathbb{P}(Y_{n+1} - Y_n = j | Y_n = m) = \frac{p_j(m + j + \mathbb{E}_{m+j} < \tau_0^-)}{\sum_k p_k(m + k + \mathbb{E}_{m+k} < \tau_0^-)}.$$

We will call Y_n a *random walk conditioned to stay positive.* Let $Y_{N,\delta}(t)$ denote the process such that $Y_{N,\delta}(t) = \frac{Y_{Nt}}{\sqrt{N}}$ with linear interpolation in between where where $Y_0 = \delta\sqrt{N}$. Let $\mathbf{Y}_\delta(t)$ denote $\mathbf{D}(\frac{\sigma^2}{y}, \sigma^2)$ process started at δ. Thus $\mathbf{Y}$ is a Bessel process of dimension 3.

THEOREM 4.3. *([4]) As $N \to \infty$ $Y_{N,0}(t) \Rightarrow \mathbf{Y}_\delta(t)$.*

Let $Y_{N,\delta,\alpha,\beta}(t)$ denote the process $Y_{N,\delta}(t)$ stopped at $\min(\tau^-_{\alpha\sqrt{N}}, \tau^+_{\beta\sqrt{N}})$. $\alpha\sqrt{N}$ or $\beta\sqrt{N}$. Let $\mathbf{Y}_{\delta,\alpha,\beta}$ denote $\mathbf{Y}(t)$ stopped when reaching either α or β.

LEMMA 4.4. *For each $\delta > 0$, $0 < \alpha < \delta < \beta$ $Y_{N,\delta,\alpha,\beta}(t) \Rightarrow \mathbf{Y}_{\delta,\alpha,\beta}(t)$.*

COROLLARY 4.5. *For each $\delta > 0$, $Y_{N,\delta}(t) \Rightarrow \mathbf{Y}_\delta(t)$.*

PROOF. Since Bess_3 does not visit 0 by Corollary 7.21 given $\varepsilon > 0$ there exists α, β such that $\mathbb{P}_\delta(\min(\tau_\alpha, \tau_\beta) < T) < \varepsilon$. Now by Lemma 4.4 for large N $\mathbb{P}(\min(\tau_{\alpha\sqrt{N}}, \tau_{\beta\sqrt{N}}) < T) < 2\varepsilon$. Consider a continuous functional $\Phi : C[0,T] \to \mathbb{R}$. By the foregoing discussion

$$|\mathbb{E}(\Phi(\mathbf{Y}_\delta(t)) - \mathbb{E}(\Phi(\mathbf{Y}_{\delta,\alpha,\beta}(t))| \leq \varepsilon||\Phi||, \quad |\mathbb{E}(\Phi(Y_{N,\delta}(t)) - \mathbb{E}(\Phi(Y_{N,\delta,\alpha,\beta}(t))| \leq 2\varepsilon||\Phi||.$$

Finally by Lemma 4.4 for large N

$$|\mathbb{E}(\Phi(Y_{N,\delta,\alpha,\beta}(t)) - \mathbb{E}(\Phi(\mathbf{Y}_{\delta,\alpha,\beta}(t))| \leq \varepsilon.$$

□

PROOF OF LEMMA 4.4. First we prove tightness. Observe that by Lemma 4.1.

$$\frac{\mathbb{P}(\Delta_n = j | Y_n)}{p_j} = 1 + \frac{j}{Y_n} + O\left(\frac{1}{Y_n^2}\right) = \exp\left(\frac{j}{Y_n} + O\left(\frac{1}{Y_n^2}\right)\right).$$

Let $\bar{\Delta}_n = \Delta_n - \mathbb{E}_Y(\Delta_n)$. Note that the correction term $\mathbb{E}_Y(\Delta_n) = O(\frac{1}{Y_n})$. Consequently if $\mathbb{Q}$ denote the law of the original walk with the same starting point and the same stopping rules as for Y_n then

$$\frac{d\mathbb{Q}}{d\mathbb{P}}(Y) \exp\left(-\sum_{n=1}^{TN} \frac{\bar{\Delta}_n}{Y_n} + O(1)\right).$$

Note that $M_k = \sum_{n=1}^k \frac{\bar{\Delta}_n}{Y_n}$ is a martingale with quadratic variation

$$\sum_{n=1}^k \left(\frac{\sigma^2}{Y_n^2} + O\left(\frac{1}{Y_n^3}\right)\right) = O\left(\frac{k}{N}\right).$$

Therefore Theorem 7.6 implies that given ε there exists C such that

$$\mathbb{P}\left(\frac{1}{C} \leq \frac{d\mathbb{Q}}{d\mathbb{P}} < C\right) \leq \varepsilon.$$

On the other hand since X_n converges to the Brownian Motion after diffusive rescaling there is a compact set $K \subset C[0,T]$ such that $\mathbb{Q}(Y_{N,\delta,\alpha,\beta} \notin K) \leq \frac{\varepsilon}{C}$. Then

$$\mathbb{P}(Y_{N,\delta,\alpha,\beta} \notin K) \leq \frac{\varepsilon}{C}C + \mathbb{P}(\frac{d\mathbb{P}}{d\mathbb{Q}} > C) \leq 2\varepsilon.$$

Since ε is arbitrary, tightness follows.

Next take $f \in \mathcal{D}$. Note that

$$\mathbb{E}(Y_{n+1} - Y_n|Y_n) = \frac{\sum_j p_j(Y_n + c + j)j}{Y_n + c} + O\left(\theta^{\sqrt{N}}\right) = \frac{\sigma^2}{Y_n + c} + O(\theta^{\sqrt{N}}),$$

$$\mathbb{E}(Y_{n+1} - Y_n|Y_n) = \sigma^2 + O\left(\frac{1}{Y_n}\right).$$

Accordingly

$$\mathbb{E}\left(f\left(\frac{Y_{n+1}}{\sqrt{N}}\right) - f\left(\frac{Y_n}{\sqrt{N}}\right)|\mathcal{F}_n\right)$$

$$= f'\left(\frac{Y_n}{\sqrt{N}}\right)\frac{\sigma^2}{(Y_n + c)\sqrt{N}} + \frac{1}{2}f''\left(\frac{Y_n}{\sqrt{N}}\right)\frac{\sigma^2}{N} + O\left(\frac{1}{N^{3/2}}\right).$$

Summation over n from 0 till $\min(\tau^-_{\alpha\sqrt{N}}, \tau^+_{\beta\sqrt{N}}, NT)$ shows that if $\mathbb{Y}$ is a limiting point of $Y_{N,\delta,\alpha,\beta}$ then

$$\mathbb{E}\left(f(\mathbb{Y}(T)) - f(\mathbb{Y}(0)) - \int_0^{\min(T,\tau_\alpha,\tau_\beta)}\left[\frac{\sigma^2}{\mathbb{Y}(t)}f'(\mathbb{Y}(t)) + \frac{1}{2}f''(\mathbb{Y}(t))\right]dt\right) = 0.$$

Similarly summing between $\min(\tau^-_{\alpha\sqrt{N}}, \tau^+_{\beta\sqrt{N}}, NT_1)$ and $\min(\tau^-_{\alpha\sqrt{N}}, \tau^+_{\beta\sqrt{N}}, NT_2)$ shows that

$$\mathbb{E}\left(f(\mathbb{Y}(T_2)) - f(\mathbb{Y}(T_1)) - \int_{T_1}^{\min(T_2,\tau_\alpha,\tau_\beta)}\left[\frac{\sigma^2}{\mathbb{Y}(t)}f'(\mathbb{Y}(t)) + \frac{1}{2}f''(\mathbb{Y}(t))\right]dt|\mathcal{F}_{T_1}\right) = 0.$$

Thus $M^f(t)$ is martingale for each $f \in \mathcal{D}$ so by Theorem 7.13 $\mathbb{Y} = \mathbf{Y}$. □

LEMMA 4.6. *There exists $\rho < 1$ such that for each r*

$$\mathbb{P}(\tau^+_m < m^2 r) \leq \rho^r.$$

Lemma 4.6 implies Theorem 4.3. Indeed by definition the $\mathbf{Y}_0$ is the limit of $\mathbf{Y}_\delta$ as $\delta \to 0$. By Corollary 4.5 $\frac{Y_{\tau^+_{\delta\sqrt{N}}+tN}}{\sqrt{N}} \Rightarrow \mathbf{Y}_\delta$ while by Lemma 4.6 $\tau_{\delta\sqrt{N}}/N \to 0$ as $\delta \to 0$.

Proof of Lemma 4.6. Define $\sigma_0 = 0, m_0 = 1$ and let σ_{j+1} be the first time after σ_j such that $Y_k \geq 2^{m_j+1}$ or $Y_k \leq 2^{m_j+1}$. In the first case let $m_{j+1} = m_j + 1$, in the second case let $m_{j+1} = m_j - 1$. Recall (see (7.15)) that $1/\mathbf{Y}$ is a martingale. Accordingly if j is large then

$$\mathbb{P}(m_{j+1} - m_j = 1) \approx \frac{2}{3}.$$

Lemma 4.7. *(a) Let m_j be a process such that $m_{j+1} - m_j = \pm 1$ and there exists $M > 0$ such that*

$$\mathbb{P}(m_{j+1} - m_j = 1 | \mathcal{F}_j) \geq p \text{ if } m_j \geq M.$$

Let Z_j be a simple random walk such that $\mathbb{P}(Z_{j+1} - Z_j = 1) = p$ started from m. Then if $m_0 \geq M$ then there is a coupling between m and Z such that

$$m_j \geq Z_j \text{ on the set } \min_{k \leq j} Z_j \geq M.$$

(b) Consequently if $p > \frac{1}{2}$ then

$$\mathbb{P}(m_j > M \text{ for all } j > 0) \geq \mathbb{P}(Z_j > M \text{ for all } j > 0) = 1 - \frac{q}{p}.$$

Proof. The coupling is constructed as follows. Let U_j be iid random variables having uniform distribution on $[0, 1]$. We let $Z_{j+1} - Z_j = 1$ iff $U_j \leq p$ and $m_{j+1} - m_j = 1$ iff $U_j \geq \mathbb{P}(m_{j+1} - m_j = 1 | \mathcal{F}_j)$. □

Lemma 4.7 implies that each time $\ln Y_{\sigma_j}$ visits a site there is a positive probability that it will never return there. Also by Lemma 4.4 the distribution of $\frac{\sigma_{j+1} - \sigma_j}{4^{m_j}}$ is asymptotically independent of m_j so

$$\mathbb{P}(\sigma_{j+1} \leq \sigma_j + 2^{m_j}, m_{j+1} = m_j + 1 \text{ and } \ln Y_{\sigma_j} \text{ never returns to } 4^{m_j})$$

is uniformly bounded from below. In other words there exists $\rho < 1$ such that

$$\mathbb{P}(Y_{n+l} > 2^k \text{ for } l > 4^k | Y_n \in [2^{k-1}, 2^k]) \geq \rho.$$

Hence

$$\mathbb{P}(Y \text{ spends time } s4^k \text{ inside } [2^{k-1}, 2^k]) \leq (1 - \rho)^s.$$

Next if $\tau_m^+ > m^2 r$ then there exists k such that $2^k < m$ and Y spends at least time $\frac{r}{C}(N - k + 1)4^k$ inside $[2^{k-1}, 2^k]$. The probability of such an event is $O(\theta^P r(N - k + 1)/C)$. Summation over k proves the lemma. □

Exercise 4.2. Let $X \sim \mathbf{D}(a, b)$ on $[\alpha, \beta]$. Let Y denote the law of X conditioned on hitting β before α. Show that Y is a diffusion process and compute its drift and diffusion coefficient.

EXERCISE 4.3. Let X_N denote SqBess_δ, $\delta < 2$ conditioned on reaching N before 0. Show that as $N \to \infty$, X_N converges to $\mathrm{SqBess}_{4-\delta}$.

EXERCISE 4.4. Consider a Markov chain where the particle moves from n to $n+1$ with probability $\frac{1}{2}+\frac{c}{n}$ and to $n-1$ with probability $\frac{1}{2}-\frac{c}{n}$. Suppose that $X_0 = N$ and the particle is stopped then it reaches either αN or βN. Let $Y^N(t) = \frac{X_{N^2 t}}{N}$. Show that as $N \to \infty$ $Y^N \Rightarrow \mathbf{D}(\frac{2c}{y}, 1)$. Thus the limiting process in Bess_{4c+1}.

CHAPTER 5

Excited random walk.

1. Results.

Our exposition follows [**22, 23**].

We consider the following model. Fix $M\in\mathbb{N}$. For each site $m\in\mathbb{Z}$ pick M numbers $p_{m1}, p_{m2}\dots p_{mM}$ between 0 and 1. Let X_n be a walk on $\mathbb{Z}$ such that $X_{n+1}-X_n=\pm 1$ and if X_n visits m for jth time then

$$\mathbb{P}(X_{n+1}-X_n=1)=\begin{cases} p_{mj} & j\leq M \\ \frac{1}{2} & j>M. \end{cases}.$$

Denote $q_{mj}=1-p_{mj}$.

We assume that $(p_{m1}, p_{m2}\dots p_{mM})$ are iid random vectors even though the case where $p_{mj}\equiv p$ is already non-trivial and presents most of the difficulties of the general case. Let $\delta_m=\sum_{j=1}^M (p_j-q_j)$ be the total drift stored at site m and let $\boldsymbol{\delta}=\mathbb{E}(\delta_m)$. By replacing X by $-X$ if necessary we may assume that $\boldsymbol{\delta}\geq 0$.

THEOREM 5.1. *([**22**]) If $\boldsymbol{\delta}\leq 0$ then X_n is recurrent in the sense that it visits every site infinitely many times. If $\boldsymbol{\delta}>1$ then $X_n\to+\infty$ almost surely.*

In case $\boldsymbol{\delta}>1$, X_n looks increasing after appropriate rescaling. Therefore one can study τ_N the time it takes to reach site N.

THEOREM 5.2. *([**1, 23**])*

(a) If $1<\boldsymbol{\delta}<2$ then $\frac{\tau_N}{N^{2/\boldsymbol{\delta}}}$ converges to a stable law.

(b) If $\boldsymbol{\delta}=2$ then there exists a constant u such that $\frac{\tau_N-uN\ln N}{N}$ converges to a stable law.

(c) If $2<\boldsymbol{\delta}<4$ then there is a constant u such that $\frac{\tau_N-Nu}{N^{2/\boldsymbol{\delta}}}$ converges to a stable law.

(d) If $\boldsymbol{\delta}=4$ then there are constants u and D such that

$$\frac{\tau_N-Nu}{\sqrt{N\ln N}}\Rightarrow\mathcal{N}(0,D).$$

(e) If $\boldsymbol{\delta} > 4$ then there are constants u and D such that

$$\frac{\tau_N - Nu}{\sqrt{N}} \Rightarrow \mathcal{N}(0, D).$$

To describe the limiting distribution in the recurrent case we need the following fact.

THEOREM 5.3. *(*[**5, 6, 29**]*) Let $|\alpha|, |\beta| < 1$. Then almost surely there exists unique solution to the equation*

$$Y(t) = W(t) + \alpha S(t) + \beta I(t) \tag{5.1}$$

where $W(t)$ is a standard Brownian Motion, $I(t) = \inf_{0 \le s \le t} Y(s)$, $S(t) = \sup_{0 \le s \le t} Y(s)$.

We shall call Y (α, β)*–perturbed Brownian Motion* and write $Y \sim \mathrm{BM}(\alpha, \beta)$.

THEOREM 5.4. *If $0 \le \boldsymbol{\delta} < 1$ then $\frac{X_{Nt}}{\sqrt{N}} \Rightarrow Y(t)$ where Y is $(\boldsymbol{\delta}, -\boldsymbol{\delta})$-perturbed Brownian Motion.*

2. Ray-Knight Theorems.

The proofs of Theorems 5.1, 5.2 and 5.4 rely on extensions of Ray-Knight Theorems from Section 2.4 to excited random walks.

Recall (see Section 2.4) that the trajectory of X_n can be uniquely determined if for each m, j we decide where the walker goes when he visits site m for jth time. Let $\tau_{0,N}$ be the time of the N-th visit to 0. Then the number of forward crossings of edge $[0, 1]$ equals to $\tilde{\rho}_0 = \rho' + \zeta$ where $\rho' \sim \mathrm{Bin}(N - M, \frac{1}{2})$ and ζ is the number of right steps during the first M visits to 0. Likewise let $\tilde{\rho}_m$ be the number of forward crossings of $[m, m+1]$ before $\tau_{0,N}$. Then the number of backward crossings is also $\tilde{\rho}_m$ and so

$$\tilde{\rho}_{m+1} = \sum_{j=1}^{\tilde{\rho}_m - M} \xi_{mj} + \tilde{\zeta}_m \tag{5.2}$$

where ξ_{jm} are iid having $\mathrm{Geom}(\frac{1}{2})$ distribution and on the event $\tilde{\rho}_m > M$ $\tilde{\zeta}_m$ is independent of ξs and

$$\mathbb{E}(\zeta_m | \mathcal{F}_m) = M + \boldsymbol{\delta}. \tag{5.3}$$

Indeed if $\rho^{\pm}_{m,k}$ is the number of right (left) steps during first k visits to site m then

$$\rho^{+}_{m,k} - \rho^{-}_{m,k} - \sum_{j=1}^{k} (p_{m,j} - q_{m,j})$$

is a martingale and so if k_M is the first time $\rho^+_{m,k} = M$

$$\mathbb{E}_\omega(\zeta_m) - M = \boldsymbol{\delta}_m$$

which implies (5.3).

Similarly if τ_N is the first time when the walker reaches site N then denoting $\hat{\rho}_m$ the number of backward crossings of $[m-1, m]$ we have $\hat{\rho}_N = 0$ and

$$\hat{\rho}_m = \sum_{j=1}^{\hat{\rho}_{m+1}-M+1} \zeta_{mj} + \hat{\zeta}_m \tag{5.4}$$

where -1 in $\hat{\rho}_{m+1} + M - 1$ appears because the number of forward crossing of each each is by one larger than the number of backward crossings and

$$\mathbb{E}(\zeta_m|\mathcal{F}_m) = M - \boldsymbol{\delta}.$$

Note that in contrast to simple symmetric random walk there is no guarantee that τ_N and $\tau_{0,N}$ are finite. However the following statement can be proven by straightforward induction

Proposition 5.5. *(a) Let $\tilde{\rho}_m$ be defined by* (5.2)*. Let $\bar{\tilde{\rho}}_m$ be the number of forward crossings of the edge $[m, m+1]$ by the time* $\min(\tau_{0,N}, \infty)$*. Then $\bar{\tilde{\rho}}_m \leq \tilde{\rho}_m$ and we have the equality if $\tau_{0,N}$ is finite almost surely.*

(b) Let $\hat{\rho}_m$ be defined by (5.4)*. Let $\bar{\hat{\rho}}_m$ be the number of forward crossings of the edge $[m, m+1]$ by the time* $\min(\tau_{0,N}, \infty)$*. Then $\bar{\hat{\rho}}_m \leq \hat{\rho}_m$ and we have the equality if $\tau_{0,N}$ is finite almost surely.*

In case of $\hat{\rho}$ it is convenient to change the direction of time and let

$$\rho_m = \sum_{j=1}^{\rho_m - M+1} + \zeta_m$$

so that $\hat{\rho}_m = \rho_{N-m}$ and

$$\mathbb{E}(\zeta_m|\mathcal{F}_m) = M - \boldsymbol{\delta} \text{ if } \rho_m \geq M.$$

Theorem 5.6. *(a) Suppose that $\tilde{\rho}_{m_0} = n$. Let $\widetilde{\mathfrak{R}}^n_{\alpha,\beta}(t) = \frac{\tilde{\rho}_{m_0+n^2t}}{n}$ stopped when it reaches either αn or βn. Then $\widetilde{\mathfrak{R}}^n(t) \Rightarrow \tilde{Y}(t)$ where $\tilde{Y} =$ $\mathbf{D}(\boldsymbol{\delta}, 2y)$ stopped when it reaches either α or β. Moreover if $\psi(\tilde{Y}(t))$ is a martingale then*

$$\mathbb{E}\left(\frac{\tilde{\rho}_{\min(\tau^-_{\alpha n}, \tau^+_{\beta n})}}{n}\right) = \phi(\frac{\tilde{\rho}_0}{n}) + O\left(n^- \sigma\right).$$

(b) Suppose that $\rho_{m_0} = n$. *Let* $\mathfrak{R}^n_{\alpha,\beta}(t) = \frac{\rho_{m_0+n^2 t}}{n}$ *stopped when it reaches either* αn *or* βn. *Then* $\mathfrak{R}^n(t) \Rightarrow Y(t)$ *where* $Y = \mathbf{D}(1-\boldsymbol{\delta}, 2y)$ *stopped when it reaches either* α *or* β.

Thus $\tilde{Y} \sim \mathrm{SqBess}_{2\boldsymbol{\delta}}$ and $Y \sim \mathrm{SqBess}_{2-2\boldsymbol{\delta}}$. We now use the information about square Bessel processes to obtain analogous results about the branching processes. Recall that SqBess_d tends to ∞ almost surely if $d > 2$ and it takes arbitrary small values otherwise.

THEOREM 5.7. $\tilde{\rho}_m$ *dies with probability 1 if* $\boldsymbol{\delta} \le 1$ *and it survives with positive probability if* $\boldsymbol{\delta} > 1$. *Moreover the probability of survival tends to* 1 *as* $\tilde{\rho}_0 \to \infty$.

Next recall that 0 is either regular or entrance boundary for SqBess_d if $d > 0$.

THEOREM 5.8. *If* $\boldsymbol{\delta} < 1$ *then* $\frac{\rho_{tN}}{N} \Rightarrow Y(t)$ *where* $Y \sim \mathbf{D}(1-\boldsymbol{\delta}, 2y)$ *started from* 0.

Recall Lemmas 7.23 and 7.24. Let $\tau_0 = 0$ and τ_j be the first time after τ_{j-1} when $\rho_\tau < M$. Let $\tilde{\tau}_0 = 0$ and $\tilde{\tau}_j$ be the first time after $\tilde{\tau}_{j-1}$ when $\rho_{\tilde{\tau}} < 0$. Let $\sigma_j = \tau_j - \tau_{j-1}$, $\tilde{\sigma}_j = \tilde{\tau}_j - \tilde{\tau}_{j-1}$, $A_j = \sum_{m=\tau_{j-1}+1}^{\tau_j} \rho_m$.

LEMMA 5.9. *(a) If* $\boldsymbol{\delta} > 0$ *then there are constants* C_1 *and* C_2 *such that*

$$t^{\boldsymbol{\delta}} \mathbb{P}(\sigma > t) \to C_1, \quad t^{\boldsymbol{\delta}/2} \mathbb{P}(A > t) \to C_2.$$

(b) There are constants C_3 *such that for each* $k < M$ *we have*

$$t^{\boldsymbol{\delta}} \mathbb{P}(\tilde{\sigma}_j > t | \rho_{\tilde{\sigma}_{j-1}} = k) > C_3.$$

3. Recurrence and transience.

PROOF OF THEOREM 5.1. (a) Let $0 \le \boldsymbol{\delta} \le 1$. By Theorem 5.7 the sequence defined by (5.2) dies out almost surely. Therefore by Proposition 5.5 X_n visits only finitely many sites in $\mathbb{Z}_+$ before $\min(\tau_{0,N}, \infty)$. Similarly Proposition 5.5 implies that X_n visits only finitely many sites in $\mathbb{Z}_-$ before $\min(\tau_{0,N}, \infty)$. (Note that changing the direction of the space axis replaces $\boldsymbol{\delta}$ by $-\boldsymbol{\delta}$ so to have finite number of visits to $\mathbb{Z}_+$ we need $2\boldsymbol{\delta} \le 2$ while to have finite number of visits to $\mathbb{Z}_+$ we need $-2\boldsymbol{\delta} \le 2$). Hence $\tau_{0,N}$ is finite almost surely, that is X visits 0 infinitely many times. Next fix $m \in \mathbb{Z}$. By Theorem 5.6 $\frac{\tilde{\rho}_m(\rho_{0,N})}{N} \Rightarrow 1$ and so for each L

$$\mathbb{P}(X \text{ visits } m \text{ less than } L \text{ times before } \tau_{0,N}) \to 0, \text{ as } L \to \infty.$$

Thus X visits m infinitely often almost surely.

(b) Let $\boldsymbol{\delta} > 1$. By Proposition 5.5 conditioned on the event $\tau_{-N} < \infty$ we have that $\frac{\hat{\rho}_0}{N}$ converges to a distribution which has no atoms at 0. Accordingly for each L

$$\mathbb{P}(\hat{\rho}_0 < L | \tau_{-N} < \infty) \to 0 \text{ as } L \to \infty.$$

On the other hand by Theorem 5.7

$$\mathbb{P}(\tilde{\rho}_m \text{ dies out} | \tilde{\rho}_m > L) \to 0 \text{ as } L \to \infty.$$

In other words

$$\mathbb{P}(X \text{ visits finitely many sites before } \tau_{-N}) \to 0 \text{ as } N \to \infty.$$

Hence X_n does not visit $-N$ if N is large enough. However if X visited any other site m infinitely many times then it would visit $-N$ almost surely since there are constants M and ρ such that

$$\mathbb{P}(X_{n+k} = -N \text{ for some } k \leq N | X_n = m) \geq \rho.$$

Therefore X visits every site only finitely many times and hence $X_n \to +\infty$. □

Proof of Theorem 5.7. Consider three cases.

(a) $\boldsymbol{\delta} > 1$. Let $\tilde{\rho}_0 = 2^{m_0}$ (to prove the result we can assume without the loss of generality that m_0 is an integer, however the proof below will not rely on this assumption). Let $T_0 = 0$ and let T_{j+1} be the first time after T_j such that $\tilde{\rho}_T \geq 2^{m_j+1}$ or $\tilde{\rho}_T \leq 2^{m_j-1}$. In the first case we set $m_{j+1} = m_j + 1$, in the second case we set $m_{j+1} = m_j - 1$. We wish to compare $\ln \tilde{\rho}_{T_j}$ to a random walk. However we need to keep in mind that ρ_{T_j} may be far from 2^{m_j} (even though this happen with a small probability). To take care of this issue we say that $\tilde{\rho}$ has a long jump at time n if $|\tilde{\rho}_n - \tilde{\rho}_{n-1}| > \tilde{\rho}_{n-1}^{3/4}$.

Exercise 5.1. Show that

$$\mathbb{P}(|\tilde{\rho}_n - \tilde{\rho}_{n-1}| > \tilde{\rho}_{n-1}^{3/4} | \mathcal{F}_n) \leq e^{-c\sqrt{\tilde{\rho}_n}}.$$

Let $\mathcal{G}_j$ be the σ-algebra of events which has happened before time T_j. Since $\boldsymbol{\delta} > 1$ there exist $p > \frac{1}{2}, M$ such that on $\rho_{T_j} \geq 2^M$ we have

$$\mathbb{P}(m_{j+1} = m_j + 1 | \mathcal{G}_j) \geq p,$$

$$\mathbb{P}(\tilde{\rho} \text{ has long jump on } [T_j + 1, T_{j+1}]) \leq \exp\left(-c2^{m_j-1}\right).$$

Now let z_j be the random walk gets killed at each step with probability $\exp\left(-c2^{m_j-1}\right)$ and on the event of survival z goes to the right with probability p and to the left with probability $1-p$. We claim that

$$\mathbb{P}(z \text{ survives and } z_j \to \infty) \to 1 \text{ as } z_0 \to \infty. \tag{5.5}$$

Indeed if we disregard killing then given ε there exist constants a, A such that

$$\mathbb{P}(z_j \geq z_0 + aj - A \text{ for all } j) \geq 1 - \varepsilon$$

(in fact one can take $a = \frac{2p-1}{2}$ independently of ε). On the other hand

$$\mathbb{P}(z_j \text{ gets killed}|z_j \geq z_0 + aj - A \text{ for all } j) \leq \sum_{j=0}^{\infty} e^{-c2^{z_0+aj-A}} \to 0 \text{ as } z_0 \to \infty.$$

This proves (5.5). Next arguing as in Lemma 4.7 we can construct a coupling between m_j and z_j such that

$$\{\tilde{\rho} \text{ has a long jump}\} \subset \{z_j \text{ is killed}\}$$

and if z_j survives and $z_j \geq M$ then $m_j \geq z_j$. Now the result follows from (5.5).

(b) $\boldsymbol{\delta} < 1$. In this case we let $T_0 = 0$ and let T_{j+1} be the first time after T_j such that $\tilde{\rho}_T \geq 2\tilde{\rho}_{T_j}$ or $\tilde{\rho}_T \leq \frac{\tilde{\rho}_{T_j}}{2}$. We compare $\ln \tilde{\rho}_{T_j}$ with the sequence $\tilde{z}_j$ defined as follows. Set $z_0 = \ln \tilde{\rho}_0$. If $\tilde{\rho}_{T_{j+1}} \leq \frac{\tilde{\rho}_{T_j}}{2}$ we set $\tilde{z}_{j+1} = \tilde{z}_j - 1$. If $\tilde{\rho}_{T_{j+1}} \geq 2\tilde{\rho}_{T_j}$ we let $\tilde{z}_{j+1} = \tilde{z}_j + 1 + \varepsilon$ if $\tilde{\rho}$ does not have a long jump at time T_{j+1}. If $\tilde{\rho}$ has a long jump then we set $\tilde{z}_{j+1} = \tilde{z}_j + k$ where k is the smallest number such that $\tilde{\rho}_{T_{j+1}} \leq 2^k \tilde{\rho}_{T_j}$.

By induction $\tilde{\rho}_j \geq 2^{\tilde{z}_j}$ and there are exist constants $L, p \geq \frac{1}{2}$ and γ such that on $\rho_{T_j} \geq L$ we have

$$\mathbb{P}(\tilde{z}_{j+1} - \tilde{z}_j = -1|\mathcal{G}_j) \geq p, \quad \mathbb{P}(\tilde{z}_{j+1} - \tilde{z}_j = k|\mathcal{G}_j) \leq e^{\gamma 2^k L} \text{ for } k \geq 2.$$

Thus we can compare $\ln \tilde{\rho}_{T_j}$ with a random walk z_j such that

$$\mathbb{P}(z_{j+1} - z_j = -1|\mathcal{G}_j) = p, \quad \mathbb{P}(z_{j+1} - z_j = k|\mathcal{G}_j) = e^{\gamma 2^k L} \text{ for } k \geq 2.$$

That is we can construct a coupling between $\tilde{\rho}$ and z such that $\rho_{T_j} \leq z_j$ on the event $\max_{k \leq j} z_k \geq L$. If L is large and ε is small then z is recurrent implying that $\tilde{\rho}$ falls below L infinitely many times. Since every time $\tilde{\rho}$ falls below L the probability of its dying at the next step is uniformly bounded from below we conclude that $\tilde{\rho}$ dies almost surely.

(c) $\boldsymbol{\delta} = 1$. Let τ_j be the consecutive times when either $\tilde{\rho}_\tau \leq \frac{\tilde{\rho}_{\tau_j}}{2}$ or $\tilde{\rho}$ has a long jump. Define m_j and z_j as in part (a).

EXERCISE 5.2. Prove that

$$\mathbb{P}(z_j \text{ is killed at site } m_0 + k \text{ before visiting } m_0 - 1) \leq e^{-c2^{k+m_0}}.$$

Using Exercise 5.2 we can compare $\ln \tilde{\rho}_{\tau_j}$ with a random walk Z which moves k units to the left with probability $e^{-c2^{k+m_0}}$ and moves 1 unit to the left with probability

$$1 - \sum_{k=0}^{\infty} e^{-c2^{k+m_0}}.$$

Since Z is recurrent we can finish the proof as in part (b). □

4. Limit Theorems in transient case.

Proof of Theorem 5.2. To fix our ideas we give the proof for in case $2 < s < 4$. Other cases are similar.

Let L^- be total number of visits to $\mathbb{Z}_-$. Since $X_n \to +\infty$ we have $\frac{L^-}{a_n} \to 0$ for any sequence a_n tending to ∞. Therefore we may restrict the attention to visit to $\mathbb{Z}_+$. Since the number of forward crossing of each edge is one more than the number of backward crossings we have

$$\tau_N = L^- + 2\sum_{m=1}^{N} \hat{\rho}_m - N$$

so it suffices to obtain the limiting distribution for $\sum_{m=0}^{N-1} \hat{\rho}_m$. Since $\sum_{m=0}^{N-1} \hat{\rho}_m = \sum_{m=1}^{N} \rho_m$ we can restrict out attention to the later sum. Let

$$R_k = \sum_{j=1}^{k} A_j, \quad Q_k = \sum_{j=1}^{k} \sigma_j.$$

By Theorem 7.10(c) Q_k satisfies the Central Limit Theorem. Accordingly for any sequence $\Gamma_k \to \infty$ we have

$$\mathbb{P}(|Q_k - k\mathbb{E}\sigma| < \Gamma_k\sqrt{k}) \to 1. \tag{5.6}$$

Let k_N be the first time when $Q_k > N$. Then $A_{k_N-1} < \tau_N \leq A_{k_N}$. By (5.6)

$$\mathbb{P}\left(\left|k_N - \frac{N}{\mathbb{E}(\sigma)}\right| < \Gamma_N\sqrt{N}\right) \to 0 \text{ if } \Gamma_N \to \infty.$$

Hence with probability close to 1

$$R_{N_-} \leq \tau_N \leq R_{N_+} \text{ where } N_\pm = \frac{N}{\mathbb{E}(\sigma)} \pm \Gamma_N\sqrt{N}.$$

We claim that if $\frac{\Gamma_N\sqrt{N}}{N^{2/\boldsymbol{\delta}}} \to 0$ then

$$\max_{N-\leq j\leq N_+} \frac{R_j - R_{N_-}}{N^{2/\boldsymbol{\delta}}} \Rightarrow 0. \tag{5.7}$$

Indeed (5.7) follows from the fact that if $D_M \to \infty$ then

$$\max_{j\leq M} \frac{R_j}{MD_M} \Rightarrow 0. \tag{5.8}$$

But (5.8) is an easy consequence of the Law of Large Numbers.

Accordingly the limiting distribution of

$$\frac{\sum_{m=0}^{N-1} \rho_m - \frac{\mathbb{E}(A)}{\mathbb{E}(\sigma)} N}{N^{2/\boldsymbol{\delta}}}$$

is the same as the limiting distribution of

$$\frac{R_{N_-} - \mathbb{E}(A) N_-}{N^{2/\boldsymbol{\delta}}}.$$

Now the result follows from (7.8). □

EXERCISE 5.3. Work out the proof in case $\boldsymbol{\delta} \geq 4$ or $1 < \boldsymbol{\delta} \leq 2$.

PROOF OF LEMMA 5.9. The proof is similar to the proofs of Lemmas 7.23 and 7.24. See [**23**] for details. □

5. Limit Theorem in recurrent case.

Let $S_n = \max_{k \leq n} X_n$, $I_n = \min_{k \leq n} X_n$. First of all we show that by time n the walker consumes almost all the drift between I_n and S_n.

LEMMA 5.10. *Given* $\gamma_1 > \boldsymbol{\delta}$ *there exists* $\gamma_2 > 0$ *such that for each* $0 \leq n \leq n^*$ *we have*

$$\mathbb{P}(\text{Card}(n^* - n \leq m \leq n^* : L_m(\tau_{n^*}) < M) \geq n^{\gamma_1}) \leq \theta^{n^{\gamma_2}} \tag{5.9}$$

and

$$\mathbb{P}(\text{Card}(-n^* \leq m \leq -(n^* - n) : L_m(\tau_{n^*}) < M) \geq n^{\gamma_1}) \leq \theta^{n^{\gamma_2}} \tag{5.10}$$

PROOF. We will prove (5.9), the proof of (5.10) is the same. We consider the case where $0 < \boldsymbol{\delta} \leq 1$. By translation invariance we may assume that $n = n^*$. Let $0 = \tilde{\tau}_0 < \tilde{\tau}_1 < \cdots < \tilde{\tau}_k \ldots$ be consecutive times when $\rho_\tau < M$. Let $\tilde{\sigma}_j = \tilde{\tau}_j - \tilde{\tau}_{j-1}$. Let j^* be the first time when $\tilde{\sigma}_{j^*} > n$. By Lemma 5.9 $\mathbb{P}(\tilde{\sigma} > n) \geq \frac{C_3}{n^{\boldsymbol{\delta}}}$. Hence

$$\mathbb{P}\left(j^* > n^{\gamma_1}\right) \leq \left(1 - \frac{C_3}{n^{\boldsymbol{\delta}}}\right)^{n^{\gamma_1}}.$$

Since $\text{Card}(0 \leq m \leq n : L_m(\tau_n) < M) \leq j^*$ the result follows. □

EXERCISE 5.4. Prove (5.9) in case $1 \leq \boldsymbol{\delta} \leq 0$.

Hint. Construct a branching process with migration ρ_n^* such that $\rho_n^* \geq \rho_n$ and $\mathbb{E}(\rho_{n+1}^* - \rho_n^*) = \boldsymbol{\delta}^*$ for some $0 < \boldsymbol{\delta}^* < 1$.

Next we show that $\sqrt{N}$ is a correct scaling in Theorem 5.4.

LEMMA 5.11. *For each $\gamma_3 < 2$ there exists C such that*

$$\mathbb{P}\left(\tau_N \leq \frac{N^2}{L}\right) \leq \frac{C}{L^{\gamma_3}}, \quad \mathbb{P}\left(\tau_{-N} \leq \frac{N^2}{L}\right) \leq \frac{C}{L^{\gamma_3}}.$$

PROOF. We prove the first inequality, the second is similar. Let T_1 be the first time $\rho_{T_1} \geq \frac{N}{L^\varepsilon}$ and T_2 be the first time after T_1 such that $\rho_T \leq \frac{N}{2L^\varepsilon}$. Then

$$\mathbb{P}(\tau_N \leq \frac{N^2}{L}) \leq \mathbb{P}\left(T_1 \geq N - \frac{N}{L^{1-\varepsilon}}\right) + \mathbb{P}\left(T_2 - T_1 \leq \frac{N}{L^{1-\varepsilon}}\right).$$

In view of Theorem 5.8 there exists $\rho > 0$ such that

$$\mathbb{P}\left(\rho_{n+\frac{N}{L^\varepsilon}} \leq \frac{N}{L^\varepsilon} | \rho_n \leq \frac{N}{L^\varepsilon}\right) \leq 1 - \rho.$$

Accordingly by induction we have that

$$\mathbb{P}\left(\rho_{\frac{jN}{L^\varepsilon}} \leq \frac{N}{L^\varepsilon} \text{ for all } j \leq k\right) \leq (1-\rho)^k$$

so

$$\mathbb{P}(T_1 \geq N - \frac{N}{L^{1-\varepsilon}}) \leq (1-\rho)^{cL^\varepsilon}.$$

On the other hand if

$$T_2 - T_1 \leq \frac{N}{L^{1-\varepsilon}}$$

then

$$\sup_{m \in [T_1, T_1 + \frac{N}{L^{1-\varepsilon}}]} \rho_m - \rho_{T_1} \geq \frac{N}{2L^\varepsilon}$$

and similarly to (2.11)

$$\mathbb{P}\left(\sup_{m \in [T_1, T_1 + \frac{N}{L^{1-\varepsilon}}]} \rho_m - \rho_{T_1} \geq \frac{N}{2L^\varepsilon}\right) \leq C\left(\frac{N}{L^{1-\varepsilon}}\right)^2 \left(\frac{N}{L^\varepsilon}\right)^2 : \left(\frac{N}{L^\varepsilon}\right)^4 = CL^{4\varepsilon-2}.$$

Since ε is arbitrary this proves the result. □

Let $\Delta_k = X_{k+1} - X_k$. Split $X_n = B_n + C_n$ where

$$B_n = \sum_{k=0}^{n-1} \Delta_k - \mathbb{E}_\omega(\Delta_k | \mathcal{F}_k), \quad C_n = \sum_{k=0}^{n-1} \mathbb{E}_\omega(\Delta_k | \mathcal{F}_k).$$

Denote $X^{(N)}(t) = \frac{X_{Nt}}{\sqrt{N}}$, $B^{(N)}(t) = \frac{B_{Nt}}{\sqrt{N}}$, $C^{(N)}(t) = \frac{C_{Nt}}{\sqrt{N}}$. Then B_n is a martingale and by Lemma 5.11 its quadratic variation is $n + O(\sqrt{n})$. Thus by Theorem 7.7 $B^{(N)}$ converges to a standard Brownian Motion. Therefore we need to relate C_n to $r_n = S_n + I_n$.

LEMMA 5.12. *For each* $\varepsilon > 0$

$$\mathbb{P}(\sup_{n\leq N} \frac{C_n - \boldsymbol{\delta} r_n}{\sqrt{N}} > \varepsilon) \to 0.$$

PROOF. Observe that $|C_n| \leq M r_n$ so we can assume that $r_n > \frac{\varepsilon\sqrt{N}}{M+1}$. By Lemma 5.11 for each ε' there exists K such that $\mathbb{P}(r_N > K\sqrt{N}) < \varepsilon'$, so we can assume that

$$|I_N| < K\sqrt{N}, \quad S_N \leq K\sqrt{N}.$$

Note that

$$C_n = \sum_{m=I_n}^{S_n} \delta_m - \sum_{m=I_n}^{S_n} 1_{L_m(n)<M} \sum_{j=1}^{L_m(n)+1} (p_m - q_m). \tag{5.11}$$

By the strong law of large numbers

$$\frac{\sum_{m=I_n}^{S_n} \delta_m - \boldsymbol{\delta} r_n}{r_n} \to 0.$$

On the other hand the second term in (5.11) is less than

$$M\text{Card}(m \in [I_n, S_n] : L_m(n) < M).$$

Divide the interval $[I_n, S_n]$ into subintervals of length $N^{1/4}$. By Lemma 5.10 with probability at least $1 - \theta^{N^{\gamma_2/4}} K N^{1/4}$ all the intervals except the two extreme one have at most $N^{\gamma_1/4}$ points which are visited less than M times. Thus

$$\mathbb{P}(\text{Card}(m \in [I_n, S_n] : L_m(n) < M) \geq N^{(1+\gamma_1)/4} + 2N^{1/4}) \leq \theta^{N^{\gamma_2/4}} K N^{1/4}.$$

□

The last result we need to complete the proof of Theorem 5.4 is the following.

LEMMA 5.13. *The family* $\{X^{(N)}(t)\}$ *is tight.*

PROOF OF THEOREM 5.4. Since $\{X^{(N)}(t)\}$is tight we can assume by choosing a subsequence that $X^{(N)} \Rightarrow X(t)$. Next $B^{(N)}(t) \Rightarrow W(t)$– the standard Brownian Motion while by Lemma 5.11

$$C^{(N)}(t) \Rightarrow \boldsymbol{\delta}[S(t, X) - I(t, X)].$$

□

PROOF OF LEMMA 5.13. By Theorem 7.3 it suffices to estimate $\mathbb{P}(\cup_{k<2^l} \Omega_{N,k,l})$ where

$$\Omega_{N,k,l} = \left\{ \left| X^{(N)}\left(\frac{k+1}{2^l}\right) - X^{(N)}\left(\frac{k}{2^l}\right) \right| \geq 2^{-l/8} \right\}.$$

Denote $n_1 = \frac{kN}{2^l}$, $n_2 = \frac{(k+1)N}{2^l}$, $\bar{n} = n_2 - n_1$. Then

$$\Omega_{N,k,l} = \{|X_{n_2} - X_{n_1}| \geq N^{3/8}\bar{n}^{1/8}\}.$$

We have $\Omega_{N,k,l} = \Omega^+_{N,k,l} \bigcup \Omega^-_{N,k,l}$ where

$$\Omega^+_{N,k,l} = \{X_{n_2} > X_{n_1} + N^{3/8}\bar{n}^{1/8}\}, \quad \Omega^-_{N,k,l} = \{X_{n_2} < X_{n_1} - N^{3/8}\bar{n}^{1/8}\}.$$

We shall deal with $\Omega^+_{N,k,l}$, $\Omega^-_{N,k,l}$ is similar.

Consider two cases:

(1) $S_{n_1} \geq X_{n_1} + \frac{1}{2}N^{3/8}\bar{n}^{1/8}$. Let

$$B^+_n = \sum_{k=n_1}^{n-1} [\Delta_k - \mathbb{E}_\omega(\Delta_k|\mathcal{F}_k)] 1_{X_k \in [X_{n_1}, X_{n_1} + \frac{1}{2}N^{3/8}\bar{n}^{1/8}]},$$

$$C^+_n = \sum_{k=n_1}^{n-1} \mathbb{E}_\omega(\Delta_k|\mathcal{F}_k) 1_{X_k \in [X_{n_1}, X_{n_1} + \frac{1}{2}N^{3/8}\bar{n}^{1/8}]}.$$

Note that $\Omega^+_{N,k,l}$ implies either

$$\Omega^{B+}_{N,k,l} = \{\sup_{j\in[n_1,n_2]} B^+_n \geq \frac{1}{4}N^{3/8}\bar{n}^{1/8}]\} \text{ or}$$

$$\Omega^{C+}_{N,k,l} = \{\sup_{j\in[n_1,n_2]} C^+_n \geq \frac{1}{4}N^{3/8}\bar{n}^{1/8}]\}.$$

By Theorem 7.6

$$\mathbb{P}(\Omega^{B+}_{N,k,l}) \leq C\frac{\bar{n}^2}{\bar{n}^{1/2}N^{3/2}} = C\left(\frac{\bar{n}}{N}\right)^{3/2}.$$

so

$$\mathbb{P}(\bigcup_k \Omega^{B+}_{N,k,l}) \leq C\left(\frac{\bar{n}}{N}\right)^{1/2}.$$

Next

$$\max_k C^+_k \leq M\text{Card}(m \in [n_1, n_1 + \frac{1}{2}\bar{n}^{1/8}N^{3/8}] : L_k(n_1 + \frac{1}{2}\bar{n}^{1/8}N^{3/8}) \leq M)$$

and by Lemma 5.10

$$\mathbb{P}(\text{Card}(m \in [n_1, n_1 + \frac{1}{2}\bar{n}^{1/8}N^{3/8}] : L_k(n_1 + \frac{1}{2}\bar{n}^{1/8}N^{3/8}) \leq M) \geq \frac{\bar{n}^{1/8}N^{3/8}}{4M}) \leq \theta^{cN^{3\gamma_2/8}n^{\gamma_2/4}}$$

so that

$$\mathbb{P}(\bigcup_k \Omega^{C+}_{N,k,l}) \ll \mathbb{P}(\bigcup_k \Omega^{B+}_{N,k,l}).$$

This completes the analysis of case (1).

(2) $S_{n_1} < X_{n_1} + \frac{1}{2}N^{3/8}\bar{n}^{1/8}$. In this case $\Omega^+_{N,k,l}$ implies that

$$\tau_{S_{n_1} + \frac{1}{2}\bar{n}^{1/8}N^{3/8}} - \tau_{S_{n_1}+1} < \bar{n}.$$

Recall that

$$\bar{n} = \frac{N}{2^{l}}, \quad \frac{1}{2}\bar{n}^{1/8}N^{3/8} = \frac{\sqrt{N}}{2^{\bar{l}+1}}$$

where $\bar{l} = \frac{l}{8}$. Hence there exists m such that

$$\left[\frac{m\sqrt{N}}{2^{\bar{l}+2}}, \frac{(m+1)\sqrt{N}}{2^{\bar{l}+2}}\right] \subset [S_{n_1}+1, S_{n_1}+\frac{1}{2}\bar{n}^{1/8}N^{3/8}].$$

So we need to estimate $\mathbb{P}(\bigcup_{m\leq 2^{\bar{l}+2}K}\Omega^{\dagger}_{m,\bar{l},N})$ where

$$\Omega^{\dagger}_{m,\bar{l},N} = \left\{\tau_{\frac{(m+1)\sqrt{N}}{2^{\bar{l}+2}}} - \tau_{\frac{m\sqrt{N}}{2^{\bar{l}+2}}} \leq \frac{N}{2^{8\bar{l}}}\right\}.$$

By Lemma 5.11

$$\mathbb{P}\left(\Omega^{\dagger}_{m,\bar{l},N}\right) \leq \frac{C}{2^{6\gamma_3\bar{l}}}$$

so that

$$\mathbb{P}\left(\Omega^{\dagger}_{m,\bar{l},N}\right) \leq \frac{C}{2^{(\gamma_3-1)\bar{l}}}.$$

This completes the analysis of case (2) establishing Lemma 5.13. □

6. Branching processes with migration and Bessel processes.

Here we prove Theorem 5.6. To prove parts (a) and (b) simultaneously we consider general branching processes with migration. Namely let

$$V_{m+1} = \sum_{j=1}^{V_m-M}\xi_{m,j} + \zeta_m$$

where $\xi_{m,j}$ are iid, $\mathbb{E}(\xi_{m,j}) = 0$ and conditionally on $V_m > M$ ζ_m are independent of $\zeta_1, \zeta_2 \dots \zeta_{m-1}$ and of ξs. We also assume that $E(e^{r\xi}) < \infty$, $E(e^{r\zeta}) < \infty$ for some $r > 0$.

Let $V_0 = N$ and let $V^{(N)}(t) = \frac{V_{Nt}}{N}$. We stop V at time $\min(\tau^-_{\alpha N}, \tau\beta N^+)$.

THEOREM 5.14. *As $N\to\infty$ $V^{(N)}(t) \Rightarrow Y(t)$ where $Y(t) \sim \mathbf{D}(\mathbb{E}(\zeta) - M, \mathrm{Var}(\xi)y)$ on $[\alpha,\beta]$ stopped when it reaches the boundary. Moreover if $\psi(Y)$ is a martingale then*

$$\mathbb{E}(\phi\left(\frac{V_{\min(\tau^-_{\alpha N},\tau^+_{\beta N})}}{N}\right) = \phi(1) + O(N^{-1/2}). \tag{5.12}$$

PROOF. Tightness can be proven similarly to (2.9). Writing

$$V_{m+1} - V_m = \sum_{j=1}^{V_m-M}(\xi_{m,j}-1) + (\zeta_m - M)$$

we compute

$$\mathbb{E}(V_{m+1}-V_m|\mathcal{F}_m)=\mathbb{E}(\zeta)-M,$$
$$\mathbb{E}((V_{m+1}-V_m)^2|\mathcal{F}_m)=(V_m-M)\mathrm{Var}(\xi)+\mathbb{E}((\zeta-M)^2).$$

Let f be a test function. Then

$$\mathbb{E}\left(f\left(\frac{V_{m+1}}{N}\right)-f\left(\frac{V_m}{N}\right)|\mathcal{F}_m\right)$$
$$=\frac{1}{N}\left[f'\left(\frac{V_m}{N}\right)(\mathbb{E}(\xi)-M)+\frac{1}{2}f''\left(\frac{V_m}{N}\right)\frac{V_m}{N}\right]+O\left(\frac{1}{N^{3/2}}\right).$$

Hence

$$\mathbb{E}\left(f\left(\frac{V_{m+1}}{N}\right)-f\left(\frac{V_m}{N}\right)\right) \tag{5.13}$$
$$=\frac{1}{N}\left[\mathbb{E}\left(f'\left(\frac{V_m}{N}\right)\right)(\mathbb{E}(\xi)-M)+\frac{1}{2}\mathbb{E}\left(f''\left(\frac{V_m}{N}\right)\frac{V_m}{N}\right)\right]+O\left(\frac{1}{N^{3/2}}\right).$$

Thus if $Y(t)$ is a limit point of $V^{(N)}(t)$ then

$$\mathbb{E}(f(Y(t))-f(Y(0)))=\mathbb{E}\left(\int_0^t\left[f'(Y)(\mathbb{E}(\zeta)-M)+\frac{f''(Y)}{2}Y\mathrm{Var}(\xi)\right](s)ds\right).$$

A similar argument shows that for each s

$$\mathbb{E}\left(f(Y(t))-f(Y(s))-\int_0^t\left[f'(Y)(\mathbb{E}(\zeta)-M)+\frac{f''(Y)}{2}Y\mathrm{Var}(\xi)\right](s)ds|\mathcal{F}_s\right)=0.$$

Thus $Y\sim\mathbf{D}(\mathbb{E}(\zeta)-M,\mathrm{Var}(\xi)y)$. To prove (5.12) observe that by already proven first part of the theorem there exists $\zeta>0$ such that

$$\mathbb{P}(\min(\tau^-_{\alpha N},\tau^+_{\beta N})>N)\leq(1-\zeta)$$

and by induction

$$\mathbb{P}(\min(\tau^-_{\alpha N},\tau^+_{\beta N})>kN)\leq(1-\zeta)^k.$$

Now (5.12) follows from the fact that the error term in (5.13) is $O(N^{-3/2})$.

□

COROLLARY 5.15. *Theorem 5.14 remains valid if we stop V_n only at time $\tau^-_{\alpha N}$. The limiting process is $\mathbf{D}(\mathbb{E}(\zeta)-M),\mathrm{Var}(\xi)y)$ stoped at α.*

PROOF. Let $\Phi:C[0,\mathbf{T}]\to\mathbb{R}$ be a continuous functional. Since ∞ is inacessible given ε we can find β such that

$$|\mathbb{E}(\Phi(Y_{\alpha,\beta}))-\mathbb{E}(\Phi(Y_\alpha)|\leq\varepsilon.$$

By Theorem 5.14 for large N

$$|\mathbb{E}(\Phi(V^{(N)}_{\alpha,\beta}))-\mathbb{E}(\Phi(Y_{\alpha,\beta})|\leq\varepsilon.$$

Fianally

$$|\mathbb{E}(\Phi(V_{\alpha,\beta}^{(N)})) - \mathbb{E}(\Phi(V_{\alpha}^{(N)})| \leq ||\Phi||\mathbb{P}(V_{\alpha,\beta}^{(N)} \neq V_{\alpha}^{(N)})$$

and since ∞ is inacessible the last probability can be made as small as we wish by choosing large β. □

Theorem 5.8 will follow from the following more general result.

THEOREM 5.16. *Suppose that* $\mathbb{E}(\zeta) > M$. *Let* $V_0 = 0$ *and* $V^{(N)}(t) = \frac{V_{Nt}}{N}$. *Then as* $N \to \infty$ $V^{(N)}(t) \Rightarrow Y(t)$ *where*

$$Y \sim \mathbf{D}(\mathbb{E}(\zeta) - M, \mathrm{Var}(\xi)y)$$

started at 0.

To prove this theorem we need to investigate the behavior of our process near 0. Let $d = \frac{4(\mathbb{E}(\zeta)-M)}{\mathrm{Var}(\xi)}$, $\nu = 1 - \frac{d}{2}$.

LEMMA 5.17. *(a) If* $d < 2$ *then there exists* C *such that for each* $x \leq n/2$

$$\mathbb{P}_x(\tau_n^+ < \tau_{x/2}^-) \geq C\left(\frac{x}{n}\right)^{-\nu}.$$

(b) If $d < 2$ *then for each* $\varepsilon > 0$ *there exists* C *such that for each* $x \leq n/2$

$$\mathbb{P}_x(\tau_n^+ < \tau_{x/2}^-) \geq C\left(\frac{x}{n}\right)^{-\varepsilon}.$$

(c) If $d > 2$ *then there exists* p *such that for each* $x \leq n$

$$\mathbb{P}_x(\tau_n^+ < \tau_{x/2}^-) \geq p.$$

PROOF OF THEOREM 5.16. We will give a proof in case $d < 2$, other cases are easier.

Fix $\delta > 0$. We say that $[n_1, n_2]$ is a δ-*excursion* if

$$V_{n_1} \geq \delta N, \quad V_{n_1-1} < \delta N, \quad V_{n_2} \leq \delta N.$$

Similarly we define δ-excursions for Y. Let A_j be the length of j-th δ excursion for V and B_j be the time between j-th and $(j-1)$-st excursions. Let a_j and b_j denote the corresponding quantities for Y. We define δ-prunnings Y^δ and $V(N), \delta$ of the processes Y and $V^{(N)}$ respectively by replacing the intervals between consequtive δ-excursions by lines with slope 1 joining $\frac{\delta}{2}$ and δ. Note that Y^δ consists of a sequence of independent copies square Bessel process started from δ and stopped at $\frac{\delta}{2}$ separated by straight line segments. Accordingly Theorem 5.14 implies that

$$V^{(N),\delta} \Rightarrow Y^\delta \text{ as } N \to \infty. \tag{5.14}$$

EXERCISE 5.5. Give a detailed derivation of (5.14).

Therefore Theorem 5.16 will follow once we show that if δ is small then Y^δ is close to Y and $V^{(N),\delta}$ is close to $V^{(N)}$ with probability close to 1.

EXERCISE 5.6. Show that there exists c such that $\mathbb{P}_\delta(\tau_{\delta/2} > t\delta) \geq ct^{-\nu}$.

Hint. Show that

$$\mathbb{P}_\delta(\tau_{\delta/2} > t\delta) = \mathbb{P}_1(\tau_{1/2} > t) \geq c\mathbb{P}_1(\tau_{1/2} > \tau_t).$$

Also by scaling invariance of square Bessel process the distribution of $\frac{\tau_\delta}{\delta}$ is independent of δ. Accordingly there exists t_0 such that for each $x < \delta$

$$\mathbb{P}_x(\tau_\delta > t_0) \leq \frac{1}{2}.$$

By induction

$$\mathbb{P}_x(\tau_\delta > t_0) \leq \left(\frac{1}{2}\right)^k. \tag{5.15}$$

Next let j^* be the first time when $a_{j^*} > \mathbf{T}$. Note that the number of δ-excursions on $[0, \mathbf{T}]$ is less than j^*. By Execrise 5.6

$$\mathbb{P}(j^* > j) \leq (1 - c\delta^\nu)^j.$$

Hence given ε there exists K such that

$$\mathbb{P}(j^* > K\delta^{-\nu}) \leq \varepsilon. \tag{5.16}$$

Let η_j be iid random variables, such that $\eta \sim \text{Geom}(\frac{1}{2})$. Combining (5.15) and (5.16) we get

$$\mathbb{P}\left(\sum_j b_j \geq \kappa\right) < \mathbb{P}\left(t_0\delta \sum_{j=1}^{K\delta^\nu} \eta_j \leq \kappa\right) + \varepsilon.$$

The first term here tends to 0 as $\delta \to 0$ since $\nu < 1$. Therefore the compliment to $\delta-$excursions has small measure with probability close to 1. That is Y^δ is close to Y.

The proof that $V^{(N),\delta}$ is close to $V^{(N)}$ proceeds along similar lines.

Let J^* be the first time when $A_j > \mathbf{T}N$. Note that if $x_N/N \to \delta$ then

$$\mathbb{P}_{x_N}(A_j > \mathbf{T}N) \geq \mathbb{P}_{x_N}(\tau^-\delta N/2 > \tau_N^+)\mathbb{P}_N(\tau_{N/2} > N) \geq (C_1\delta^{-\nu})C_2$$

where Theorem 5.14 was used to estimate both factors in the last inequality. Accordingly given ε there exists K such that

$$\mathbb{P}(J^* > K\delta^{-\nu}) \leq \varepsilon$$

so to complete the proof it remains to show that there exists t_0 such that for each $x \leq \delta N$

$$\mathbb{P}_x(\tau^+_{\delta N} > t_0 \delta N) \leq \frac{1}{2}. \tag{5.17}$$

Let $N_1 = \delta N$. Let $n(m)$ be the m-th time when $V_n > M$. Note that by large deviation bound there exists $K > 0, \theta < 1$ such that

$$\mathbb{P}(n(m) > Km) < \theta^m)$$

so it suffices to prove (5.17) for $V_{n(m)}$ instead of V_m. Denote

$$\bar{V}_m = V_{\min(n(m), \tau^+_{N_1})}, \quad \eta_k = \mathbb{E}(\bar{V}_k - \bar{V}_{k-1} | \bar{\mathcal{F}}_{k-1})$$

where $\bar{\mathcal{F}}_k$ is the σ-algebra generated by $\bar{V}_1 \dots \bar{V}_k$. Split $\bar{V}_m = M_m + H_m$ where $H_m = \sum_{k=1}^m \eta_k$. Then M_m is a martingale and since $\mathbb{E}((\eta_k)^2) \leq CN_1$ its quadratic variation Γ_m satisfies $\Gamma_m \leq CN_1 m$. On the other hand on $\tau^+_{N_1} \geq m$ we have

$$H_m \geq (\mathbb{E}(\zeta) - M)m.$$

Take $m = t_0 N_1$ then on $\tau^+_{N_1} > m$ we have $M_m \geq (ct_0 - 1)N_1$. But

$$\mathbb{P}(M_m \geq (ct_0 - 1)N_1) \leq \bar{C} \frac{CN_1^2 t_0}{(ct_0 - 1)^2 N_1^2} = \bar{C} \frac{Ct_0}{(ct_0 - 1)^2}$$

and this expression can be made as small as we wish by choosing t_0 large.

This completes the proof of (5.17) and hence the proof of Theorem 5.16 in case $d < 2$.

Exercise 5.7. Prove Theorem 5.16 for $d \geq 2$.

□

7. Perturbed Brownian Motion.

In this section we shall give the proof of Theorem 5.3. We consider only the case where α and β are sufficiently small (that is, $\frac{|\alpha\beta|}{(1-\alpha)(1-\beta)} < 1$) referring to [**6, 29**] for the general case. We need the following fact.

Lemma 5.18. (Skorokhod Reflection Lemma). *Given a continuous function $y(t)$ such that $y(0) \geq 0$ there exists unique decomposition pair $(z(t), a(t))$ such that*

(i) $z(t) = y(t) + a(t)$
(ii) $z(t) \geq 0$ and
(iii) $a(t)$ is non decreasing and grows only on the set where $z(s) = 0$.

Moreover

$$a(t) = \sup_{s \le t} \max(-(y(s), 0). \tag{5.18}$$

PROOF. Define a by (5.18), $z = y + a$. Then $a(t) \ge -y(t)$ and a grows only if $-y(t) = sup_{s \le t} y(s) = a(t)$. This proves existence.

Next let (z_1, a_1) and (z_2, a_2) be two pairs having the required properties. Since $z_2 - z_1 = a_2 - a_1$ has bounded variation we have

$$0 \le (z_2 - z_1)^2(t) = \int_0^t (z_2 - z_1)(s) d(z_2 - z_1)(s) = \int_0^t (z_2 - z_1)(s) d(a_2 - a_1)(s)$$

$$= - \int_0^t z_2(s) da_1(s) - \int_0^t z_1(s) da_2(s) \le 0.$$

Hence, in fact, $z_2(t) \equiv z_1(t)$. This proves uniqueness. □

PROOF OF THEOREM 5.3. Equality

$$Y(t) + I(t) = W(t) + \alpha S(t) + (\beta - 1) I(t)$$

and Lemma 5.18 yield

$$(\beta - 1) I(t) = \sup_{s \le t} -(W(s) + \alpha S(s)). \tag{5.19}$$

Likewise

$$(1 - \alpha) S(t) = \sup_{s \le t} (W(s) + \beta I(s)).$$

Hence

$$S(t) = \frac{1}{1 - \alpha} \sup_{s \le t} \left[W(s) + \frac{\beta}{1 - \beta} \sup_{u \le s} -[W(u) + \alpha S(u)] \right].$$

Define a mapping $\Gamma : C[0, \mathbf{T}] \to C[0, \mathbf{T}]$ by

$$(\Gamma S)(t) = \frac{1}{1 - \alpha} \sup_{s \le t} \left[W(s) + \frac{\beta}{1 - \beta} \sup_{u \le s} -[W(u) + \alpha S(u)] \right].$$

Then

$$||\Gamma S_1 - \Gamma S_2||_\infty \le \frac{|\alpha \beta|}{(1 - \alpha)(1 - \beta)} ||S_1 - S_2||_\infty.$$

Therefore by Contraction Mapping Principle there exists unique S such that $\Gamma(S) = S$. Thus if (Y_1, S_1, I_1) and (Y_2, S_2, I_2) two solutions of (5.1) then $S_1 = S_2$. Likewise $I_1 = I_2$. Then (5.1) implies that $Y_1 = Y_2$. This proves uniqueness.

To prove existence let S satisfy $\Gamma(S) = S$, I satisfy (5.19) and Y satisfy (5.1). Combing (5.19) and (5.1) we get

$$(\beta - 1) I(t) = \beta I(t) + \sup_{s \le t} Y(s),$$

that is $I(t) = \inf_{s\le t} Y(s)$. Next substituting (5.19) into $\Gamma(S) = S$ we get

$$(1-\alpha)S(t) = \sup_{s\le t}(W(s) + \beta I(s)) = \sup_{s\le t}(Y(s) - \alpha I(s)).$$

Thus $S(t)$ has the following properties:

(i) $(1-\alpha)S(t) \ge Y(t) - \alpha S(t)$, that is $S(t) \ge Y(t)$;

(ii) $S(t)$ is non-decreasing;

(iii) S grows only if

$$Y(t) - \alpha S(t) = \sup_{s\le t}(Y(s) - \alpha S(s)) = (1-\alpha)S(t)$$

that is if $S - Y = 0$.

By Lemma (5.18) $S = \sup_{s\le t} Y(s)$. □

CHAPTER 6

Random walk in random environment.

1. Recurrence and transience.

Consider stationary inhomogeneous walk where if the walker is at site n then he moves to $n+1$ with probability p_n and to $n-1$ with probability $q_n = 1 - p_n$. Denote $\alpha_n = \frac{q_n}{p_n}$. We state have the following criterion for recurrence and transience of the walk.

THEOREM 6.1.

$$(a) \quad \mathbb{P}_1(X \textit{ reaches } 0) = 1 \textit{ iff } \sum_n \alpha_1 \alpha_2 \dots \alpha_n = \infty$$

$$(b) \quad \mathbb{P}_{-1}(X \textit{ reaches } 0) = 1 \textit{ iff } \sum_n \frac{1}{\alpha_{-1}\alpha_{-2}\dots\alpha_{-n}} = \infty.$$

PROOF. We start by looking for a function $\phi : [0, \infty[\to \mathbb{R}$ such that $\phi(X_n)$ is a martingale. We have

$$\phi(n) = p_n \phi(n+1) + q_n \phi(n-1)$$

which can be rewritten as

$$p_n[\phi(n+1) - \phi(n)] = q_n[\phi(n) - \phi(n-1)].$$

Therefore normalizing our function by conditions $\phi(0) = 0, \phi(1) = 1$ and introducing $\psi_n = \phi(n) - \phi(n-1)$ we get

$$\psi_{n+1} = \alpha_n \psi_n \quad \psi_1 = 1$$

which implies $\psi_n = \alpha_n \alpha_{n-1} \dots \alpha_1$ so that

$$\phi(n) = \sum_{j=1}^{n-1} \alpha_1 \alpha_2 \dots \alpha_j.$$

Now the optimal stopping theorem says that

$$\mathbb{P}_1(X \text{ reaches } 0 \text{ before } n) = \frac{1}{\phi(n)}.$$

Letting $n \to \infty$ we see that

$$\mathbb{P}_1(X \text{ reaches } 0) = 1 \text{ iff } \phi(n) \to \infty, n \to \infty.$$

This proves part (a). Part (b) is similar. □

Let $\omega = \{p_n\}, i \in \mathbb{Z}$ be an i.i.d. sequence of random variables, $0 < p_n < 1$. The sequence ω is called *environment* (or *random environment*). Let $(\Omega, \mathbf{P})$ be the corresponding probability space with Ω being set of all environments and $\mathbf{P}$ the probability measure on Ω. The expectation with respect to this measure will be denoted by $\mathbf{E}$. Given an ω we define a random walk $X = \{X_n,\ n \geq 0\}$ on $\mathbb{Z}$ in the environment ω by setting $X_0 = 0$ and

$$\mathbb{P}_\omega(X_{n+1} = X_n + 1 | X_0 \dots X_n) = p_{X_n} \quad \mathbb{P}_\omega(X_{n+1} = X_n - 1 | X_0 \dots X_n) = q_{X_n}$$

where $q_n = 1 - p_n$. Thus, a fixed (quenched) environment ω provides us with a conditional probability measure $\mathbb{P}_\omega$ on the space of trajectories starting from 0; the corresponding expectation will be denoted by $\mathbb{E}_\omega$. In turn, these two measures naturally generate the so called *annealed* measure on the direct product of Ω and the space of trajectories which, formally speaking, is a semi-direct product of $\mathbf{P}$ and $\mathbb{P}_\omega$. However, with a very slight abuse of notation, $\mathbf{P}$ and $\mathbf{E}$ will also denote the latter measure and the corresponding expectation; the exact meaning of the corresponding probabilities and expectations will always be clear from the context. The term *annealed walk* will be used to discuss properties of the above random walk with respect to the annealed probability.

We assume that p_n satisfy ellipticity condition

$$\varepsilon_0 < p_n < 1 - \varepsilon_0. \tag{6.1}$$

Let $\gamma = \mathbb{E}(\ln(p_n/q_n))$.

COROLLARY 6.2. [**35**] *(a) If $\gamma > 0$ then $X_n \to +\infty$ almost surely.*
(b) If $\gamma < 0$ then $X_n \to -\infty$ almost surely.
(c) If $\gamma = 0$ then X is recurrent.

PROOF. Assume that $\gamma > 0$. Then the strong law of large numbers tells us that there is a constant $C(\omega)$ such that

$$\frac{1}{\alpha_{-1}\alpha_{-2}\dots\alpha_{-n}} \geq e^{n\sigma/2} C^{-1}(\omega).$$

It follows that

$$\sum_n \frac{1}{\alpha_{-1}\alpha_{-2}\dots\alpha_{-n}} = \infty$$

so that if the walker moves to the left then he returns to his original position with probability 1. Likewise

$$\alpha_1\alpha_2\dots\alpha_n \leq e^{-n\sigma/2} C(\omega)$$

so that

$$\sum_n \alpha_1\alpha_2\dots\alpha_n < \infty.$$

Therefore after moving to the right the walker never returns to his original position with positive probability. Hence the walker visits every site only finitely many times eventually moving to the right. This proves (a). The proof of (b) is similar. Finally if $\gamma = 0$ then using recurrence of zero mean walk we see that both $\alpha_1 \dots \alpha_n > 1$ and $\alpha_{-1} \dots \alpha_{-n} > 1$ infinitely often which cause both

$$\sum_n \alpha_1 \alpha_2 \dots \alpha_n \text{ and } \sum_n \frac{1}{\alpha_{-1}\alpha_{-2}\dots\alpha_{-n}}$$

to diverge so that the walk is recurrent. □

In fact the proof gives an explicit estimate on the probability of backtracking which will be useful in the sequel.

LEMMA 6.3. *If $\gamma > 0$ then there exist $C > 0, \beta < 1$ such that*

$$\mathbf{P}(X \text{ visits } n \text{ after } n+m) \leq C\beta^m.$$

PROOF. By translation invariance we may assume that $n = 0$. By Optional Stopping Theorem

$$\mathbb{P}_{m,\omega}(X \text{ reaches } 0 \text{ before } n) = 1 - \frac{\phi(n)}{\phi(m)} = \frac{\sum_{j=m}^{n-1} \alpha_1 \dots \alpha_j}{\sum_{j=1}^{n-1} \alpha_1 \dots \alpha_j} \leq \frac{\sum_{j=m}^{n-1} \alpha_1 \dots \alpha_j}{\varepsilon_0}.$$

By the Large Deviation Bound

$$\mathbf{P}\left(\text{There exists } j \geq m \sum_{k=1}^{j} \ln \alpha_j > -\frac{\gamma j}{2}\right) \leq \theta^m.$$

On the other hand if $\alpha_1 \dots \alpha_j \leq e^{-\gamma j/2}$ then $\sum_{j=m}^{n-1} \alpha_1 \dots \alpha_j \leq \frac{e^{-\gamma m/2}}{1-e^{-\gamma/2}}$. □

2. Transient walks: the results.

In this section we consider transient walk. That is we assume that

(6.2) $$\mathbf{E}(\ln(p/q)) > 0.$$

In this case we have $X_n \to +\infty$ almost surely. Let s be the positve solution of the equation

$$\mathbf{E}((q/p)^s) = 1.$$

If the above equation has no solution we let $s = +\infty$.

Since X_n is transient it looks monotonically increasing on a large scale and hence it makes sense to study the hitting time $\tilde{T}_N := \min(n :$

$X_n = N$) which can roughly be viewed as the inverse function of X_n. The following condition will be useful in the sequel

(6.3) the distribution of $\ln(p/q)$ is non arithmetic

THEOREM 6.4. *([**20**]) Assume that either $s > 2$ or (6.3) holds. Then the annealed random walk X_n has the following properties:*

(a) If $s < 1$ then the distribution of $\frac{\tilde{T}_N}{N^{1/s}}$ converges to a stable law with index s.

(b) If $1 < s < 2$ then there is a constant u such that the distribution of $\frac{\tilde{T}_N - Nu}{N^{1/s}}$ converges to a stable law with index s.

(c) If $s > 2$ then there is a constant u such that the distribution of $\frac{\tilde{T}_N - Nu}{N^{1/2}}$ converges to a normal distribution.

(d) If $s = 1$ then there is a sequence $u_N \sim cN \ln N$ such that the distribution of $\frac{\tilde{T}_N - u_N}{N}$ converges to a stable law with index 1.

(e) If $s = 2$ then there is a constant u such that the distribution of $\frac{\tilde{T}_N - Nu}{\sqrt{N \ln N}}$ converges to a normal distribution.

The following definitions and notations will be used. The *occupation time* T_N of the interval $[0, N)$ is the total time the walk X_n starting from 0 spends on this (semi-open) interval during its life time. In other words, $T_N = \#\{n : \ 0 \le n < \infty, \ 0 \le X_n \le N - 1\}$

We thus use the following convention: starting from a site j counts as one visit of the walk to j. The occupation time of a site j is defined similarly and is denoted by ξ_j. Observe that T_N (and ξ_j) is equal to the number of visits by the walk to $[0, N)$ (respectively, to site j). Since our random walk is transient to the right, both T_N and ξ_j are, $\mathbf{P}$-almost surely, finite random variables. It is clear from these definitions that

$$T_N = \sum_{j=0}^{N-1} \xi_j.$$

The following lemma shows that T_N and the hitting time $\tilde{T}_N$ have the same asymptotic behaviour.

LEMMA 6.5. *For any $\varepsilon > 0$*

$$\mathbf{P}\left(\frac{|T_N - \tilde{T}_N|}{N^{1/s}} > \varepsilon \right) \to 0 \ \textit{as} \ N \to \infty.$$

PROOF. It is easy to see that

$$\tilde{T}_N = \#\{n : \ 0 < n \le \tilde{T}_N, \ X_n \in [0, N-1]\} + \#\{n : \ 0 < n \le \tilde{T}_N, \ X_n < 0\}$$

and

$$T_N = \#\{n:\ 0 \le n \le \tilde{T}_N,\ X_n \in [0, N-1]\} + \#\{n:\ n > \tilde{T}_N,\ X_n \in [0, N-1]\}.$$

Since the first terms in these formulae are equal, $|T_N - \tilde{T}_N|$ can be estimated above by a sum of two random variables: the number of visits to the left of 0 and the number of visits to the left of N after $\tilde{T}_N$:

$$|T_N - \tilde{T}_N| \le \#\{n:\ n \ge 0,\ X_n < 0\} + \#\{n:\ n > \tilde{T}_N,\ X_n < N\}$$

The first term in this estimate is bounded for $\mathbf{P}$-almost all ω. Since $\tilde{T}_N$ is a hitting time, the second term has, for a given ω, the same distribution as $\#\{n:\ n > 0,\ X_n < N \,|\, X_0 = N\}$ (due to the strong Markov property). Finally, the latter is a stationary sequence with respect to the annealed measure and therefore is stochastically bounded. Hence the Lemma. $\square$

From now on we shall deal mainly with $\mathfrak{t}_N$ which is the normalized version of T_N:

$$\mathfrak{t}_N = \begin{cases} \frac{T_N}{N^{1/s}} & \text{if } 0 < s < 1, \\ \frac{T_N - \mathbb{E}_\omega(T_N)}{N^{1/s}} & \text{if } 1 \le s < 2, \\ \frac{T_N - \mathbb{E}_\omega(T_N)}{\sqrt{N \ln N}} & \text{if } s = 2, \\ \frac{T_N - \mathbb{E}_\omega(T_N)}{\sqrt{N}} & \text{if } s < 2. \end{cases}$$

It is also important and natural to have control over the $\mathbb{E}_\omega(T_N)$. The corresponding normalized quantity is defined as follows:

$$\mathfrak{u}_N = \begin{cases} \frac{\mathbb{E}_\omega(T_N)}{N^{1/s}} & \text{if } 0 < s < 1, \\ \frac{\mathbb{E}_\omega(T_N) - u_N}{N^{1/s}} & \text{if } s = 1, \\ \frac{\mathbb{E}_\omega(T_N) - \mathbf{E}(T_N)}{N^{1/s}} & \text{if } 1 < s < 2, \\ \frac{\mathbb{E}_\omega(T_N) - \mathbf{E}(T_N)}{\sqrt{N \ln N}} & \text{if } s = 2, \\ \frac{\mathbb{E}_\omega(T_N) - \mathbf{E}(T_N)}{\sqrt{N}} & \text{if } s > 2. \end{cases}$$

Set

$$F_N^\omega(x) = \mathbb{P}_\omega\left(\mathfrak{t}_N \le x\right).$$

Then $x \to F_N^\omega(x)$ is a sequence of random processes. The limiting behaviour of the sequence $\mathfrak{t}_N$ can be described in terms of a point Poisson process which we shall now introduce.

Given a $\mathbf{c} > 0$, let $\Theta = \{\Theta_j\}$ be a Poisson process on $(0, \infty)$ with intensity $\frac{\mathbf{c}}{\theta^{1+s}}$. For a given collection of points $\{\Theta_j\}$ let $\{\Gamma_j\}$ be a collection of i.i.d. random variables with mean 1 exponential distribution labeled by these points.[1] Let

$$Y = \begin{cases} \sum_j \Theta_j \Gamma_j & \text{if } 0 < s < 1 \\ \sum_j \Theta_j \left(\Gamma_j - 1\right) & \text{if } 1 \le s < 2 \end{cases}.$$

Observe that Y is finite almost surely. Indeed, there are only finitely many points with $\Theta_j \ge 1$. Next, if $0 < s < 1$ let

$$\tilde{Y} = \sum_{\Theta_j < 1} \Theta_j \Gamma_j .$$

Then

$$\mathbf{E}\tilde{Y} = \int_0^1 \frac{\mathbf{c}\theta d\theta}{\theta^{1+s}} = \frac{\mathbf{c}}{1-s} < \infty.$$

In case $1 \le s < 2$ let

$$\tilde{Y}_\delta = \sum_{\delta < \Theta_j < 1} \Theta_j (\Gamma_j - 1).$$

Then $\mathbf{E}(\tilde{Y}_\delta) = 0$ and

$$\mathrm{Var}(\tilde{Y}_\delta) = \int_\delta^1 \frac{\mathbf{c}\theta^2 d\theta}{\theta^{1+s}} = \frac{\mathbf{c}}{2-s} \left(1 - \delta^{2-s}\right).$$

THEOREM 6.6. *Suppose that* (6.3) *holds For* $0 < s < 2$ *and a* $\delta > 0$ *there is a sequence* $\Omega_{N,\delta} \subset \Omega$ *such that* $\mathbf{P}(\Omega_{N,\delta}) \to 0$ *as* $N \to \infty$ *and a sequence of random point processes*

$$(\Theta^{(N,\delta)}, \Gamma^{(N,\delta)}) = (\{\Theta_j^{(N,\delta)}(\omega), \Gamma_j^{(N,\delta)}(\omega, X)\})$$

defined for $\omega \in \Omega_{N,\delta}$ *and having the following asymptotic properties as* $N \to \infty$*:*
(i) The component $\Theta^{(N,\delta)}$ *converges weakly to a point Poisson process* $\Theta^{(\delta)}$ *on* $[\delta, \infty)$ *with intensity* $\frac{\mathbf{c}}{\theta^{1+s}}$ *(with some constant* $\mathbf{c} > 0$*)*
(ii) The component $\Gamma^{(N,\delta)}$ *converges weakly to a point process* $\Gamma = \{\Gamma_j\}$ *whose realizations are collections of mutually independent mean 1 exponential random variables labeled by the points of* Θ*. Moreover* Γ_j *are asymptotically independent of* ω *in the following sense: given* $k, x_1, x_2 \dots x_k$ *let* $\Omega_{N,\delta,k} = \Omega_{N,\delta} \cap \mathrm{Card}(\Theta_j \ge k)$ *then given* $\varepsilon > 0$ *there is a subset* $\tilde{\Omega}_{N,k,\delta} \subset \Omega_{N,k,\delta}$ *such that* $\mathbf{P}(\Omega_{N,\delta,k} - \tilde{\Omega}_{N,\delta,k}) \to 0$ *and for each* $\omega \in \tilde{\Omega}_{N,\delta,k}$

$$|\mathbb{P}(\Gamma_1 \ge x_1, \Gamma_2 \ge x_2 \dots \Gamma_k \ge x_k) - \exp(-(x_1 + x_2 + \dots x_k)| < \varepsilon.$$

(iii) The $\mathfrak{t}_N$ *and* $\mathfrak{u}_N$ *can be presented in the following form:*

[1]Strictly speaking we should write $\{\Gamma_{\Theta_j}\}$ instead of $\{\Gamma_j\}$. However, we believe that the meaning of the concise notation will always be clear from the context.

(a) If $0 < s < 1$ *then for* $\omega \in \Omega_{N,\delta}$

$$\mathfrak{t}_N = \sum_j \Theta_j^{(N,\delta)} \Gamma_j^{(N,\delta)} + R_N, \quad \textit{where} \;\; R_N \geq 0 \;\; \textit{and} \;\; \mathbf{E}(R_N) = O(\delta^{1-s})$$

$$\mathfrak{u}_N = \sum_j \Theta_j^{(N,\delta)} + O(\delta^{1-s})$$

(b) If $s = 1$ *then for* $\omega \in \Omega_{N,\delta}$

$$\mathfrak{t}_N = \sum_j \Theta_j^{(N,\delta)} (\Gamma_j^{(N,\delta)} - 1) + R_N, \quad \textit{where} \;\; \mathbf{E}(|R_N|) = O(\delta^{1/2})$$

$$\mathfrak{u}_N = \sum_j (\Theta_j^{(N,\delta)} - \mathbf{E}) + \bar{R}_N, \quad \textit{where} \;\; \mathbf{E}(|\bar{R}_N|) = O(\delta^{1/2})$$

(c) If $1 < s < 2$ *then for* $\omega \in \Omega_{N,\delta}$

$$\mathfrak{t}_N = \sum_j \Theta_j^{(N,\delta)} (\Gamma_j^{(N,\delta)} - 1) + R_N, \quad \textit{where} \;\; \mathbf{E}(|R_N|) = O(\delta^{1-s/2})$$

$$\mathfrak{u}_N = \sum_j (\Theta_j^{(N,\delta)} - \mathbf{E}) + \bar{R}_N, \quad \textit{where} \;\; \mathbf{E}(|\bar{R}_N|) = O(\delta^{1-s/2})$$

Given a Θ let F^Θ be the conditional distribution function of Y. The following statements are easy corollaries of Theorem 6.6.

(a) If $0 < s < 2$, $s \neq 1$ then F_N^ω converges weakly to F^Θ.

(b) If $1 < s < 2$ then $\left(F_N^\omega, \frac{\mathbb{E}_\omega(T_N) - \mathbf{E}(T_N)}{N^{1/s}}\right)$ converges weakly to

$$\left(F^\Theta, \left(\sum_{\Theta_j > \delta} \Theta_j - \frac{\bar{c}}{(s-1)\delta^{s-1}}\right)\right).$$

(c) If $s = 1$ then there exists $u_N \sim \bar{c} N \ln N$ such that $\left(F_N^\omega, \frac{\mathbb{E}_\omega(T_N) - u_N}{N}\right)$ converges weakly to

$$\left(F^\Theta, \left(\sum_{\Theta_j > \delta} \Theta_j - E(\delta)\right)\right)$$

where $E(\delta) = \mathbf{E}\left(\sum_{\delta < \Theta_j < 1} \Theta_j\right)$. We complete the picture by stating the result for the case $s \geq 2$.

THEOREM 6.7. *(a) If* $s = 2$ *and* (6.3) *holds then there are constants* D_1, D_2 *such that*

$$\left(\frac{T_N - \mathbb{E}_\omega(T_N)}{\sqrt{N \ln N}}, \frac{\mathbb{E}_\omega(T_N) - \mathbf{E}(T_N)}{\sqrt{N \ln N}}\right)$$

converge weakly to $(\mathcal{N}_1, \mathcal{N}_2)$ *where* $\mathcal{N}_1$ *is the distribution of the Gaussian random variable with zero mean and variance* D_1 *and* $\mathcal{N}_2$ *is the Gaussian random variable with zero mean and variance* D_2 *independent of* $\mathcal{N}_1$*. Moreover* $\frac{T_N - \mathbb{E}_\omega(T_N)}{\sqrt{N \ln N}}$*, is asymptotically independent of the environment in the sense that there is a set* Ω_N *such that* $\mathbf{P}(\Omega_N) \to 1$ *and*

$$F_N^\omega(x) \to \mathbf{P}(\mathcal{N}_1 \leq x) \text{ uniformly for } \omega \in \Omega_N.$$

(b) If $s > 2$ *then there are constants* D_1, D_2 *such that*

$$\left(\frac{T_N - \mathbb{E}_\omega(T_N)}{\sqrt{N}}, \frac{\mathbb{E}_\omega(T_N) - \mathbf{E}(T_N)}{\sqrt{N}}\right)$$

converge weakly to $(\mathcal{N}_1, \mathcal{N}_2)$ *where* $\mathcal{N}_1$ *is the distribution of the Gaussian random variable with zero mean and variance* D_1 *and* $\mathcal{N}_2$ *is the Gaussian random variable with zero mean and variance* D_2 *independent of* $\mathcal{N}_1$*. Moreover* $\frac{T_N - \mathbb{E}_\omega(T_N)}{\sqrt{N}}$*, is asymptotically independent of the environment in the sense that there is a set* Ω_N *such that* $\mathbf{P}(\Omega_N) \to 1$ *and*

$$F_N^\omega(x) \to \mathbf{P}(\mathcal{N}_1 \leq x) \text{ uniformly for } \omega \in \Omega_N.$$

The reason why the hitting times do not always satisfy the Central Limit Theorem is the presence of traps which slow down the particle. It will be seen in the proofs that Theorems 6.6 and 6.7 state that if traps are ordered according to the expected time the walker spends inside the trap then the asymptotic distribution of traps is Poissonian with intensity $\frac{\mathbf{c}}{\theta^{1+s}}$. This result holds regardless of the value of s. However, if $s \geq 2$ then the time spent inside the traps is smaller than the time spent outside of the traps.

Proof of Theorem 6.4. If $0 < s < 1$ then our result follows from Theorem 6.7(a), Lemma 7.8(c) and Lemma 7.11(a).

If $1 < s < 2$ let

$$Y'_\delta = \sum_{\Theta_i > \delta} \Theta_i \Gamma_i, \quad Y''_\delta = \sum_{\Theta_i > \delta} \Theta_i, \quad Y_\delta = Y'_\delta - Y''_\delta.$$

Observe that $\mathbf{E}(Y'_\delta) = \mathbf{E}(Y''_\delta)$. By Theorem 6.7(b) $T_N - \mathbf{E}(T_N)$ is asymptotically distributed as

$$Y_\delta + (Y''_\delta - \mathbf{E}Y''_\delta) = (Y'_\delta - \mathbf{E}Y'_\delta).$$

Therefore the result follows by Lemma 7.8(c) and Lemma 7.11(b).

The proofs in case $s = 1$ is similar.

If $s \geq 2$ then the result follows from Theorem 6.7 and the fact that the sum of independent Gaussian variables is Gaussian. □

3. Preliminaries.

3.1. Occupation times. Recurrence relation. As before, let ξ_n be the number of visits to the site n and $\rho_n = \mathbb{E}_\omega \xi_n$.

LEMMA 6.8. *If $X_0 = 0$ then for $n \geq 0$*

$$\rho_n = p_n^{-1} q_{n+1} \rho_{n+1} + p_n^{-1} = p_n^{-1}(1 + \alpha_{n+1} + \alpha_{n+1}\alpha_{n+2} + \ldots), \tag{6.4}$$

where $\alpha_j = \frac{q_j}{p_j}$.

PROOF. Let η_n^+ and η_n^- be the number of passages of the edge $[n, n+1]$ in the forward, respectively, backward direction. Denote $\sigma_n^\pm = \mathbb{E}_\omega \eta_n^\pm$. We have

$$\rho_n = \sum_j \mathbb{P}_\omega(X_j = n) \text{ and } \sigma_n^+ = \sum_j \mathbb{P}_\omega(X_j = n, X_{j+1} = n+1).$$

Thus $\sigma_n^+ = \rho_n p_n$. Likewise $\sigma_n^- = \rho_{n+1} q_{n+1}$. Since $\xi_n \to +\infty$ we have that $\eta_n^+ - \eta_n^- = 1$ for $n \geq 0$. Hence

$$\rho_n p_n - \rho_{n+1} q_{n+1} = 1.$$

This implies the first relation in (6.4). The second one is obtained by iterating the first one. □

For future references, we shall introduce here several elementary but useful relations. We start with a direct corollary of (6.4):

$$\rho_{n-k} = p_{n-k}^{-1} \alpha_{n-k+1} \ldots \alpha_{n-1} q_n \rho_n + (1 + \alpha_{n-k+1} + \cdots + \alpha_{n-k+1} \ldots \alpha_{n-1}) p_{n-k}^{-1}. \tag{6.5}$$

Observe that ξ_n has geometric distribution with parameter $1/\rho_n$. Next, we introduce

$$\begin{aligned} z_n &:= 1 + \alpha_{n+1} + \alpha_{n+1}\alpha_{n+2} + \ldots + \alpha_{n+1} \ldots \alpha_{n+m} + \ldots. \\ &= 1 + \alpha_{n+1} + \alpha_{n+1}\alpha_{n+2} + \ldots + \alpha_{n+1} \ldots \alpha_{n+m} z_{n+m} \end{aligned} \tag{6.6}$$

It is clear that $z_n = 1 + \alpha_{n+1} z_{n+1}$, where α_{n+1} and z_{n+1} are independent random variables and the sequence $\{z_n\}_{-\infty<n<\infty}$ considered backward in time forms a Markov chain. Obviously, $\rho_n = p_n^{-1} z_n$ is a function on the phase space of a Markov chain $\{p_n, z_n\}$ (where p_n and z_n are independent). Since $\mathbf{E}(\ln \alpha) < 0$ the series in (6.4) and (6.6) converge $\mathbf{P}$-almost surely and the distributions of z_n and of (p_n, z_n) are the stationary measures of the respective processes. The following heavy tail property of these stationary measures plays a very important role in the sequel.

THEOREM 6.9. *([**19**]) If* (6.3) *holds then there exists c and $c^* > 0$ such that*

$$\lim_{x\to+\infty} x^s \mathbf{P}(z_n > x) = c, \quad \lim_{x\to+\infty} x^s \mathbf{P}(\rho_n > x) = c^*,$$

Note that here the second relation is a simple corollary of the first one because $\mathbf{P}(\rho_n > x) = \mathbf{E}(\mathbf{P}(z_n > xp_n|p_n) \sim \mathbf{E}(cx^{-s}p^{-s}) = cx^{-s}\mathbf{E}(p^{-s})$. We also see that $c^* = c\mathbf{E}(p^{-s})$.

To prove Theorem 6.9 we need a number of auxiliary estimates.

Call $b_j = \ln \alpha_j$, $y_j = \sum_{k=1}^j b_j$. By Exercise 2.3

$$\mathbf{P}(y_n \geq an) \sim \frac{C}{\sigma(a)\sqrt{n}} e^{-\gamma(a)n}.$$

Let $a_0 = \arg\min \frac{\gamma(a)}{a}$. We have $s = \frac{\gamma(a_0)}{a_0}$.

LEMMA 6.10. *(a)* $\mathbf{P}(\min y_{-n} \geq Y) \leq Ce^{-sY}$.
(b) $\mathbf{P}(\exists n_1, n_2 : n_2 > n_1 + k$ *and* $y_{-n_j} \geq Y$ *for* $j = 1, 2) \leq Ce^{-(sY+\tilde{s}k)}$.

PROOF. By Exercise 2.3

$$\mathbf{P}(y_n \geq Y) \approx \frac{C}{\sqrt{n}} \exp\left[-\left(\frac{\gamma(Y/n)}{Y/n}\right) Y\right]$$

$$\leq \frac{C}{\sqrt{n}} \exp[-sY] \exp\left[-c\left(\frac{Y}{n} - \alpha_0\right)^2 Y\right]. \tag{6.7}$$

The main contribution to this sum comes from $n \approx Y/a_0$. For those n

$$c\left(\frac{Y}{n} - a_0\right)^2 Y \leq \tilde{c}\frac{(Y - a_0 n)^2}{Y}.$$

Since

$$\sum_n \frac{1}{\sqrt{Y}} \exp\left[-\tilde{c}\frac{(Y - a_0 n)^2}{Y}\right] \leq \text{Const}$$

(a) follows.

To prove (b) we use the foregoing argument and the Markov property to conclude that

$$\sum_{m_1, m_2 : m_2 - m_1 > k} \tilde{\mu}(y_{m_1} \geq Y, y_{m_2} \geq Y) \geq \text{Const} e^{-(sY+\tilde{s}k)}.$$

□

LEMMA 6.11. *Suppose that $n, Y \to \infty$ so that $\frac{n - \frac{Y}{a_0}}{\sqrt{Y}} \to \beta$ Denote $\Omega_n = \{y_n \geq Y, y_m < Y$ for all $0 < m < n\}$. Let $g : \mathbb{R}^{2N+2} \to \mathbb{R}$ be a bounded continuous function. Then the following limits exist*

$$(a) \qquad \lim_{n\to\infty} \mathbf{P}(\Omega_n)\sqrt{Y}e^{sY};$$

$$(b) \qquad \lim_{n\to\infty} \mathbf{P}(1_{\Omega_n} g(y_n - Y, \alpha_{n-N}, \alpha_{n-N+1} \dots \alpha_n \dots \alpha_{n+N}))\sqrt{Y}e^{sY}.$$

Moreover both limits are bounded by $\mathrm{Const} e^{-c\beta^2}$.

PROOF. (a) By the argument of Lemma 6.10 it is enough to show that for each k the following limit exists

$$\lim_{n\to\infty} \tilde{\mu}(y_n \geq Y, y_m < Y \text{ for } n-k < m < n).$$

This can be reduced to computation of the limits

$$\mathbf{P}(\alpha_n \in I_n, \alpha_{n-1} \in I_{n-1} \dots \alpha_{n-k} \in I_{n-k}, y_{n-k} \in J)\sqrt{Y}e^{sY}. \tag{6.8}$$

Therefore part (a) follows from Exercise 2.3 and the Markov property. Likewise part (b) also reduces to computing limits (6.8). Finally the fact that above limits are $\mathcal{O}\left(e^{-c\beta^2}\right)$ follows from the estimates in the proof of Lemma 6.10 (see (6.7)). □

PROOF OF THEOREM 6.9. We have $z_0 = \sum_{n=0}^{\infty} e^{y_n}$. Let $Y = \ln t$. Take $M \gg 1$. We claim that terms with $y_n \leq Y - M$ can be ignored. Indeed for terms with $Y - M - 1 < y_n < Y - M$ to make a contribution greater than $e^{M/2}$ there should be at least $e^{M/2}/C$ such terms. By Lemma 6.10 the probability of such an event is

$$\mathcal{O}\left(\exp - [s(Y-M) + \tilde{s}\exp(M/2)]\right)$$

which establishes our claim. Therefore for large M and k we have

$$\sum_n \mu(1_{\Omega_{n,M}} 1_{B^-_{n,k,\varepsilon}})t^s - \varepsilon \leq \mathbf{P}(z_0 \geq t)t^s \leq \sum_n \mu(1_{\Omega_{n,M}} 1_{B^+_{n,k,\varepsilon}})t^s + \varepsilon$$

where

$$\Omega_{n,M} = \{y_n \geq Y - M, y_m < Y - M \text{ for } 0 \leq m < n\},$$

$$B^{\pm}_{n,k} = \sum_{|m-n|<k} e^{y_m} \geq t \mp \varepsilon.$$

The fact that the last sum has limit follows from Lemma 6.11(b). □

Next we derive a number of consequences of Theorem 6.9.

LEMMA 6.12. *There exist* $\varepsilon_1 > 0, \varepsilon_2 > 0, 0 < \beta < 1$ *such that for any* $\delta > 0$ *there are* N_δ *and* $C = C_\delta > 0$ *such that for* $N > N_\delta$ *one has:*
(a) If $k \leq \varepsilon_1 \ln N$ *then*

$$\mathbf{P}(\rho_n \geq \delta N^{1/s}, \rho_{n-k} \geq \delta N^{1/s}) \leq \frac{C\beta^k}{N};$$

(b) If $k \geq \varepsilon_1 \ln N$ then

$$\mathbf{P}(\rho_n \geq \delta N^{1/s}, \rho_{n-k} \geq \delta N^{1/s}) \leq CN^{-(\varepsilon_2+1)}.$$

PROOF. (a) It follows from (6.5) that

$$\begin{aligned}\rho_{n-k} &\leq p_{n-k}q_n\rho_n\alpha_n \dots \alpha_{n-k+1} + \bar{C}k\varepsilon_0^{-k} \\ &\leq \bar{C}\rho_n\alpha_{n-1}\dots\alpha_{n-k} + \bar{C}\varepsilon_1(\ln N)\varepsilon_0^{-\varepsilon_1 \ln N} \\ &\leq \bar{C}\rho_n\alpha_{n-1}\dots\alpha_{n-k} + \bar{C}N^{\frac{1}{2s}} = \bar{C}\rho_n A_{n,k} + \bar{C}N^{\frac{1}{2s}},\end{aligned}$$

where ε_0 is from (6.1), ε_1 is chosen so that $-\varepsilon_1 \ln \varepsilon_0 \leq \frac{1}{3s}$, N is sufficiently large, and $A_{n,k} := \alpha_{n-1}\dots\alpha_{n-k}$. We note that $A_{n,k}$ is independent of ρ_n. (Here the $\bar{C}$ does not depend on δ; the $n-k+1$ was replaced by $n-k$ just for convenience.) Next, there exist $\beta_1, \beta_2 < 1$ such that

$$\mathbf{P}(\alpha_{n-1}\dots\alpha_{n-k} \geq \beta_1^k) \leq \beta_2^k.$$

Indeed, if $0 < h < s$ and β_1 is such that $\mathbf{E}(\alpha^h) < \beta_1^h < 1$ then it follows from the Markov's inequality that

$$\mathbf{P}(\alpha_{n-1}\dots\alpha_{n-k} \geq \beta_1^k) \leq \frac{(\mathbf{E}(\alpha^h))^k}{\beta_1^{hk}} \equiv \beta_2^k.$$

We can now choose N_δ so that for $N > N_\delta$ we shall have

$$\begin{aligned}\mathbf{P}(\rho_n \geq \delta N^{1/s}, \rho_{n-k} \geq \delta N^{1/s}) &\leq \mathbf{P}(\rho_n \geq \delta N^{1/s}, \bar{C}\rho_n A_{n,k} + \bar{C}N^{\frac{1}{2s}} \geq \delta N^{1/s}) \\ &\leq \mathbf{P}(\rho_n \geq \delta N^{1/s}, \bar{C}\rho_n A_{n,k} \geq \frac{\delta}{2}N^{1/s})\end{aligned}$$

Finally, the right hand side in the above inequality is estimated as follows:

$$\begin{aligned}&\mathbf{P}(\rho_n \geq \delta N^{1/s},\ \bar{C}\rho_n A_{n,k} \geq \frac{\delta}{2}N^{1/s}) \\ &= \mathbf{P}(\rho_n \geq \delta N^{1/s},\ \bar{C}\rho_n A_{n,k} \geq \frac{\delta}{2}N^{1/s},\ A_{n,k} \leq \beta_1^k) \\ &+ \mathbf{P}(\rho_n \geq \delta N^{1/s},\ \bar{C}\rho_n A_{n,k} \geq \frac{\delta}{2}N^{1/s},\ A_{n,k} > \beta_1^k) \\ &\leq \mathbf{P}\left(\rho_n \geq \frac{\beta_1^{-k}\delta N^{1/s}}{2\bar{C}}\right) + \mathbf{P}(\rho_n > \delta N^{1/s} \text{ and } A_{n,k} > \beta_1^k) \leq \text{Const}\frac{\beta_1^{ks} + \beta_2^k}{N},\end{aligned}$$

where the last step makes use of Theorem 6.9 (hence the dependence of the Const on δ) and of independence of ρ_n and $A_{n,k}$.

(b) For any $\varepsilon_3 > 0$ we can write

(6.9)

$$\begin{aligned}
&\mathbf{P}(\rho_n \geq \delta N^{1/s},\, \rho_{n-k} \geq \delta N^{1/s})\\
&= \mathbf{P}(\delta N^{1/s} \leq \rho_n \leq \delta N^{\frac{1+\varepsilon_3}{s}},\, \rho_{n-k} \geq \delta N^{1/s}) + \mathbf{P}(\rho_n > \delta N^{\frac{1+\varepsilon_3}{s}},\, \rho_{n-k} \geq \delta N^{1/s})\\
&\leq \mathbf{P}(\delta N^{1/s} \leq \rho_n \leq \delta N^{\frac{1+\varepsilon_3}{s}},\, \rho_{n-k} \geq \delta N^{1/s}) + \mathbf{P}(\rho_n > \delta N^{\frac{1+\varepsilon_3}{s}})\\
&\leq \frac{\bar{\bar{C}}}{N^{1+\varepsilon_3}} + \mathbf{P}(\delta N^{1/s} \leq \rho_n \leq \delta N^{\frac{1+\varepsilon_3}{s}},\, \rho_{n-k} \geq \delta N^{1/s}),
\end{aligned}$$

where the last step follows from Theorem 6.9. It follows from (6.5) that

$$\rho_{n-k} \leq \bar{C} A_{n,k} \rho_n + \bar{C} B_{n,k},$$

where $B_{n,k} = 1 + \alpha_{n-k+1} + \cdots + \alpha_{n-k+1} \ldots \alpha_{n-1}$. The following inequalities are self-explanatory:

(6.10)

$$\begin{aligned}
&\mathbf{P}(\delta N^{1/s} \leq \rho_n \leq \delta N^{\frac{1+\varepsilon_3}{s}},\, \rho_{n-k} \geq \delta N^{1/s})\\
&\leq \mathbf{P}(\delta N^{1/s} \leq \rho_n \leq \delta N^{\frac{1+\varepsilon_3}{s}},\, \bar{C} A_{n,k} \rho_n + \bar{C} B_{n,k} \geq \delta N^{1/s})\\
&\leq \mathbf{P}(\delta N^{1/s} \leq \rho_n \leq \delta N^{\frac{1+\varepsilon_3}{s}},\, \bar{C} A_{n,k} \delta N^{\frac{1+\varepsilon_3}{s}} + \bar{C} B_{n,k} \geq \delta N^{1/s})\\
&= \mathbf{P}(\delta N^{1/s} \leq \rho_n \leq \delta N^{\frac{1+\varepsilon_3}{s}})\, \mathbf{P}(\bar{C} A_{n,k} \delta N^{\frac{1+\varepsilon_3}{s}} + \bar{C} B_{n,k} \geq \delta N^{1/s}),
\end{aligned}$$

where the last step is due to the independence of ρ_n and $(A_{n,k}, B_{n,k})$. Next, let $1 > h > 0$ be such that $\bar{\beta} = \mathbf{E}(\alpha^h) < 1$, then $\mathbf{E}(B_{n,k}^h) \leq (1 - \bar{\beta})^{-1}$. By Markov's inequality

(6.11)

$$\begin{aligned}
&\mathbf{P}(\bar{C} A_{n,k} \delta N^{\frac{1+\varepsilon_3}{s}} + \bar{C} B_{n,k} \geq \delta N^{1/s}) \leq \bar{C}^h \frac{\mathbf{E}(\delta^h N^{\frac{1+\varepsilon_3}{s} h} A_{n,k}^h + B_{n,k}^h)}{\delta^h N^{h/s}}\\
&\leq \bar{\bar{C}} N^{\frac{\varepsilon_3 h}{s}} \bar{\beta}^k + \bar{\bar{C}} N^{\frac{-h}{s}}.
\end{aligned}$$

Since $k \geq \varepsilon_1 \ln N$, we have that $N^{\frac{\varepsilon_3 h}{s}} \bar{\beta}^k \leq N^{\frac{\varepsilon_3 h}{s} + \varepsilon_1 \ln \bar{\beta}} = N^{-\bar{\varepsilon}}$ (with ε_3 sufficiently small so that to make $\bar{\varepsilon}$ strictly positive). Finally, it follows from Theorem 6.9, (6.10) and (6.11) that

(6.12)

$$\mathbf{P}(\delta N^{1/s} \leq \rho_n \leq \delta N^{\frac{1+\varepsilon_3}{s}},\, \rho_{n-k} \geq \delta N^{1/s}) \leq \text{Const} N^{-1-\min(\bar{\varepsilon},\, h/s)}.$$

This together with (6.9) implies the proof of (b). □

Next we need the fact that ρ_n is exponentially mixing. To prove this we use (6.5). Conditions (6.2) and (6.1), imply that there exist $\beta_1, \beta_2 < 1$ such that

$$\mathbf{P}\left(\max_{k>L} \alpha_n \alpha_{n-1} \ldots \alpha_{n-k-1} \geq \beta_1^L\right) \leq \beta_2^L.$$

Therefore for typical realization of α the dependence on ρ_n decays exponentially. We formulate this statement as follows. Let $\hat\rho_n$ be a random variable with distribution ν independent of $\{\alpha_j\}_{j\le n}$. For $k>0$ define

(6.13)
$$\hat\rho_{n-k} = p_{n-k}^{-1}\hat\rho_n q_n\alpha_{n-1}\dots\alpha_{n-k+1} + (\alpha_{n-1}\dots\alpha_{n-k+1}+\dots+1)p_{n-k}^{-1}.$$

Observe that $\{\hat\rho_{n-k}\}$ is a stationary sequence.

LEMMA 6.13. *Let $\hat\rho_{n-k}$ be defined by* (6.13) *with $\hat\rho_n$ having the stationary distribution ν and independent of $\{\alpha_j\}_{j\le n}$. Then there exist $K>0$ and β_1, $\beta_2<1$ such that for $k>K\ln\rho_n$*

$$\mathbf{P}\left(|\rho_{n-k}-\hat\rho_{n-k}|\ge\beta_1^k|\rho_n\right)\le\beta_2^k.$$

3.2. Occupation times. Correlations. We will make use of several elementary equalities and inequalities concerned with a Markov chain $Y=\{Y_t,\ t\ge0\}$ with a phase space of 3 sites and transition matrix

$$\begin{pmatrix}\bar p & \bar q & 0\\ \bar{\bar q} & \bar{\bar p} & \varepsilon\\ 0 & 0 & 1\end{pmatrix}. \tag{6.14}$$

Namely, let $\bar\eta$ and $\bar{\bar\eta}$ be the total numbers of visit to the first and the second site respectively. Set $U_1=E(\bar\eta|Y_0=1)$, $U_2=E(\bar\eta|Y_0=2)$, $V_1=E(\bar{\bar\eta}|Y_0=1)$, $V_2=E(\bar{\bar\eta}|Y_0=2)$. It follows easily from the standard first step analysis that

$$U_1=\frac{\varepsilon+\bar{\bar q}}{\varepsilon\bar q},\quad U_2=\frac{\bar{\bar q}}{\varepsilon\bar q},\quad V_1=V_2=\frac{1}{\varepsilon}. \tag{6.15}$$

Next, set $W_i=E(\bar\eta\bar{\bar\eta}|Y_0=i)$, where $i=1,2$. Once again, by the first step analysis, one easily obtains that

$$W_1=\bar pW_1+\bar qW_2+V_1,\quad W_2=\bar{\bar q}W_1+\bar{\bar p}W_2+U_2. \tag{6.16}$$

Solving (6.17) gives

$$W_1=V_1(U_1+U_2),\quad W_2=U_2(V_1+V_2) \tag{6.17}$$

and hence

$$\mathrm{Cov}(\bar\eta,\bar{\bar\eta}|Y_0=1)=\mathrm{Cov}(\bar\eta,\bar{\bar\eta}|Y_0=2)=V_1U_2. \tag{6.18}$$

It is a standard fact that $\bar\eta$ conditioned on $Y_0=1$ has geometric distribution whose parameter is thus U_1^{-1}. If our Markov chain starts from 1 it must visit 2 before being absorbed by 3. Hence the distribution of $\bar{\bar\eta}$ conditioned on $Y_0=1$ is the same as the distribution of $\bar{\bar\eta}$ conditioned on $Y_0=2$ and is geometric with parameter $V_2^{-1}=\varepsilon$. We therefor have that $\mathrm{Var}(\bar\eta|Y_0=1)=U_1^2-U_1$ and $\mathrm{Var}(\bar{\bar\eta}|Y_0=1)=V_2^2-V_2$. We can now compute the correlation coefficient of $\bar{\eta}$ and $\bar{\bar{\eta}}$ which, taking into account (6.15), can be presented as follows:

(6.19)

$$\mathrm{Corr}(\bar{\eta},\bar{\bar{\eta}}|Y_0=1)=\frac{V_1U_2}{\sqrt{(U_1^2-U_1)(V_2^2-V_2)}}=\frac{\bar{\bar{q}}}{\bar{\bar{q}}+\varepsilon}(1-U_1^{-1})^{-\frac{1}{2}}(1-V_2^{-1})^{-\frac{1}{2}}.$$

This formula implies lower and upper bounds for correlations in two different regimes: (a) when $\bar{\bar{q}}/\varepsilon \to 0$ and (b) when $\varepsilon \to 0$ while $\bar{q}$, $\bar{\bar{q}}$ remain separated from 0. Here is the precise statement we need.

LEMMA 6.14. *(a) Suppose that $U_1 \geq 1+c$, $V_2 \geq 1+c$, where $c > 0$. Then*

$$\mathrm{Corr}(\bar{\eta},\bar{\bar{\eta}}|Y_0=1) \leq \mathrm{Const}\,\frac{\bar{\bar{q}}}{\varepsilon} \equiv \mathrm{Const}\,\bar{\bar{q}}V_1. \tag{6.20}$$

(b) If $\bar{\bar{q}} \geq c$ and $\bar{q} \geq c$ for some $c > 0$ then for ε small enough, or, equivalently, U_1 large enough

$$\mathrm{Corr}(\bar{\eta},\bar{\bar{\eta}}|Y_0=1) \geq 1-\frac{\varepsilon}{c}, \quad \mathrm{Corr}(\bar{\eta},\bar{\bar{\eta}}|Y_0=1) \geq 1-\frac{1}{cU_1}. \tag{6.21}$$

PROOF. (a) Inequality (6.20) is an immediate corollary of (6.19).
(b) (6.19) can be written as

$$\mathrm{Corr}(\bar{\eta},\bar{\bar{\eta}}|Y_0=1)=\frac{\bar{\bar{q}}}{\bar{\bar{q}}+\varepsilon}\left(1-\frac{\varepsilon\bar{q}}{\bar{\bar{q}}+\varepsilon}\right)^{-\frac{1}{2}}(1-\varepsilon)^{-\frac{1}{2}}.$$

If $\frac{\varepsilon}{\bar{\bar{q}}}<1$ then it follows from here that

$$\mathrm{Corr}(\bar{\eta},\bar{\bar{\eta}}|Y_0=1)=1-\left(1-\frac{\bar{q}+\bar{\bar{q}}}{2}\right)\frac{\varepsilon}{\bar{\bar{q}}}+\mathcal{O}\left(\left(\frac{\varepsilon}{\bar{\bar{q}}}\right)^2\right). \tag{6.22}$$

Due to (6.15) and conditions of the Lemma we have $\varepsilon=\frac{\bar{\bar{q}}}{\bar{q}}\left(U_1^{-1}+\mathcal{O}(U_1^{-2})\right)$ and hence

$$\mathrm{Corr}(\bar{\eta},\bar{\bar{\eta}}|Y_0=1)=1-\left(1-\frac{\bar{q}+\bar{\bar{q}}}{2}\right)\frac{1}{\bar{q}U_1}+\mathcal{O}(U_1^{-2}). \tag{6.23}$$

(6.21) is now a simple corollary of (6.22) and (6.23). □

LEMMA 6.15. *There is a $C > 0$ such that for* **P***-almost all ω and $n \geq 0$*

$$\mathrm{Corr}_\omega(\xi_n,\xi_{n+1}) \geq 1-\frac{C}{\rho_n}. \tag{6.24}$$

PROOF. Let ω be such that the random walk X runs away to $+\infty$ with $\mathbb{P}_\omega$ probability 1 (which is the case for **P**-almost all ω). For a given $n \geq 0$ consider a Markov chain $Y=\{Y_t,\ t\geq 0\}$, with the state space $\{n,\ n+1,\ as\}$, where $n,\ n+1$ are sites on $\mathbb{Z}$ and as is an absorbing state. Let $k_0 < k_1 < ... < k_\tau$ be the sequence of all moments such that $X_{k_j} \in \{n, n+1\}$; we set $Y_t = X_{k_t}$ if $t \le \tau$ and $Y_t = as$ if $t > \tau$. It easy to see that the transition matrix of Y is as in (6.14) with transition probabilities given by

$$\bar{p} = q_n, \quad \bar{q} = p_n, \quad \bar{\bar{q}} = q_{n+1},$$

$$\bar{\bar{p}} = \mathbb{P}_\omega\{X_k \text{ starting from } n+1 \text{ returns to } n+1 \text{ before visiting } n\},$$

$$\varepsilon = \mathbb{P}_\omega\{X_t \text{ starting from } n+1 \text{ never returns to } n+1\}.$$

Also, in this context, $\bar{\eta} = \xi_n$, $\bar{\bar{\eta}} = \xi_{n+1}$ and hence $V_1 = \rho_n$. Next, $\bar{q}$, $\bar{\bar{q}}$ are separated from 0 because of condition (6.1). All conditions of Lemma 6.14 are thus satisfied and hence, for ρ_ns which are sufficiently large, (6.24) follows from (6.21). □

LEMMA 6.16. *(a) There exist sets Ω_N, $K > 0$ such that $\mathbf{P}(\Omega_N^c) \le N^{-100}$ and if $\omega \in \Omega_N$ then for all $0 \le n_1, n_2 \le N$ such that $n_2 > n_1 + K \ln N$ we have*

$$\mathrm{Corr}_\omega(\xi_{n_1}, \xi_{n_2}) \le N^{-100}.$$

(b) If K is sufficiently large then for each N there exist random variables $\{\bar{\xi}_n\}_{n=0}^N$ such that for each $\omega \in \Omega_N$ for any sequence $0 \le n_1 < n_2 \cdots < n_k \le N$ such that $n_{j+1} > n_j + K \ln N$, the variables $\{\bar{\xi}_{n_j}\}_{j=0}^k$ are mutually independent and

$$\mathbf{P}(\bar{\xi}_n = \xi_n \text{ for } n = 0, \ldots, N) \ge 1 - \frac{C}{N^{100}}. \tag{6.25}$$

PROOF. (a) Consider a Markov chain Y which is defined as in the proof of Lemma 6.15 with the difference that its state space is $\{n_1, n_2, as\}$ and that $\bar{\eta} = \xi_{n_1}$, $\bar{\bar{\eta}} = \xi_{n_2}$. Then by (6.20)

$$\mathrm{Corr}_\omega(\xi_{n_1}, \xi_{n_2}) \le \mathrm{Const}\, \bar{\bar{q}} \rho_{n_2}.$$

But, by Theorem 6.9, $\rho_n \le N^{\frac{103}{s}}$ except for the set of measure $\mathcal{O}(N^{-103})$. Now Lemma 6.3 guarantees that we can choose K so that if the sites are separated by $K \ln N$ then $\bar{\bar{q}} < N^{-(101+103/s)}$ except for the set of measure $\mathcal{O}(N^{-103})$. This proves (a) for fixed n_1, n_2 on a set of measure $\ge 1 - \mathcal{O}(N^{-103})$ which in turn implies the wanted result.

(b) Let $\bar{\xi}_n$ be the number of visits to the site n before the first visit to $n + \frac{K \ln N}{2}$. It follows from this definition that $\{\bar{\xi}_{n_j}\}_{j=0}^k$ are mutually independent. Next,

$$\mathbf{P}(\bar{\xi}_n = \xi_n) \le \mathbf{P}(X \text{ visits } n \text{ after } n + 0.5K \ln N)$$

Now (6.25) follows from Lemma 6.3. □

4. Convergence to the stable law.

In this section we will prove Theorem 6.6. Our goal is to show that the main contribution to T_N comes from the terms where ρ_n is large. However, the set where ρ_n is large has an additional structure. Namely, if ρ_n is large the same is true for $\rho_{n\pm 1}$ and more generally for ρ_{n_1} and ρ_{n_2} when n_1 and n_2 are in a sense close to n; this implies that the corresponding ξ_{n_1} and ξ_{n_2} are strongly correlated. But if n_1 and n_2 are far apart then ρ_{n_1} and ρ_{n_2}, and also ξ_{n_1} and ξ_{n_2}, are almost independent. In the arguments below we need to take care about this additional structure.

But first we show that terms where $\rho_n < \delta N^{1/s}$ can be neglected.

LEMMA 6.17. *Let $\delta > 0$. Then there is N_δ (which depends also on s) such that for $N > N_\delta$ the following holds:*

(a) If $0 < s < 1$ then

$$\mathbf{E}\left(\sum_{\rho_n < \delta N^{1/s}} \xi_n\right) \leq \mathrm{Const} N^{1/s} \delta^{1-s}.$$

(b) If $1 < s < 2$ then there is an $\tilde{\Omega}_{N,\delta}$ such that $\mathbf{P}(\tilde{\Omega}_{N,\delta}) \to 1$ as $N \to \infty$ and for $\omega \in \tilde{\Omega}_{N,\delta}$

$$\mathbb{E}_\omega \left(\sum_{\rho_n < \delta N^{1/s}} (\xi_n - \rho_n)\right)^2 \leq \mathrm{Const} N^{2/s} \delta^{2-s}.$$

(c) If $0 < s < 1$ then

$$\mathbf{E}\left(\sum_{\rho_n < \delta N^{1/s}} \rho_n\right) \leq \mathrm{Const} N^{1/s} \delta^{1-s}.$$

(d) If $1 < s < 2$ then

$$\mathbf{P}\left(\sum_{\rho_n < \delta N^{1/s}} (\rho_n - \mathbf{E}(\rho)) > \varepsilon N^{1/s}\right) \to 0.$$

(e) If $s = 1$ then

$$\mathbf{P}\left(\sum_{\rho_n < \delta N} (\xi_n - \rho_n) > \varepsilon N\right) \to 0 \tag{6.26}$$

$$\mathbf{P}\left(\sum_{\rho_n < \delta N} (\rho_n - \mathbf{E}(\rho I_{\rho < \delta N})) > \varepsilon N\right) \to 0 \tag{6.27}$$

PROOF. (a) Denote $Y_\delta = \sum_{\rho_n < \delta N^{1/s}} \xi_n$. Then

$$\mathbf{E}(Y_\delta) = N\mathbf{E}(\rho I_{\rho < \delta N^{1/s}}).$$

By Theorem 6.9 this expectation is bounded by $\text{Const} N^{1/s}\delta^{1-s}$ proving our claim.

(b) Denote $\tilde{Y}_\delta = \sum_{\rho_n < \delta N^{1/s}} (\xi_n - \rho_n)$. Then $\mathbb{E}_\omega(\tilde{Y}_n) = 0$ and so it suffices to show that $\text{Var}_\omega(\tilde{Y}_\delta) = o(N^{2/s})$ except for a set of small probability. Due to Lemma 6.16 for most ωs we have

(6.28)

$$\begin{aligned}\text{Var}_\omega(\tilde{Y}_\delta) &= \left| o(1) + \sum_{n_2 - K\ln N < n_1 < n_2} 2\text{Cov}_\omega\left(\xi_{n_1}\xi_{n_2}\right) + \sum_n \text{Var}_\omega\left(\xi_n\right) \right| \\ &\leq 1 + \text{Const} \sum_{n_2 - K\ln N < n_1 \leq n_2} \rho_{n_1}\rho_{n_2}\end{aligned}$$

where the summation is over pairs with $\rho_{n_i} < \delta N^{1/s}$. The last step uses Cauchy-Schwartz inequality and the fact that ξ_n has geometric distribution, namely $|\text{Cov}_\omega\left(\xi_{n_1}\xi_{n_2}\right)| \leq \sqrt{\text{Var}_\omega\left(\xi_{n_1}\right)\text{Var}_\omega\left(\xi_{n_2}\right)} \leq \rho_{n_1}\rho_{n_2}$.

Next, we estimate the expectation of the last sum in (6.28). Set $\chi_n = I_{\rho_n < \delta N^{1/s}}$ and $\beta = \mathbf{E}(\alpha) < 1$; these concise notations will be used only within this proof. Using (6.5) we can write

$$\rho_{n-k}\rho_n = p_{n-k}^{-1}\rho_n^2 q_n \alpha_{n-1}\dots\alpha_{n-k+1} + (\alpha_{n-1}\dots\alpha_{n-k+1} + \dots + 1)p_{n-k}^{-1}\rho_n.$$

Since ρ_n and $\{\alpha_j,\ j < n\}$ are independent we obtain

$$\mathbf{E}\left(\rho_{n-k}\rho_n\chi_n\right) \leq \text{Const}\left[\beta^{k-1}\mathbf{E}\left(\rho_n^2\chi_n\right) + \mathbf{E}\left(\rho_n\chi_n\right)\sum_{j=0}^{k-2}\beta^j\right]$$

Thus

$$\mathbf{E}\left(\sum_{k=0}^{K\ln N}\left(\rho_{n-k}\rho_n\chi_n\right)\right) \leq \text{Const}\left[\mathbf{E}\left(\rho_n^2\chi_n\right) + \ln N\mathbf{E}\left(\rho_n\chi_n\right)\right]. \tag{6.29}$$

Hence

$$\begin{aligned}\mathbf{E}\left(\sum_{n_2 - K\ln N < n_1 < n_2}\rho_{n_1}\rho_{n_2}\chi_{n_1}\chi_{n_2}\right) &\leq \mathbf{E}\left(\sum_{n_2 - K\ln N < n_1 < n_2}\rho_{n_1}\rho_{n_2}\chi_{n_2}\right) \\ \leq \text{Const}\sum_{n_2}\left[\mathbf{E}\left(\left(\rho_{n_2}\right)^2\chi_{n_2}\right) + \ln N\mathbf{E}(\rho_{n_2})\right] &\leq \text{Const}\delta^{2-s}N^{2/s}.\end{aligned}$$

(c) The proof of (c) is the same as proof of (a).

(d) We assume first that

$$\nu([\delta N^{1/s} - N^{-100}, \delta N^{1/s} + N^{-100}]) \leq N^{-50}. \tag{6.30}$$

In view of Theorem 6.9 it is enough to estimate

$$\boldsymbol{\sigma} = \sum_{\rho_n < \delta N^{1/s}} \left(\rho_n - \mathbf{E}(\rho_n I_{\rho_n < \delta N^{1/s}})\right)$$

(recall that we can take $\delta \ll \varepsilon$). To estimate $\boldsymbol{\sigma}$ we compute its variance. Lemma 6.13 shows that if (6.30) holds then for $|n_2 - n_1| \geq \tilde{K} \ln N$

$$\mathrm{Cov}\left(\rho_{n_1} I_{\rho_{n_1} < \delta N^{1/s}}, \rho_{n_2} I_{\rho_{n_2} < \delta N^{1/s}}\right) < \frac{1}{N^3}$$

provided that $\tilde{K}$ is large enough. Hence

$$|\mathrm{Var}(\boldsymbol{\sigma})| \leq 1 + \left| \sum_{|n_1 - n_2| < \tilde{K} \ln N} \mathrm{Cov}\left(\rho_{n_1} I_{\rho_{n_1} < \delta N^{1/s}}, \rho_{n_2} I_{\rho_{n_2} < \delta N^{1/s}}\right) \right|.$$

provided that $\tilde{K}$ is sufficiently large. The estimate of the last sum is exactly the same as in part (b). This completes the proof of part (d) in the case when (6.30) holds. In case (6.30) fails we can repeat the computation below with δ replaced by δ' and δ'' where δ', δ'' satisfy (6.30) and such that

$$\delta < \delta' < \delta + N^{-(50+1/s)}, \quad \delta - N^{-(50+1/s)} < \delta'' < \delta.$$

The bound for δ' will allow us to estimate the sum of part (d) from above and the bound for δ'' will allow us to estimate the sum of part (d) from below.

(e) We prove (6.26), (6.27) is similar. In view of (6.28) it suffices to show that for a set of ω of measure close to 1 we have

$$\sum_{n_1 \leq n_2 < n_1 + K \ln N, \rho_{n_1} < \delta N, \rho_{n_2} < \delta N} \rho_{n_1} \rho_{n_2} \leq \varepsilon N.$$

Take $\kappa < 1$. We have

$$\mathbf{E}\left(\sum_{n_1 \leq n_2 < n_1 + K \ln N, \rho_{n_1} < \delta N, \rho_{n_2} < \delta N} \rho_{n_1} \rho_{n_2} \right)^{\kappa}$$

$$\leq \sum_{n_1 \leq n_2 < n_1 + K \ln N} \mathbf{E}\left((\rho_{n_1} \rho_{n_2})^{\kappa} I_{\rho_{n_1} < \delta N} I_{\rho_{n_2} < \delta N} \right).$$

Using that $\mathbf{E}(\alpha^{\kappa}) < 1$ we can proceed as in part (b) to estimate the last sum by

$$C \sum_n \mathbf{E}\left((\rho_n)^{\kappa} I_{\rho_n < \delta N} \right) = \tilde{C} N^{2\kappa} \delta^{2\kappa}.$$

$\square$

Lemma 6.17 allows us to concentrate on sites where $\rho_n > \delta N^{1/s}$. In view of Theorem 6.9 for each fixed δ we expect to have finitely many such points on $[0, N]$ (namely the expected number of points is $\mathcal{O}(\delta^{-s})$).

Let $M = M_N := \ln \ln N$. We shall say that n is a *massive* site if $\rho_n \geq \delta N^{1/s}$. A site $n \in [0, N-1]$ is *marked* if it is massive and $\rho_{n+j} < \delta N^{1/s}$ for $1 \leq j \leq M$. For n marked the interval $[n-M, n]$ is called the *cluster* associated to n.

It may happen that not all massive sites belong to one of the clusters. This situation is controlled by the following

LEMMA 6.18.

$$\mathbf{P}\left(\rho_n \geq \delta N^{1/s} \textit{ and } n \textit{ is not in a cluster}\right) \leq \mathrm{Const}\frac{\beta^M}{N}. \tag{6.31}$$

PROOF. Suppose that n is a massive point which is not in a cluster. Then consider all massive points such that $n < n_1 < ... < n_k < n+M$. Note that such points exist because otherwise n would have been a marked point. Let now $n^* > n_k$ be the nearest to n_k massive point. Then by construction $n^* \geq n+M$. Also $n^* \leq n+2M$ because otherwise n_k would have been a marked point and n would belong to the n_k-cluster. Hence the event

$$\{n \text{ is massive and not in a cluster}\} \subset \bigcup_{n' \in [n+M, n+2M]} \{\rho_n \geq \delta N^{1/s},\ \rho_{n'} \geq \delta N^{1/s}\}.$$

By Lemma 6.12(a) we obtain

$$\begin{aligned} &\mathbf{P}\left(n \text{ is massive and not in a cluster}\right) \\ &\leq \sum_{n'=n+M}^{n+2M} \mathbf{P}\left(\rho_n \geq \delta N^{1/s},\ \rho_{n'} \geq \delta N^{1/s}\right) \leq \mathrm{Const}\frac{\beta^M}{N} \end{aligned}$$

which proves our statement. □

We shall now turn to the analysis of the properties of clusters. The next lemma is the main technical result of the paper. It will be proved in Section 5.

We need one more definition. For each marked point n, we set

$$a_n = \rho_n / \delta N^{1/s}, \quad b_n = \frac{\Sigma_{j=0}^M \rho_{n-j}}{\rho_n} \text{ and } m_n = \Sigma_{j=0}^M \rho_{n-j} = \delta N^{1/s} a_n b_n. \tag{6.32}$$

We call m_n the mass of the cluster.

LEMMA 6.19. *For a given $\delta > 0$ the following holds:*

(a) The point process $\{(\frac{n}{N}, a_n, b_n) : \quad n \quad$ *is marked* $\}$ *converges as* $N \to \infty$ *to a point process* $\{(t_j, \tilde{a}_j, \tilde{b}_j)\}$ *where* t_j *form a Poisson process with a constant intensity* $\tilde{c}\delta^{-s}$.

(b) For a given (finite) collection $\{t_j\}$ *the corresponding collection* $\{(\tilde{a}_j, \tilde{b}_j)\}$ *consist of i.i.d. random variables which are independent of* $\{t_j\}$ *(except that both collections have the same cardinality). The distributions of the pair* $(\tilde{a}, \tilde{b})$ *does not depend on* δ.

(c) Consequently[2] $\{\left(\frac{n}{N}, \frac{m_n}{N^{1/s}}\right)\}$ *converges to a Poisson process* $\Lambda^\delta = \{(t_j, \Theta_j)\}$ *on* $[0,1] \times [\delta, \infty)$.

We claim that Λ^δ has a limit as $\delta \to 0$ in the following sense. Let Φ be a continuous function whose support is a compact set disjoint from the segment $\{\theta = 0\}$. Then

$$\lim_{\delta \to 0} \mathbf{E}_{\Lambda^\delta} \left(\sum_j \Phi(t_j, \Theta_j) \right)$$

exists. Indeed let $\tilde{\delta} < \delta$. Consider again the converging sequence $\{\left(\frac{n}{N}, \frac{m_n}{N^{1/s}}\right)\}$ corresponding to $\tilde{\delta}$. We may have more clusters corresponding to $\tilde{\delta}$ but for any fixed b it is unlikely that one of those new clusters will have mass greater than $bN^{1/s}$ since this would mean that all points in that cluster would have

$$\rho_n < \delta N^{1/s}, \text{ but } \sum_{j=0}^{M} \rho_{n-j} > \frac{bN^{1/s}}{\delta}, \tag{6.33}$$

where the sum is over n in the additional cluster. But (6.33) is unlikely in view of parts (c) and (d) of Lemma 6.17.

The second distinction between Λ^δ and $\Lambda^{\tilde{\delta}}$ is the following. Consider a δ-cluster $\bar{C}$ and a $\tilde{\delta}$-cluster $\bar{\bar{C}}$ intersecting it. Then $\bar{C}$ and $\bar{\bar{C}}$ are shifted with respect to each other so they have different masses. However, with probability close to 1 the masses of all such pairs of clusters differ by a relatively small amount. Indeed $\bar{\bar{C}} \setminus \bar{C}$ always contains only sites with $\rho_n < \delta N^{1/s}$ and $\bar{C} \setminus \bar{\bar{C}}$ is unlikely to contain sites where $\rho_n \geq \delta N^{1/s}$ due to Lemma 6.18. On the other hand, from Lemma 6.17 we know that terms with $\rho_n < \delta N^{1/s}$ are unlikely to make a large contribution.

Let $\Lambda = \lim_{\delta \to 0} \Lambda^\delta$. Let $\{\Theta_j\}$ be the projection of Λ into the second coordinate. By Lemma 7.8(a), $\{\Theta_j\}$ is a Poisson process.

LEMMA 6.20. *There exists* $\mathbf{c}$ *such that the intensity of* $\{\Theta_j\}$ *equals to* $\frac{\mathbf{c}}{\theta^{1+s}}$.

[2]part (c) of Lemma 6.19 follows from parts (a) - (b) and Lemma 7.8.

Proof. For each κ, we have $\Lambda = \lim_{\delta\to 0} \Lambda^{\kappa\delta}$. Λ^{δ} depends on δ in two ways. First its intensity is proportional to δ^{-s}. Second, $\Theta_j/\delta = \tilde{a}_j\tilde{b}_j$. Recall that the distribution of $\tilde{a}_j\tilde{b}_j$ is independent of δ. Therefore replacing δ by $\kappa\delta$ replaces $\Theta \to \kappa\Theta$ and multiplies the intensity by κ^{-s}. In other words re-scaling $\{\Theta_j\}$ by κ amounts to multiplying its intensity by κ^{-s}. Now the result follows from Lemma 7.8(a). □

We are now in a position to finish the proof of Theorem 6.6. We shall do that in the case $0 < s < 1$. In all other cases the proof is similar.

Present the time spent by the walk in $[0, N)$ as

$$T_N = \sum_{n=0}^{N-1} \xi_n = S_1 + \ S_2 + \ S_3, \tag{6.34}$$

where

$$S_1 = \sum_{n\colon \rho_n < \delta N^{1/s},\, n \notin \text{any cluster}} \xi_n$$

$$S_2 = \sum_{n\colon \rho_n \geq \delta N^{1/s},\, n \text{ is not in a cluster}} \xi_n$$

$$S_3 = \sum_{n\colon n \text{ is in a cluster}} \xi_n.$$

By Lemma 6.17, (a) we have that $\mathbf{E}(S_1) \leq \text{Const} N^{1/s}\delta^{1-s}$. Next by Lemma 6.18 we have that $\mathbf{P}(S_2 > 0) \to 0$ as $N \to \infty$. We readily have that for $\omega \notin \bar{\Omega}^{\delta}_N$

$$\mathfrak{t}_N = N^{-\frac{1}{s}} S_3 + N^{-\frac{1}{s}} S_1 = N^{-\frac{1}{s}} S_3 + R_N,$$

where $R_N := N^{-\frac{1}{s}} S_1$ and satisfies the requirements of (a), Theorem 6.6.

Next, consider S_3 which comes from the sum over the clusters and is the main contribution to T_N. Let us present it as follows:

$$N^{-\frac{1}{s}} S_3 = \sum_{n\colon n \text{ is marked}} N^{-\frac{1}{s}} \sum_{j=0}^{M} \xi_{n-j}.$$

In turn

$$\sum_{j=0}^{M} \xi_{n-j} = \sum_{j=1}^{M} \left(\frac{\xi_{n-j}}{\rho_{n-j}} - \frac{\xi_n}{\rho_n} \right) \rho_{n-j} + \frac{\xi_n}{\rho_n} \sum_{j=0}^{M} \rho_{n-j}$$

Next, using Lemma 6.15 and the fact that ξ_n is a geometric random variable and therefore $\mathrm{Var}_\omega(\xi_n) = \rho_n^2 - \rho_n$ one obtains

$$\left\| \frac{\xi_{n-j}}{\rho_{n-j}} - \frac{\xi_n}{\rho_n} \right\| \leq \sum_{k=n-j}^{n-1} \left\| \frac{\xi_k}{\rho_k} - \frac{\xi_{k+1}}{\rho_{k+1}} \right\| \leq \text{Const} \sum_{k=n-j}^{n-1} \frac{1}{\sqrt{\rho_k}}.$$

Here and below $\|f\| := \sqrt{\mathbb{E}_\omega(|f|^2)}$ with f being a function on the space of trajectories of the walk.

For $n-j$ belonging to a cluster, that is $(n-j) \in [n-M, n]$ we have that $\rho_{n-j} \geq c\varepsilon_0^M \rho_n \geq cN^{-\bar{\varepsilon}}\rho_n$. (Remember that if ε_1 in Lemma 6.12 is small enough then $\bar{\varepsilon}$ can be made very small which is what we shall use in this proof.) Thus

$$\left\| \sum_{j=1}^{M} \left(\frac{\xi_{n-j}}{\rho_{n-j}} - \frac{\xi_n}{\rho_n} \right) \rho_{n-j} \right\| \leq \text{Const} \frac{N^{\bar{\varepsilon}/2}}{\sqrt{\rho_n}} \sum_{j=1}^{M} \rho_{n-j}$$

If for n marked we set

$$\zeta_n = m_n^{-1} \sum_{j=1}^{M} \left(\frac{\xi_{n-j}}{\rho_{n-j}} - \frac{\xi_n}{\rho_n} \right) \rho_{n-j}$$

then $\|\zeta_n\| \leq \text{Const} \frac{N^{\bar{\varepsilon}/2}}{\sqrt{\rho_n}} \to 0$ as $N \to \infty$ and we have

$$\frac{\sum_{j=0}^{M} \xi_{n-j}}{N^{1/s}} = \left(\frac{\xi_n}{\rho_n} + \zeta_n \right) \frac{m_n}{N^{1/s}},$$

Next ξ_n/ρ_n is asymptotically exponential with mean 1 since ξ_n is geometric with parameter $1/\rho_n$. Also by Lemma 6.16 $\{\frac{\xi_n}{\rho_n}\}_{n \text{ is marked}}$ are asymptotically independent. On the other hand $\{\frac{m_n}{N^{1/s}}\}_{n \text{ is marked}}$ are asymptotically Poisson by Lemma 6.19. In other words, we proved the statement (a) of Theorem 6.6 with

$$(\Theta^{N,\delta}, \Gamma^{N,\delta}) = \left(\left\{ \frac{m_n}{N^{1/s}}, \frac{\xi_n}{\rho_n} + \zeta_n \right\}_{n \text{ is marked}} \right).$$

5. Poisson Limit for expected occupation times.

In this section we prove lemma 6.19.

To understand the asymptotic properties of the distribution of a_n defined in (6.32) we need the following

Lemma 6.21. *(a) For each $m > 0$ and $y \geq 1$*

(6.35)
$$\mu^m(y) := \lim_{N\to\infty} N\mathbf{P}\left(\frac{\rho_n}{N^{1/s}} \geq \delta y, \rho_{n+1} < \delta N^{1/s}, \dots, \rho_{n+m} < \delta N^{1/s}\right)$$
$$= \delta^{-s} c\mathbf{E}[(D_0^s y^{-s} - \max_{1\leq j\leq m} D_j^s) I_{\max_{1\leq j\leq m} D_j < D_0 y^{-1}}],$$

where $D_j := p_{n+j}^{-1}\alpha_{n+j+1}...\alpha_{n+m}$ (and by convention $D_m := p_{n+m}^{-1}$).
(b) There exists $\mu^\infty(y) = \lim_{m\to\infty} \mu^m(y)$.
(c) $\mu^\infty(1) > 0$.

Proof. (a) We shall make use of (6.6) and the relation $\rho_n = p_n^{-1} z_n$. For a fixed m and $0 \leq j \leq m$ we can write

$$\rho_{n+j} = p_{n+j}^{-1}\alpha_{n+j+1}...\alpha_{n+m} z_{n+m} + \mathcal{O}(1).$$

Set $D_j := p_{n+j}^{-1}\alpha_{n+j+1}...\alpha_{n+m}$ (with a convention that $D_m := p_{n+m}$). The inequalities $\rho_n/N^{1/s} \geq \delta y$ and $\rho_{n+j} < \delta N^{1/s}$ in (6.35) are equivalent to

$$z_{n+m} \geq N^{1/s}\delta(D_0^{-1}y + \mathcal{O}(N^{-1/s})) \text{ and } z_{n+m} < N^{1/s}\delta(D_j^{-1} + \mathcal{O}(N^{-1/s}))$$

respectively. Thus

$$\mathbf{P}\left(\frac{\rho_n}{N^{1/s}} \geq \delta y, \rho_{n+1} < \delta N^{1/s}, \dots, \rho_{n+m} < \delta N^{1/s}\right) =$$
$$\mathbf{P}\left(N^{1/s}\delta(D_0^{-1}y + \mathcal{O}(N^{-1/s})) \leq z_{n+m} < N^{1/s}\delta \min_{1\leq j\leq m}(D_j^{-1} + \mathcal{O}(N^{-1/s}))\right).$$

Since z_{n+m} and $\{p_{n+j}\}_{j\leq m}$ are independent, we can compute the following limit by conditioning on $\{p_{n+j}\}_{j\leq m}$ and using Theorem 6.9:

$$\lim_{N\to\infty} N\mathbf{P}\left(\rho_n N^{-1/s} \geq y, \rho_{n+1} < \delta N^{1/s}, \dots, \rho_{n+m} < \delta N^{1/s} | \{p_{n+j}\}_{j\leq m}\right) =$$
$$\delta^{-s} c(D_0^s y^{-s} - \max_{1\leq j\leq m} D_j^s) I_{\max_{1\leq j\leq m} D_j < D_0 y^{-1}}.$$

To compute the limit (6.35), it remains to take the expectation with respect to $\{p_{n+j}\}_{j\leq m}$:

$$\lim_{N\to\infty} N\mathbf{P}\left(\rho_n N^{-1/s} \geq y, \rho_{n+1} < \delta N^{1/s}, \dots, \rho_{n+m} < \delta N^{1/s}\right) =$$
$$\delta^{-s} c\mathbf{E}[(D_0^s y^{-s} - \max_{1\leq j\leq m} D_j^s) I_{\max_{1\leq j\leq m} D_j < D_0 y^{-1}}].$$

This completes the proof of part (a).
(b) The probability $\mathbf{P}\left(\frac{\rho_n}{N^{1/s}} \in [c, d], \rho_{n+1} \leq \delta N^{1/s}, \dots, \rho_{n+m} \leq \delta N^{1/s}\right)$ is a monotonically decaying function of m. Hence the proof.
(c) If $\mu^\infty = 0$ then $N\mathbf{P}(n \text{ is marked}) \to 0$ as $N \to \infty$. Consequently for each fixed j $N\mathbf{P}(n \text{ is marked and } \rho_{n-j} \geq \delta N^{1/s}) \to 0$ as $N \to \infty$.

Thus the Dominated Convergence Theorem implies via Lemma 6.12(a) that

(6.36)

$$\mathbf{E}(\text{Card}(n \le N : \rho_n \ge \delta N^{1/s} \text{ and } n \text{ belongs to a cluster})) \to 0.$$

On the other hand by Lemma 6.18

(6.37)

$$\mathbf{E}(\text{Card}(n \le N : \rho_n \ge \delta N^{1/s} \text{ and } n \text{ does not belong to a cluster})) \to 0.$$

Combining (6.36) and (6.37) we would get

$$\mathbf{E}(\text{Card}(n \le N : \rho_n \ge \delta N^{1/s})) \to 0$$

contradicting Theorem 6.9. This proves (c). □

REMARK. The measure μ_∞ depends on δ. We do not emphasize this dependence in our notation, however, it is easy to see from the proof that

$$\mu_\infty([\delta c, \delta d]) = \delta^{-s} \bar{\mu}_\infty([c, d])$$

where the measure $\bar{\mu}_\infty$ does not depend on δ.

Lemma 6.21 gives the limiting distribution of $\tilde{a}_j$ in Lemma 6.19. Next we address the distribution of $\tilde{b}_j$.

LEMMA 6.22. *The distribution of* $\frac{\sum_{j=0}^{M} \rho_{n-j}}{\rho_n}$ *conditioned on* $\rho_n \ge \delta N^{1/s}$ *converges as* $N \to \infty$ *to the distribution of*

$$1 + p_{-1}^{-1} q_0 + p_{-2}^{-1} q_0 \alpha_{-1} + \cdots + p_{-k}^{-1} q_0 \alpha_{-1} \dots \alpha_{-k+1} + ...$$

PROOF. According to (6.5)

$$\rho_{n-j} = p_{n-j}^{-1} q_n \alpha_{n-1} \dots \alpha_{n-j+1} \rho_n + \mathcal{O}(K^M).$$

Since $K^M \ll N^{1/s}$ we see that

$$\frac{\sum_{j=0}^{M} \rho_{n-1}}{\rho_n} = 1 + p_{n-1}^{-1} q_n + p_{n-2}^{-1} q_n \alpha_{n-1} + \cdots + p_{n-M}^{-1} q_n \alpha_{n-1} \dots \alpha_{n-M+1} + o(1).$$

As $N \to \infty$, also $M = M_N \to \infty$ and so the limiting distribution of the above expression is the same as the distribution of

$$1 + p_{-1}^{-1} q_0 + p_{-2}^{-1} q_0 \alpha_{-1} + \cdots + p_{-k}^{-1} q_0 \alpha_{-1} \dots \alpha_{-k+1} + ... \tag{6.38}$$

□

PROOF OF LEMMA 6.19. Take $\varepsilon_5 < \varepsilon_4 < \varepsilon_2$ where ε_2 is from Lemma 6.12(b). Divide all possible values of a_n into intervals $I_1, I_2, \dots, I_{d_1}$. Divide $[0, N]$ into a union of long intervals L_j of length N^{ε_4} and short intervals of length N^{ε_5}. (Intervals are numbered in decreasing order). Then by Theorem 6.9 the total number of clusters originated in short intervals tends to 0 in probability. Observe that by Lemmas 6.12 and 6.21

$$\mathbf{P}\left(n \text{ is marked, } a_n \in I_m \text{ and } \rho_{n-k} \leq \delta N^{1/s}, k = M \dots N^{\varepsilon_4}\right) \sim \frac{\mu_\infty(I_m)}{N}\left(1 - \mathcal{O}\left(\beta^M\right)\right).$$

Recall that b_n is independent of a_n. Hence if we divide $[1, \infty) \times [1, \infty)$ into rectangles $J_1, J_2 \dots J_{d_1}$ then

$$\mathbf{P}\left(n \text{ is marked, } (a_n, b_n) \in J_m \text{ and } \rho_{n-k} \leq \delta N^{1/s}, k = M \dots N^{\varepsilon_4}\right)$$
$$\sim \frac{\tilde{\mu}_\infty(J_m)}{N}\left(1 - \mathcal{O}\left(\beta^M\right)\right)$$

where $\tilde{\mu}_\infty$ is a product of μ_∞ and the distribution function of $\tilde{b}_n$.

Let V_j be the vector whose m-th component is

$$\mathrm{Card}(n \in L_j : n - \mathit{marked}, (a_n, b_n) \in J_m).$$

Then

$$\mathbf{P}(V_j = e_m) \sim \tilde{\mu}(J_m) N^{\varepsilon_4 - 1}, \quad \mathbf{P}(|V_j| \geq 1) = o(N^{\varepsilon_4 - 1})$$

so

$$\mathbf{E}(\exp(i\langle v, V_j\rangle)) = 1 + N^{\varepsilon_4 - 1} \sum_m \tilde{\mu}(J_m)(e^{iv_m} - 1) + o(N^{\varepsilon_4 - 1}). \tag{6.39}$$

Next, let $W_j = \sum_{k=1}^j V_k$. We claim that

$$\ln \mathbf{E}(\exp(i\langle v, W_j\rangle)) = jN^{\varepsilon_4 - 1} \sum_m \tilde{\mu}(J_m)(e^{iv_m} - 1) + o(jN^{\varepsilon_4 - 1}). \tag{6.40}$$

This holds because V_j is almost independent of $V_1, V_2 \dots V_{j-1}$. Namely, by Lemma 6.13 the value of ρ_n at the left endpoint of L_j could only influence V_j only if ρ_{n-k} is $\beta_1^{N^{\varepsilon_5}}$-close to the boundary of I_m. However if N is large then the probability that there is $n - k \in L_j$ such ρ_{n-k} is close to the boundary of I_m is $o(N^{\varepsilon_4 - 1})$ and hence arguing as in the proof of (6.39) we obtain (6.40). Taking $j \sim N^{1-\varepsilon_4}$ we obtain Lemma 6.19. □

6. Central Limit Theorem.

Here we prove Theorem ThQLT. We split the proof of part (a) into two parts.

LEMMA 6.23. *there is a set Ω_N such that $\mathbf{P}(\Omega_N) \to 1$ and*

$$F_N^\omega(x) \to \mathbf{P}(\mathcal{N}_1 \leq x) \text{ uniformly for } \omega \in \Omega_N.$$

LEMMA 6.24.
$$\frac{\mathbb{E}_\omega(T_N)-\mathbf{E}(T_N)}{\sqrt{N\ln N}})$$
converges weakly to a centered Gaussian normal variable.

PROOF OF LEMMA LMQLTGAUSS. . We split
$$\sum_{n=1}^N(\xi_n-\rho_n)=S_L+S_M+S_H$$
where S_H corresponds to the high values of ρ_n, namely, $\rho_n>\sqrt{N}\ln^{100}N$, S_M corresponds to the moderate values of ρ_n, namely, $\frac{\sqrt{N}}{\ln^{100}N}\leq\rho_n\leq\sqrt{N}\ln^{100}N$ and S_L corresponds to the low values of ρ_n, namely, $\rho_n<\frac{\sqrt{N}}{\ln^{100}N}$. We begin by showing that high and moderate values of ρ_n can be ignored. First, by Theorem 6.9
$$\mathbb{P}(S_H\neq 0)\leq N\mathbb{P}(\rho_n>\sqrt{N}\ln^{100}N)\leq\frac{C}{\ln^{200}N}.$$
Second, arguing as in the proof of Lemma 6.17(b) we see that
$$\mathbf{E}(S_M^2)\leq\text{Const}\sum\mathbf{E}\left((\rho_n)^2\,I_{\sqrt{N}/\ln^{100}N<\rho_n<\sqrt{N}\ln^{100}N}\right)\leq\text{Const}N\ln\ln N$$
and hence $S_M/\sqrt{N\ln N}$ converges to 0 in probability.

Therefore the main contribution comes from S_L. To handle it use Bernstein's method. Divide the interval $[0,N]$ into blocks of length $L_N=\ln^{10}N$ and $l_N=\ln^2 N$ following each other. More precisely the j-th big block is
$$I_j=[j(L_N+l_N),(j+1)L_N+jl_N-1]$$
and j-th small block is
$$J_j=[(j+1)L_N+jl_N,(j+1)(L_N+l_N)-1].$$
Accordingly, we split $S_L=S_L^{big}+S_L^{small}$, where S_L^{big} (S_L^{small}) is the contribution to S_L coming from big (small) blocks. Arguing as in the proof of Lemma 6.17(b) we see that
$$\mathbf{E}(\text{Var}_\omega(S_L^{small}))\leq C\sum_n\left[\mathbf{E}((\rho_n)^2)+\mathbf{E}(\rho_n l_N)\right]\ \text{(summation is over the small blocks)}$$
$$\leq C\left(N\ln N\frac{l_N}{L_N}+N\frac{l_N^2}{L_N}\right)$$
and hence the main contribution comes from the big blocks.

Next we modify ξ_n as follows. If $n\in I_j$ let $\tilde{\xi}_n$ be the number of visits to the site n before our walk reaches I_{j+1}. Let $\tilde{\rho}_n=\mathbb{E}_\omega(\tilde{\xi}_n)$. Observe that $\tilde{\xi}_n$ corresponds to imposing absorbing boundary conditions at the beginning of I_{j+1} so $\tilde{\rho}_n = p_n^{-1} q_{n+1} \tilde{\rho}_n + p_n^{-1}$ with absorbing boundary condition at $\bar{n} := (L_N + l_N)(j+1)$. Hence

$$\rho_n - \tilde{\rho}_n = \frac{q_{\bar{n}}}{q_n} \alpha_n \dots \alpha_{\bar{n}-1} \rho_{\bar{n}}$$

and so

$$\left| \sum_n [\tilde{\rho}_n - \rho_n] \right| < 1$$

except for the set of probability tending to 0 as $N \to \infty$. Also

$$\mathbf{P}(\tilde{\xi}_n = \xi_n \text{ for } n = 0 \dots N) \to 1 \text{ as } N \to \infty.$$

Let

$$\tilde{S} = \sum_n (\tilde{\xi}_n - \tilde{\rho}_n)$$

where the sum is over big blocks. By the foregoing discussion it is enough to show that
(6.41)
with $\mathbf{P}$ probability close to 1 the quenched distribution of $\tilde{S}$ is close to normal.

We claim that the following limit exists (in probability)

$$\lim_{N \to \infty} \frac{\mathrm{Var}_\omega(\tilde{S})}{N \ln N} = D_2. \tag{6.42}$$

Before proving (6.42) let us show ho to complete the proof of (6.41). Let

$$\tilde{S}_j = \sum_{n \in I_j} (\tilde{\xi}_n - \tilde{\rho}_n)$$

be the contribution of the j-th block. Since summation is taken over n with $\rho_n < \sqrt{N}/\ln^{100} N$ and $\tilde{\xi}_n$ has geometric distribution we have for $k \in \mathbb{N}$

$$\mathbb{P}_\omega \left(\tilde{S}_j > \frac{\sqrt{N} L_N k}{\ln^{100} N} \right) \le C e^{-k} L_N. \tag{6.43}$$

Indeed $\tilde{S}_j > \frac{\sqrt{N} L_N k}{\ln^{100} N}$ implies that $\tilde{\xi}_n > \frac{\sqrt{N} k}{\ln^{100} N}$ for some n in the block. (6.42) and (6.43) show that $\sum_j \tilde{S}_j$ satisfies the Lindenberg condition. It remains to establish (6.42). To this end we prove two facts.

$$(A) \quad \forall \varepsilon > 0 \exists M : \mathbf{P} \left(\frac{\sum_{n_1 < n_2 - M} \mathrm{Cov}_\omega(\tilde{\xi}_{n_1}, \tilde{\xi}_{n_1})}{N \ln N} > \varepsilon \right) < \varepsilon \quad \text{and}$$

$$(B) \quad \forall k \quad \frac{\sum_n \mathrm{Cov}_\omega(\tilde{\xi}_n, \tilde{\xi}_{n-k})}{N \ln N} \Rightarrow \frac{\mathbf{E}(\alpha)^k c^*}{2} \text{ in probability}$$

where c^* is the constant from Theorem 6.9.

The remaining part of Section **??** is devoted to the proofs of statements (A) and (B). We will drop tildes in $\tilde{\xi}$ and $\tilde{\rho}$ in order to simplify notation.

To obtain (A) we show that

$$\mathbf{E}(|\mathrm{Cov}_\omega\,(\xi_{n-k}, \xi_n)|\,|\mathcal{F}_n) \leq C\theta^k(\rho_n)^2 \tag{6.44}$$

for some $\theta < 1$. Pick a small $\epsilon > 0$ and consider two cases

(I) $\rho_n > (1+\epsilon)^k$. Then we use that

$$|\mathrm{Cov}_\omega(\xi_{n-k}, \xi_n)| \leq \sqrt{\mathrm{Var}_\omega(\xi_{n-k})\mathrm{Var}_\omega(\xi_n)} \leq C\rho_{n_1}\rho_{n_2}$$

and that

$$\mathbf{E}(\rho_{n-k}|\mathcal{F}_n) \leq \mathbf{E}(\alpha)^k\rho_n + C.$$

(II) $\rho_n \leq (1+\epsilon)^k$. Then by (6.20)

$$|\mathrm{Cov}_\omega(\xi_{n-k}, \xi_n)| \leq C\mathrm{Var}_\omega(\xi_n)\rho_{n-k}q^*$$

where q^* is the probability to visit $n-k$ before n starting from $n-1$. Hence

$$\mathbf{E}(|\mathrm{Cov}_\omega(\xi_{n-k}, \xi_n)||\mathcal{F}_n) \leq C\rho_n\sqrt{\mathbf{E}((\rho_{n-k})^2|\mathcal{F}_n)\mathbf{E}(q^*|\mathcal{F}_n)}$$

We have

$$\mathbf{E}(\rho_{n-k})^2|\mathcal{F}_n) \leq \rho_n + Ck$$

since $s = 2$ whereas $\mathbf{E}(q^*|\mathcal{F}_n) \leq C\theta^k$ by Lemma 6.3.

Summing (6.44) over k we obtain (A).

To prove (B) observe that by Lemma 6.15 for fixed k we have

$$\mathrm{Cov}_\omega(\xi_{n-k}, \xi_n) = \rho_{n-k}\rho_n + \mathcal{O}\left(\rho_n\right)$$

where the implicit constant depends on k. Since $\mathbf{E}(\rho_{n-k}|\mathcal{F}_n) = \rho_n\mathbf{E}(\alpha)^k + C$ we get

$$\mathbf{E}(\mathrm{Cov}_\omega(\xi_{n-k}, \xi_n)) = \mathbf{E}((\rho_n)^2)\mathbf{E}(\alpha)^k + \mathcal{O}\left(\mathbf{E}\left(\rho_n\right)\right).$$

Let $Z_n = \mathrm{Cov}(\tilde{\xi}_n, \tilde{\xi}_{n-k})$. Next

$$\mathrm{Var}\left(\sum_n Z_n\right) = \sum \mathrm{Var}(Z_n) + 2\sum_{n_1<n_2} \mathrm{Cov}(Z_{n_1}, Z_{n_2}).$$

Observe that Z_{n_1} and Z_{n_2} are independent if n_1, n_2 belong to different blocks and so we can limit summation over n_1, n_2 in the same block. Since

$$Z_n = \rho_n^2\alpha_{n-k}\dots\alpha_{n-1} + \mathcal{O}(\rho_n)$$

Theorem 6.9 gives

$$\mathrm{Var}(Z_n) \leq \mathrm{Const}\frac{N}{\ln^{200} N}.$$

By Cauchy-Schwartz inequality

$$\mathrm{Cov}(Z_{n_1}, Z_{n_2}) \leq \mathrm{Const}\frac{N}{\ln^{200} N}.$$

Therefore

$$\mathrm{Var}\left(\sum_n Z_n\right) \leq \mathrm{Const}\frac{N L_N^2}{\ln^{100} N}.$$

This completes the proof of (B). □

Proof of Lemma 6.24. Let us denote $\beta = \mathbf{E}(\alpha)$, $H = \sqrt{N}\ln^{100} N$. Arguing as in the proof of Lemma 6.23 we see that it suffices to establish the convergence of $B_N/\sqrt{N \ln N}$ where $B_N = \sum_j B_{N,j}$ and

$$B_{N,j} = \sum (\tilde{\rho}_N 1_{\tilde{\rho}_N < H} - \mathbf{E}(\tilde{\rho}_N 1_{\tilde{\rho}_N < H}))$$

and the sum is over the j-th big block. Since $B_{N,j}$ are independent and satisfy Feller-Lindenberg condition (since $|B_{N,j}| \leq \sqrt{N}/N^{90}$) we need to show that $\lim_{N\to\infty} \frac{\mathrm{Var}(B_N)}{N\sqrt{N}}$ exists. To this end we show that

$$\lim_{N\to\infty} \frac{\mathrm{Var}(B_{N,j})}{\ln^{11} N} \tag{6.45}$$

exists. This we will follow from two estimates

(A) There exists constants $C > 0$ and $\theta < 1$ such that for each k we have

$$\mathrm{Cov}(\tilde{\rho}_n 1_{\tilde{\rho}_n < H}, \tilde{\rho}_{n-k} 1_{\tilde{\rho}_{n-k} < H}) \leq C\left[\theta^k \ln N + \frac{1}{H}\right].$$

(B) For each k the following limit exists

$$c_k = \lim_{N\to\infty} \frac{\mathrm{Cov}(\tilde{\rho}_n 1_{\tilde{\rho}_n < H}, \tilde{\rho}_{n-k} 1_{\tilde{\rho}_{n-k} < H})}{\ln N}.$$

Once (A) and (B) are proven (6.45) follows by the Dominated convergence Theorem. To prove (A) we note that by (6.5)

$$\mathbf{E}(\tilde{\rho}_{n-k}|\mathcal{F}_n) = \mathbf{E}(\tilde{\rho}) + \beta^{k-1}\rho_n q_n \mathbf{E}\left(p_{n-k}^{-1}\right) - \sum_{j=k}^{\infty} \beta^j \mathbf{E}(p^{-1}). \tag{6.46}$$

Next we claim that on $\rho_n < H$ we have

$$\mathbf{E}(\tilde{\rho}_{n-k} 1_{\tilde{\rho}_{n-k} < H}|\mathcal{F}_n) \leq C\left[\beta^k \rho + \theta^k + \frac{1}{H}\right] \tag{6.47}$$

To show this observe that

$$\tilde{\rho}_{n-k} 1_{\tilde{\rho}_{n-k} < H} \leq p_{n-k}^{-1} \alpha_{n-k+1} \dots \alpha_{n-1} q_n \tilde{\rho}_n + A 1_{\rho_{n-k} < H} \tag{6.48}$$

where

$$A = (1 + \alpha_{n-k+1} + \cdots + \alpha_{n-k+1} \ldots \alpha_{n-1})p_{n-k}^{-1}.$$

The conditional expectation of the first term can be estimated by $C\beta^k\tilde{\rho}_n$. Next

$$\mathbf{P}(\tilde{\rho}_{n-k} > H|\mathcal{F}_n) \leq \mathbf{P}(p_{n-k}^{-1}\alpha_{n-k+1} \ldots \alpha_{n-1}q_n\tilde{\rho}_n < H/2\mathcal{F}_n) + \mathbf{P}(A < H/2|\mathcal{F}_n)$$
$$\leq C\left[\theta^m + \frac{1}{H^2}\right].$$

Now by Theorem 6.9

$$\mathbf{P}(A > t|\mathcal{F}_n) \leq \frac{C}{t^2} \text{ so } \mathbf{E}(A1_{\hat{\Omega}}|\mathcal{F}_n) \leq C\sqrt{\mathbf{P}(\hat{\Omega}|\mathcal{F}_N)}$$

for each set $\hat{\Omega}$. Thus the second term in (6.48) is less than $C\left[\theta^{m/2} + \frac{1}{H}\right]$. This proves (6.47). Combining (6.46) and (6.47) we get

$$\mathbf{E}(\rho_n 1_{\rho_n<H}\rho_{n-k}1_{\rho_{n-k}<H}) = \mathbf{E}(\rho)\mathbf{E}(\rho 1_\rho < H) + O\left(\theta^m + \theta_m \ln N + \frac{\ln N}{H}\right).$$

This proves (A).

To prove (B) we consider two cases:

(I) $\tilde{\rho}_n > H/\ln^{1/4} N$. Since $\mathbf{P}(\tilde{\rho}_n > H/\ln^{1/4} N) < \frac{C\ln N}{H^2}$ the contribution of this case to $\mathbf{E}(\tilde{\rho}_n 1_{\tilde{\rho}_n<H}\tilde{\rho}_{n-k}1_{\tilde{\rho}_{n-k}<H}$ is less than $C\sqrt{\ln N}$.

(II) $\tilde{\rho}_n \leq H/\ln^{1/4} N$.In this case $\rho_{n-k} < H$ and so the contribution of this case to $\mathbf{E}(\tilde{\rho}_n 1_{\tilde{\rho}_n<H}\tilde{\rho}_{n-k}1_{\tilde{\rho}_{n-k}<H}$ equals to

$$\mathbf{E}(q_n\tilde{\rho}_n^2 1_{\tilde{\rho}_n<H})\beta^k\mathbf{E}\left(\frac{1}{p}\right) + O(1) = C\ln N\beta^k\mathbf{E}\left(\frac{1}{p}\right) + O(1)$$

where we have used the identity $q_n\tilde{\rho}_n^2 = \frac{q_n}{p_n^2}\tilde{z}_n$ where $\tilde{z}_n$ is $\mathcal{F}_{n+1}$ measurable due to (6.6). This proves (b). □

Lemmas 6.23 and 6.24 prove part (a) of Theorem 6.7.

The proof of part (b) proceeds along similar lines but it is simpler because there is no need to introduce the cutoffs.

EXERCISE 6.1. Prove part (b) of Theorem 6.7.

7. Favorite sites of transient walk.

Let $\xi_N^* = \max_{[0,N]} \xi_n$.

THEOREM 6.25. *If* $s > 0$ *then* $\frac{\xi_N^*}{N^{1/s}}$ *converges to* $\max_j \bar{\Theta}_j$, *where* $\bar{\Theta}$ *is a Poisson process on* $(0, \infty)$ *with intensity* $\frac{\bar{\mathbf{c}}}{\theta^{1+s}}$ *for some constant* $\bar{\mathbf{c}}$. *Accordingly*

$$\mathbf{P}\left(\xi_N^* < xN^{1/s}\right) \to \exp\left[-\frac{\bar{\mathbf{c}}}{s}x^{-s}\right].$$

PROOF. Consider the following process $\hat{\Lambda}_N^\delta = \{(\frac{n_j}{N}, \frac{\hat{m}_j}{N^{1/s}})\}$ where n_j are marked points and $\hat{m}_j$ is the maximum of ρ_n inside the j-th cluster. Similarly to Lemmas 6.19 and 6.20 we prove the following statement.

LEMMA 6.26. *(a) As $N \to \infty$ $\hat{\Lambda}_N^\delta$ converges to a Poisson process $\hat{\Lambda}^\delta = \{t_j, \mathbf{m}_j)\}$ on $[0,1] \times [\delta, \infty)$.*

(b) As $\delta \to 0$ $\hat{\Lambda}^\delta$ converges to the Poisson process $\hat{\Lambda}$.

(c) There exists a constant $\hat{c}$ such that $\hat{\Lambda}$ has the intensity $\frac{\hat{c}}{\mathbf{m}^{1+s}}$.

Next we show that the low values of ρ are unlikely to contribute to the maximal occupation times. Fix $\theta > 0$. Denote

$$\Omega_{N,k} = \{\exists n \le N : N^{1/s}2^{-(k+1)} < \rho_n \le N^{1/s}2^{-k} \text{ and } \xi_n > \theta N^{1/s}\}$$

and set

$$\Phi_{N,k,n} = \{N^{1/s}2^{-(k+1)} < \rho_n \le N^{1/s}2^{-k}\}.$$

Then by Theorem 6.9

$$\mathbf{P}(\Omega_{N,k}) \le N\mathbf{P}\left(\Phi_{N,k,n}\right)\mathbf{P}\left(\xi_n > \theta N^{1/s}|\Phi_{N,k,n}\right) \le \text{Const}2^{ks}\mathbf{P}\left(\xi_n > \theta N^{1/s}|\Phi_{N,k,n}\right)$$

Since ξ_n has a geometric distribution with parameter ρ_n^{-1} we have that

$$\mathbf{P}\left(\xi_n > \theta N^{1/s}|\Phi_{N,k,n}\right) \le (1-\rho_n^{-1})^{\theta N^{1/s}} \le \text{Const}e^{-c2^k}.$$

The first term here is $\mathcal{O}(2^{-ks})$ in view of Theorem 6.9 and Markov inequality and the second term is less than

$$4^{sk}\mathbf{P}\left(\xi_n > \theta N^{1/s}|\rho_n \le \frac{N^{1/s}}{2^k}\right) \le \text{Const}4^{sk}\beta^{2^k\theta}, \quad \beta < 1$$

since ξ_n has geometric distribution with mean ρ_n. Summing these bounds over $k \ge \log_2(1/\delta)$ we see that the points from outside of the clusters can be ignored. The rest of the proof of Theorem 6.25 is similar to the proof of Theorem 6.6. Namely Lemma 6.15 implies that the maximum occupation time inside the j-th cluster occurs at the site $\hat{n}_j$ such that $\rho_{\hat{n}_j} = \hat{m}_j$. This shows that if δ is sufficiently small then with probability close to 1 $\xi_N^* = \max_j \hat{m}_j \frac{\xi_{\hat{n}_j}}{\hat{m}_j}$ where the maximum is taken over the δ-clusters. For large N the $\frac{\xi_{\hat{n}_j}}{\hat{m}_j}$ is asymptotically exponential with mean 1. Therefore letting $N \to \infty$ and $\delta_N \to 0$ we obtain that the distribution of $\frac{\xi_N^*}{N^{1/s}}$ is asymptotically the same as that of

$$\max_j \hat{\theta}_j \Gamma_j$$

where $\hat{\Lambda} = \{(t_j, \hat{\theta}_j)\}$ and Γ_j are i.i.d random variables independent of $\hat{\Lambda}$ and having mean 1 exponential distribution. It remains to notice that by Lemma 7.8 $\{\hat{\theta}_j \Gamma_j\}$ also form a Poisson process. $\square$

8. Sinai-Golosov localization.

In this section we consider recurrent random walk in random environment, so that $\mathbf{E}(\ln p - \ln q) = 0$. Let $\sigma^2 = \mathbf{E}((\ln p - \ln q)^2)$.

THEOREM 6.27. [**16, 17**]. *There exists a random variable* Γ *and a sequence* $\Gamma_N(\omega)$ *such that*

$$\frac{\Gamma_N(\omega)}{\ln^2 N} \Rightarrow \Gamma,$$

Γ *is symmetric and there are probability distributions* $\boldsymbol{\nu}^\pm$ *on* $\mathbb{Z}$ *such that*

$$\mathbf{P}(X_N - \Gamma_N = k | \pm \Gamma_N > 0) = \boldsymbol{\nu}^\pm_k.$$

We shall see that distribution of Γ is universal in the sense that the distribution of Γ/σ does not depend on the distribution of p whereas $\boldsymbol{\nu}^\pm$ depend strongly on the distribution of p.

COROLLARY 6.28. [**33**] $\frac{X_N}{\ln^2 N} \Rightarrow \Gamma$.

We now describe Γ_n. Define

$$W_n = \begin{cases} \sum_{j=0}^{n-1} \lambda_j & \text{if } n > 0, \\ 0 & \text{if } n = 0, \\ -\sum_{j=n}^{-1} \lambda_j & \text{if } n < 0. \end{cases}$$

We say that $a < \gamma < b$ is *a valley* if

$$W_b = \max_{[\gamma,b]} W_j, \quad W_a = \max_{[a,\gamma]} W_j, \quad W_\gamma = \min_{[a,b]} W_j.$$

We define the *depth* of the valley as $\min(W_a - W_\gamma, W_b - W_\gamma)$. Let (a_N, Γ_N, b_N) be the smallest valley of depth at least $\ln N$ containing 0 (if $\min_{[a_N,b_N]}$ is achieved at several points we let Γ_N to be the point which is closest to 0).

Let $W(t)$ be the Brownian Motion with zero mean and variance $\sigma^2 t$. Let a, Γ, b be the smallest valley of depth at least 1 containing 0.

EXERCISE 6.2. As $N \to \infty$ $\frac{\Gamma_N}{\ln^2 N} \Rightarrow \Gamma$.

Fix J, δ. Let $\bar{a}_N$ be the closest number to the left on a_N such that $W_{\bar{a}_N} \geq W_{a_N} + \delta \ln N$. Let $\bar{b}_N$ be the closest number to the left on a_N such that $W_{\bar{b}_N} \geq W_{b_N} + \delta \ln N$. We say that time N is (J, δ)-good if

- $\bar{a}_N, \Gamma_N, \bar{b}_N$ is a valley;
- $\bar{b}_N - \bar{a}_N \leq J \ln^2 N$;
- If $(a, \gamma, b) \subset (\bar{a}_n, \bar{b}_n)$ is a valley with $\gamma \neq \Gamma_N$ then its depth is at most $(1 - \delta) \ln N$.

EXERCISE 6.3. Show that for each $\varepsilon > 0$ we can choose J so large and δ so small that $\mathbf{P}(N \text{ is } (J,\delta)\text{-good}) \geq 1-\varepsilon$ provided that N is large enough.

The proof of Theorem 6.27 consists of the following steps. To fix our notation we assume that $\Gamma_N > 0$.

LEMMA 6.29. *For each J, δ, ε there exists N_0 such that if $N \geq N_0$ is (J,δ)-good then*

$$\mathbb{P}_0(\tau_{\Gamma_N} < \tau_{\bar{a}_N}) \geq 1-\varepsilon.$$

Let $\tilde{X}_n$ denote the random walk moving in the same environment as X_n but with reflecting barriers at $\bar{a}_N$ and $\bar{b}_N$. We let $\widetilde{\mathbb{P}}$ be corresponding quenched distribution. Our next result shows that $\tilde{X}_n$ is unlikely to reach the boundary of $(\bar{a}_N, \bar{b}_N)$ so we can study $\tilde{X}_n$ instead of X_n.

LEMMA 6.30. *Given J, δ, ε there exists $N_0 > 0, \theta < 1$ such that if $N \geq N_0$ is (J,δ)-good then*

(a) $\max_{k\in[\bar{a}_N,\bar{b}_N]} \widetilde{\mathbb{E}}_k(\tau_{\Gamma_N}) \leq N^{1-\delta/2}$.

(b) $\widetilde{\mathbb{P}}_0(\tau_{\Gamma_N} \leq lN^{1-\delta/2}) \leq \theta^l$.

(c) $\widetilde{\mathbb{E}}_{\Gamma_N}(\tau_{\bar{a}_N}) \geq N^{1+\delta/2}$ *and* $\widetilde{\mathbb{E}}_{\Gamma_N}(\tau_{\bar{b}_N}) \geq N^{1+\delta/2}$.

(d) $\widetilde{\mathbb{P}}_0(\tau_{\bar{a}_N} \leq N^{1+\delta_4}) \leq \varepsilon$ *and* $\widetilde{\mathbb{P}}_0(\tau_{\bar{b}_N} \leq N^{1+\delta_4}) \leq \varepsilon$.

Let $\nu^{\omega,N}$ denote the invariant measure of $\tilde{X}_n$.

LEMMA 6.31. *(a) For each ε there exists M such that*

$$\mathbf{P}(\nu^{\omega,N}[\Gamma_N - M, \Gamma_N + M] \leq 1-\varepsilon) \leq \varepsilon.$$

(b) There exists a random process $\boldsymbol{\nu}^+$ on $\mathbb{Z}$ such that

$$\nu^{\Gamma_N+k} \Rightarrow \boldsymbol{\nu}_k^+ \text{ as } N \to \infty.$$

We will say that N is (J, M, δ, δ')-good if it is (J,δ)-good and $\nu^{\omega,N}([\Gamma_N - M, \Gamma_N + M]) \geq 1-\delta'$.

LEMMA 6.32. *Given J, M, δ, δ' there exists N_0 such that if $N \geq N_0$ is (J, M, δ, δ')-good then then*

(a) If N and Γ_N have the same parity then

$$\sup_k |\widetilde{\mathbb{P}}_0(\tilde{X}_N = \Gamma_N + 2k) - 2\nu^{\omega,N}_{\Gamma_N+2k}| \leq 2\delta';$$

(b) If N and Γ_N have different parity then

$$\sup_k |\widetilde{\mathbb{P}}_0(\tilde{X}_N = \Gamma_N + 2k+1) - 2\nu^{\omega,N}_{N+2k+1}| \leq 2\delta';$$

Lemmas 6.31 and 6.32 demonstrate universal behavior of X_N conditioned on the fact that Γ_N has a given parity. To complete the description of the distribution of X_N it remains to show that Γ_N is equally likely to be odd or even.

EXERCISE 6.4. Show that $\mathbf{P}(\Gamma_N > 0, \Gamma_N \text{ is even}) \to \frac{1}{4}$ as $N \to \infty$.

Hint. Construct Γ_N for the environment shifted by one unit.

9. Recurrent walk: Estimates of hitting times.

Here we prove Lemmas 6.29, 6.30 and 6.32.

PROOF OF LEMMA 6.29. In the notation of Theorem 6.1 we need to show that $\Phi(\Gamma_N) \ll \phi(\bar{a}_N)$. We have

$$\phi(\Gamma) = \sum_{j=0}^{\Gamma} e^{W_j} \leq |\Gamma| \exp(\max_{j\in[0,\Gamma]} W_j) \leq J \ln^2 N \exp(\max_{j\in[a_N,\Gamma_N} W_j)$$

while

$$\phi(\bar{a}_N) \geq e^{W_{\bar{a}_N}}.$$

Since

$$W_{\bar{a}_N} \geq \delta \ln N + \max_{j\in[a_N,\Gamma_N]} W_j$$

the result follows. $\square$

PROOF OF LEMMA 6.30. (a) We consider $k \in [\bar{a}_N, \Gamma_N]$ the other case is similar. Let $\zeta_k = \widetilde{\mathbb{E}}_k(\tau_{\Gamma_N})$. We have

$$\zeta_k = 1 + p_k \zeta_{k+1} + q_k \zeta_{k-1}.$$

Thus denoting $\psi_k = \zeta_k - \zeta_{k+1}$ we get

$$\psi_k = \frac{1}{p_k} + \alpha_k \psi_{k-1}, \quad \psi_{\bar{a}_N} = 1.$$

Therefore

$$\psi_k = \frac{1}{p_k} + \frac{\alpha_k}{p_{k-1}} + \frac{\alpha_k \alpha_{k-1}}{p_{k-2}} + \cdots + \alpha_k \alpha_{k-1} \dots \alpha_{\bar{a}_N+1}$$
$$\leq \frac{1}{\varepsilon_0} \sum_{j\in[\bar{a}_N+1,k]} e^{W_n - W_j} \leq \frac{J \ln^2 N N^{1-\delta}}{\varepsilon_0}$$

since $(\bar{a}_N, \Gamma_N)$ contains no valleys of depth greater than $(1-\delta)\ln N$. Accordingly

$$\zeta_k = \sum_{j=k}^{\Gamma_N} \psi_j \leq \frac{J \ln^4 N N^{1-\delta}}{\varepsilon_0}$$

proving (a).

(b) By part (a)

$$\widetilde{\mathbb{P}}(\tau_{\Gamma_N} \leq n + 2N^{1-\delta/2}|\mathcal{F}_n) \leq \frac{1}{2}.$$

Iterating this l times we obtain the result needed.

The proof of part (c) is similar to the proof of part (a).

EXERCISE 6.5. Prove part (c).

(d) In view of part (a) it suffices to prove that

$$\widetilde{\mathbb{P}}_{\Gamma_N}(\tau_N \leq N^{1+\delta/4}) \leq \varepsilon.$$

If this was false then by part (b) we would have that for each $k \in (\bar{a}_N, \bar{b}_N)$

$$\widetilde{\mathbb{P}}_{\Gamma_N}(\tau_N \leq 2N^{1+\delta/4}) \geq \tilde{\varepsilon}.$$

Arguing as in part (b) we would get

$$\widetilde{\mathbb{E}}(\tau_{\bar{a}_N}) \leq CN^{1+\delta/4}$$

contradicting part (c). □

PROOF OF LEMMA 6.32. Let $\tilde{X}'_n$ and $\tilde{X}''_n$ be two independent particle moving in our environment with reflecting barriers. We assume that $\tilde{X}'_n$ starts from 0 and $\tilde{X}''_n$ starts $2\mathbb{Z} \cap [\bar{a}_N, \bar{b}_N]$ and is distributed according to the restriction of $\nu^{\omega,N}$ to this set. Let ζ be the first time the particles meet. To prove the lemma it is enough to show that

$$\widetilde{\mathbb{P}}(\eta > N) \leq 2\delta'. \tag{6.49}$$

Let $\tilde{\tau}_1$ be the first time when $\tilde{X}'$ visits Γ_n and let $\tilde{\tau}_{j+1}$ be the first time after $\tilde{\tau}_j + M$ when $\tilde{X}'$ visits Γ_n. We claim that there exists $\beta < 1$ such that if

$$\widetilde{\mathbb{P}}(\zeta > \tilde{\tau}_j) > 2\delta \tag{6.50}$$

then

$$\widetilde{\mathbb{P}}(\zeta > \tilde{\tau}_{j+1}) \leq \widetilde{\mathbb{P}}(\zeta > \tilde{\tau}_j)\beta. \tag{6.51}$$

Indeed if (6.50) then

$$\widetilde{\mathbb{P}}(\tilde{X}_{\tau_j} \in [\Gamma_N - M, \Gamma_N + M]|\zeta > \tau_j)$$

$$\geq \frac{\widetilde{\mathbb{P}}(\tilde{X}_{\tau_j} \in [\Gamma_N - M, \Gamma_N + M], \zeta > \tau_j)}{\widetilde{\mathbb{P}}(\zeta > \tau_j)} \geq \frac{\delta}{2\delta} = \frac{1}{2}.$$

On the other hand if at time $\tilde{\tau}_j$ the walkers are distance less than M apart then the probability that they will meet before time $\tilde{\tau}_j + M$ is at least ε_0^M. This implies (6.51).

Iterating (6.51) we get

$$\widetilde{\mathbb{P}}(\zeta > \tilde{\tau}_j) \leq \min((1-\beta)^j, 1-2\delta').$$

On the other hand by Lemma 6.30(b)

$$\widetilde{\mathbb{P}}(\tilde{\tau}_j > t) \leq \widetilde{\mathbb{P}}\left(\sum_{k=1}^{j} \xi_k > \frac{t}{N^{1-\delta/2}}\right)$$

where $\xi_j \sim \mathrm{Geom}(\theta)$. Combining two last inequalities we obtain the lemma. □

10. Invariant measure.

Here we prove Lemma 6.31.

Exercise 6.6. Show that

$$\nu_n^{N,\omega} = \frac{1}{Z_N} \frac{\prod_{j=\bar{a}_N}^{n} p_j}{\prod_{j=\bar{a}_N}^{n} q_{j+1}}$$

where Z_N is a normalization factor.

Thus up to normalization

$$\nu_n = \frac{1}{q_{n+1}} e^{-W_n}.$$

Accordingly to establish part (a) it is enough to show that for each ε

$$\mathbf{P}\left(\sum_{|n-\Gamma_N|\geq M} \geq \varepsilon e^{-W_{\Gamma_N}}\right) \to 0 \text{ as } M \to \infty.$$

We shall show that this inequality holds even if we condition on $W_{\Gamma_N} \in [-m, -(m+R)]$ where R is the maximal downward jump of W and $m \sim \ln n$. Indeed for this condition to happen three evens should happen

(i) W_{-n} should reach $\ln n - m$ before reaching $-m$.

(ii) W_n should reach m before $\ln n - m$.

(iii) After reaching m, W_n should go $\ln n$ units up before going R units down.

Note that (i) is concerned with behavior of the process on $(-\infty, 0]$, (ii) is concerned with behavior of the process on $[0, \Gamma_N]$ and (iii) is concerned with behavior of the process on $[\Gamma_N, \infty)$ so we consider the sums over whose three intervals separately.

$$(i) \qquad \sum_{\bar{a}_N \leq n \leq 0} e^{-W_n} \leq |a_N| \exp(-\min_{[\bar{a}_N, 0]} W_n)$$

so the needed estimate follows since $|\bar{a}_N|/\ln^2 N$ has a limiting distribution while

$$\frac{\min_{[\bar{a}_N,0]} W_n + m}{\ln N}$$

conditioned on $\min_{[\bar{a}_N,0]} W_n) > -m$ has a limiting distribution which has no atom at 0.

(ii) The result follows from two estimates

$$(A) \quad \forall k_0 \quad \mathbf{P}(\tau^-_{-m+2^{k_0}} < \tau^-_{-m} - M) \to 0 \text{ as } M \to \infty.$$

(B) Let V_k be the total time W spends in $I_k := [-m+2^{k+1}, -m+2^k]$ before τ^-_m. Let $A_k = \{V_k \geq 9^k$. Then

$$\mathbf{P}\left(\bigcup_{k\geq k_0} A_k\right) \to 0 \text{ as } k_0 \to \infty.$$

Indeed on $\bigcap_{k\geq k_0} A_k^c$ we can estimate the contribution of terms with $W_n \geq -m + 2^{k_0}$ as

$$\exp(m)\left[\sum_{k=k_0}^{\infty} 9^k \exp(-2^k)\right]$$

and the second factor can be made as small as we wish by choosing k_0 large. So it suffices to prove (A) and (B).

To prove (A) note that

$$\mathbf{P}(\tau^-_{-m+2^{k_0}} < \tau^-_{-m} - M) \leq \mathbf{P}_0(\tau_{2^{k_0}+R} \geq M)$$

which tends to 0 since $2^{k_0} + R$ is a fixed number.

To prove (B) note that there a constants $\beta_1, \beta_2 > 0$ such that

$$\mathbf{P}(W_{n+4^k} \leq -m + 2^k | W_n \in I_k) \geq \beta_1 \text{ and}$$

$$\mathbf{P}(W \text{ does not return to } I_k\text{after time } n+4^k | W_{n+4^k} \leq -m+2^k) \geq \beta_2 2^{-k}.$$

Hence

$$\mathbf{P}(W \text{ does not return to } I_k\text{after time } n + 4^k) \geq \beta_1\beta_2 2^{-k}$$

and by induction

$$\mathbf{P}(V_k \geq l4^k) \leq \left(1 - \beta_1\beta_2 2^{-k}\right)^l$$

proving (B).

To handle $n \in [\tau^-_{-m}, \infty)$ we need to obtain the analogues of (A) and (B) for $n \geq \tau^-_{-m}$. However since we require that the walk reaches $-m + \ln N$ before visiting $-m + R$ the behavior of W_n on $[\tau^-_{-m}, \bar{b}_N]$ is similar to the behavior of the random walk conditioned to stay positive so (A) and (B) can be established similar to corresponding estimates in Chapter 4. This completes the proof of part (a).

To prove part (b) note that by part (a) $\nu^{N,\omega}_{\Gamma_N+k}$ can be well approximated by

$$\frac{\prod_{j=\bar{a}_N+1}^{\Gamma_N+k}\frac{p_j}{q_{j+1}}}{\sum_{k=-M}^{M-1}\prod_{j=\bar{a}_N+1}^{\Gamma_N+k}\frac{p_j}{q_{j+1}}}.$$

The last expression depends only on p_n, $n\in[\Gamma_N-M,\Gamma_N+M]$ so it is enough to show that the vector

$$(p_{\Gamma_N-M},\dots p_{\Gamma_N}\dots p_{\Gamma_N}+M)$$

has a limiting distribution. To this end let $\sigma_0=0$ and $\sigma_j^{\pm}$ be the first times after $\sigma_j^{\pm}$ when $W_{\pm n}\leq W_{\sigma_j^{\pm}}$. Let

$$H_j^{\pm}=\max_{n\in[\sigma_j^{\pm},\sigma_{j+1}^{\pm}]}(S_{\pm n}-S_{\pm\sigma_j}),$$

$$r_N^{\pm}=\min(j:H_j^{\pm}>\ln N),\quad \Gamma_N^{\pm}=\sigma_{\pm r_N^{\pm}},\quad m_N^{\pm}=S_{\sigma_{r_N^{\pm}}}.$$

Let $Z^{j,\pm}$ denotes the process

$$Z_k^{j,\pm}=W_{\pm(\sigma_j^{\pm}+k)}-W_{\pm\sigma_j^{\pm}}$$

and let $d_j^{\pm}=W_{\pm\sigma_{j+1}^{\pm}}-W_{\pm\sigma_j^{\pm}}$. Note that Z^js and hence d_j are iid and since

$$m_N^{\pm}=\sum_{j=1}^{r_N^{\pm}}d_j^{\pm}$$

the Law of Large Numbers implies that there are constants $\mathbf{a}^{\pm}$ such that $\frac{m_N^{\pm}}{r_N\pm}\to\mathbf{a}^{\pm}$. Note that $\Gamma_N=\Gamma_N^+$ if $m_N^+<m_N^-$ and $\Gamma_N=\Gamma_N^-$ otherwise. Accordingly to complete the proof it suffices to show that

$$(p_{\Gamma_N^{\pm}-M},\dots p_{\Gamma_N^{\pm}}\dots p_{\Gamma_N^{\pm}}+M)$$

have limiting distribution and that those vectors are asymptotically independent of $m_N^{\pm}$. For

$$p_{\Gamma_N^{\pm}-M},\dots p_{\Gamma_N^{\pm}}$$

this follows from the next result

EXERCISE 6.7. Let k_j be consecutive numbers such that $H_j>\ln N$. Then vectors

$$(Z^{k_j-M},Z^{k_j-M+1},Z^{k_j-1},k_j/N)$$

converges to the Poisson process with measure $\pi\times\pi\cdots\times\pi\times\mathrm{Leb}$ where π is the law of Z.

Finally, since $p_{\Gamma_N^+ +1} \dots p_{\Gamma_N^+ +M}$ depend only on $Z^{r_N +}$ and so are independent of $p_{\Gamma_N^+ -M} \dots p_{\Gamma_N}$ and of m_N^+ to complete the proof it remains to show that $\alpha_{r_N^+ +1} \dots \alpha_{r_N+M}$ approach a limiting distribution. The proof of this result is similar to the proof of (4.2) in Chapter 4.

CHAPTER 7

Appendices.

Here we review some tools which are used in the analysis of random walks.

1. Convergence of random processes.

By a (continuous) random process we mean a measure on the space $C[0, \mathbf{T}])$ of continuous functions on the interval $[0, \mathbf{T}]$. Accordingly a weak convergence of processes means the weak convergence of the corresponding measures. That is $X_n(t) \Rightarrow X(t)$ if for any continuous function $\Phi : C[0, \mathbf{T}] \to \mathbb{R}$ we have $\mathbb{E}(\Phi(X_n)) \to \mathbb{E}(\Phi(X))$. In fact the continuity can be replaced by a weaker condition

THEOREM 7.1. *If $X_n(t) \Rightarrow X(t)$ and $\Phi : C[0, \mathbf{T}] \to \mathbb{R}$ is a function such that the set of discontinuity points of Φ has zero measure with respect to the law of X then*

$$\mathbb{E}(\Phi(X_n)) \to \mathbb{E}(\Phi(X)).$$

Recall that a sequence $\{\mu_n\}$ of probability measures on a topological space $\mathcal{X}$ is *tight* if given $\varepsilon > 0$ there exists a compact set $\mathcal{K} \subset \mathcal{X}$ such that $\mu_n(\mathcal{K}) > 1 - \varepsilon$ for all n. By Arcela-Ascoli theorem a subset $\mathcal{K} \subset C[0, \mathbf{T}]$ is compact if it consists of uniformly bounded and equicontinuous functions.

THEOREM 7.2. *(Prokhorov) $X_n(t) \Rightarrow X(t)$ iff the sequence $\{X_n\}$ is tight and the finite dimensional distributions of X_n converge to the finite dimensional distributions of X.*

THEOREM 7.3. *Suppose that there are constants α, β, C such that for each l we have*

$$\mathbb{P}\left(\text{There exists } k < 2^l\mathbf{T} : \left|X^{(N)}\left(\frac{k+1}{2^l}\right) - X^{(N)}\left(\frac{k}{2^l}\right)\right| \geq 2^{-l\alpha}\right) \leq \frac{C}{2^{l\beta}}.$$

Then the family $\{X^{(N)}(t)\}$ is tight.

THEOREM 7.4. *(Kolmogorov) Let $X_n(t)$ be a family of processes such that there exists constants K, α, δ such that*

$$\mathbb{E}(|X_n(t_1) - X_n(t_2)|^\alpha) < K|t_2 - t_1|^{1+\delta}.$$

Then $\{X_n\}$ *is tight.*

2. Martingales.

Let $\mathcal{G}_n$ be an increasing sequence of σ-algebras and Z_n be a sequence of $\mathcal{G}_n$ measurable random variables. $(Z_n, \mathcal{G}_n)$ is called *martingale pair* if $\mathbb{E}(Z_{n+1}|\mathcal{G}_n) = Z_n$. Then $Y_n = Z_n - Z_{n-1}$ is called *martingale difference sequence.*

THEOREM 7.5. *(Optional Stopping Theorem)*
(a) Let τ *be a stopping time such that the sequence* $Z_{\min(n,\tau)}$ *is uniformly integrable. Then* $\mathbb{E}(Z_\tau) = E(Z_0)$.
(b) If Y_n *is uniformly bounded and* $\mathbb{E}(\tau) < \infty$ *then* $\mathbb{E}(Z_\tau) = E(Z_0)$.

THEOREM 7.6. *(Maximal Inequality) There are constants* C_1 *and* C_2 *such that for any martingale* $(Z_n, \mathcal{G}_n)$ *satisfying* $Z_0 = 0$ *the following holds. Let* $Z^* = \max_n Z_n$, $\Delta Z = \sum_n Y_n^2$. *Then*

$$\frac{1}{C_1}\mathbb{E}((\Delta Z)^2) \leq \mathbb{E}(Z^{*4}) \leq C_1\mathbb{E}((\Delta Z)^2);$$

THEOREM 7.7. *(a) Let* $(Z_n, \mathcal{G}_n)$ *be a martingale pair such that*

$$\frac{\sum_{m=1}^n \mathbb{E}(Y_m^2|\mathcal{G}_{m-1})}{n} \Rightarrow \sigma^2 \tag{7.1}$$

and for each $\varepsilon > 0$

$$\frac{1}{n}\mathbb{E}(\sum_{m=1}^n Y_m^2 1_{Y_m > \varepsilon\sqrt{n}}) \to 0$$

then $\frac{Z_n}{\sqrt{n}}$ *converges to Gaussian random variable with zero mean and variance* σ^2.
(b) In particular if $|Y_m| < C$ *with probability 1 then* (7.1) *suffices for the Central Limit Theorem.*

We shall also use the following fact. Let f_n be a $\mathcal{G}_n$-measurable sequence such that

$$\beta_n = \sum_{j=0}^{\infty} \mathbb{E}(f_{n+j}|\mathcal{G}_{n-1}) \leq \text{Const}$$

then

$$f_n = Y_n + \beta_n - \beta_{n+1} \tag{7.2}$$

where Y_n is a martingale difference sequence.

3. Poisson process.

The proofs of the facts listed below can be found in monographs [**31, 34**].

Let (X, μ) be a measure space. Recall that a Poisson process is a point process on X such that
(a) if $A \subset X$, $\mu(A)$ is finite, and $N(A)$ is the number of points in A then $N(A)$ has a Poisson distribution with parameter $\mu(A)$;
(b) if $A_1, A_2 \dots A_k$ are disjoint subsets of X then $N(A_1), N(A_2) \dots N(A_k)$ are mutually independent.

If $X \subset \mathbb{R}^d$ and μ has a density f with respect to the Lebesgue measure we say that f is the intensity of the Poisson process.

LEMMA 7.8. *(a) If $\{\Theta_j\}$ is a Poisson process on X and $\psi : X \to \tilde{X}$ is a measurable map then $\tilde{\Theta}_j = \psi(\Theta_j)$ is a Poisson process. If $X = \tilde{X} = \mathbb{R}$ and ψ is invertible then the intensity of $\tilde{\Theta}$ is*

$$\tilde{f}(\theta) = f(\psi^{-1}(\theta)) \left| \frac{d\psi}{d\theta} \right|^{-1}.$$

(b) Let (Θ_j, Γ_j) be a point process on $X \times Z$ such that $\{\Theta_j\}$ is a Poisson process on X and $\{\Gamma_j\}$ are Z-valued random variables which are i.i.d. and independent of $\{\Theta_k\}$. Then (Θ_j, Γ_j) is a Poisson process on $X \times Z$.

(c) If in (b) $X = Z = \mathbb{R}$ then $\tilde{\Theta} = \{\Gamma_j \Theta_j\}$ is a Poisson process. Its intensity is

$$\tilde{f}(\theta) = \mathbf{E}_\Gamma \left(f\left(\frac{\theta}{\Gamma}\right) \frac{1}{\Gamma} \right).$$

LEMMA 7.9. *Let Θ be Poisson process on X, $\psi : X \to \mathbb{R}$ a measurable function with $\int |\psi(\theta)| d\mu(\theta) < \infty$ then*

$$V = \sum_j \psi(\theta_j)$$

is finite with probability 1, the characteristic function of V is given by

$$\mathbf{E}(\exp(ivV)) = \exp\left[\int \left(e^{iv\psi(\theta)} - 1\right) d\mu(\theta)\right], \tag{7.3}$$

and

$$\mathbf{E}(V) = \int \psi(\theta) d\mu(\theta). \tag{7.4}$$

If in addition to the above conditions $\int \psi^2(\theta) d\mu(\theta) < \infty$ then

$$\mathrm{Var}(V) = \int \psi^2(\theta) d\mu(\theta) \tag{7.5}$$

REMARK. Proofs of the statements listed in Lemmas 7.8 and 7.9 can be found in [**31**].

4. Stable laws.

We say that U has standard positive stable law of index s if its characteristic function is equal to

(7.6)
$$\mathbb{E}(e^{i\xi U}) = \exp\left(\int_0^1 \left[e^{i\xi u} - 1 - i\xi u\right] \frac{sdu}{u^{s+1}} + \int_1^\infty \left[e^{i\xi u} - 1\right] \frac{sdu}{u^{s+1}}\right).$$

We say that U has a stable distribution of index s if

$$U = \alpha_+ U_+ - \alpha_- U_+ \alpha_0$$

where α_+, α_- and α_0 are constants and U_+ and U_- are independent identically distributed random variables having characteristic function given by (7.6).

Let X_j be iid random variables such that

$$t^s \mathbb{P}(X > t) \to C_1, \quad t^s \mathbb{P}(X < -t) \to C_2 \text{ as } t \to \infty.$$

Let $S_n = \sum_{j=1}^n X_j$.

THEOREM 7.10. *(a) If* $0 < s < 2$ *define* $a_n = \inf\{t : \mathbb{P}(|X| > t) \leq \frac{1}{n}\}$, $b_n = n\mathbb{E}(X 1_{|X| < a_n})$. *Then* $\frac{S_n - b_n}{a_n} \Rightarrow U$ *where* $U \sim St(\frac{C_1}{C_1+C_2}, \frac{C_2}{C_1+C_2}, 0)$.
(b) If $s = 2$ *then*

$$\frac{S_n - n\mathbb{E}(X)}{\sqrt{n \ln n}} \Rightarrow \mathcal{N}\left(0, \frac{C_1 + C_2}{2}\right).$$

(c) If $s > 2$ *then*

$$\frac{S_n - n\mathbb{E}(X)}{\sqrt{n}} \Rightarrow \mathcal{N}(0, \mathrm{Var}(X)).$$

Note that in part (a) $a_n \sim [(C_1 + C_2)n]^{1/s}$. Also if $s < 1$ then $b_n \sim \frac{s}{1-s}[C_1 + C_2]^{\frac{1-s}{s}}(C_1 - C_2)n^{1/s}$, if $s = 1$ then $a_n \sim (C_1 - C_2)n \ln n$, if $1 < s < 2$ then $b_n = \mathbb{E}(X) + o(1)$. Therefore if $s < 1$ then Theorem 7.10 can be restated as follows

$$\frac{S_n}{[(C_1 + C_2)n]^{1/s}} \Rightarrow \mathrm{St}\left(\frac{C_1}{C_1 + C_2}, \frac{C_2}{C_1 + C_2}, \alpha_0\right) \tag{7.7}$$

where

$$\alpha_0 = \frac{s}{1-s}[C_1 + C_2]^{\frac{1-s}{s}} \frac{C_1 - C_2}{C_1 + C_2}$$

and if $s > 1$ then Theorem 7.10 can be restated as follows

$$\frac{S_n - n\mathbb{E}(X)}{[(C_1 + C_2)n]^{1/s}} \Rightarrow \mathrm{St}\left(\frac{C_1}{C_1 + C_2}, \frac{C_2}{C_1 + C_2}, 0\right). \tag{7.8}$$

LEMMA 7.11. *(a) If $0 < s < 1$ and Θ_j is a Poisson process with intensity $\theta^{-(1+s)}$ then $\sum_j \Theta_j$ has a stable distribution of index s.*

(b) If $1 < s < 2$ and Θ_j is a Poisson process with intensity $\theta^{-(1+s)}$ then

$$\lim_{\delta\to 0}\left[\left(\sum_{\delta<\Theta_j}\Theta_j\right)-\frac{1}{(s-1)\delta^{s-1}}\right]$$

has a stable distribution of index s.

(c) If $s = 1$ and Θ_j is a Poisson process with intensity θ^{-2} then

$$\lim_{\delta\to 0}\left[\left(\sum_{\delta<\Theta_j}\Theta_j\right)-|\ln\delta|\right]$$

has a stable distribution of index 1.

EXERCISE 7.1. Derive Lemma 7.11 from (7.3).

5. Diffusion processes.

In this section we review some facts about diffusion processes. Proofs can be found in [**24**] and [**36**].

5.1. General Theory. Let a and $b > 0$ be smooth functions on $\mathbb{R}$ which have at most linear growth. A Markov process $X(t)$ with continuous path is called *diffusion process with drift a and diffusion coefficient b* if

$$\mathbb{E}_x(X(t+h)-X(t)) = a(x)h+o(h), \quad \mathbb{E}_x(X(t+h)-X(t))^2 = b(x)h+o(h).$$

In this case we shall write $X \sim \mathbf{D}(a,b)$.

THEOREM 7.12. *The law of a diffusion process is uniquely determined by a, b and the initial point.*

Let $\mathcal{D}$ denote the space of bounded continuous functions f such that $\mathcal{L}f = af' + \frac{b}{2}f''$ is bounded and continuous.

THEOREM 7.13. *Let $X(t)$ be the process with continuous paths. Then $X \sim \mathbf{D}(a,b)$ iff for each $f \in \mathcal{D}$*

$$M^f(t) = f(X(t)) - f(X(0)) - \int_0^t (\mathcal{L}f)(X_s)ds$$

is a martingale.

THEOREM 7.14. *If $X \sim \mathbf{D}(a,b)$ then for each smooth function $u(t,X)$*

$$N^u = u(t, X(t)) - u(0, X(0)) - \int_0^t \left(\frac{\partial u}{\partial t} + \mathcal{L}u \right) (X(s))ds$$

is a martingale.

A function $s(x)$ is called a *scale function* if $s(X(t))$ is a martingale. That is $\mathcal{L}s = 0$. We say that $X(t)$ is on natural scale if x is a scale function. That is $a(x) \equiv 0$. Note that $Y(t) = s(X(t))$ is on a natural scale. The equation for natural scale reads

$$as' + \frac{b}{2}s'' = 0.$$

Noticing that this is a first order equation for s' we find that the general solution is of the form $s(x) = As_0 + B$ where

$$s_0(x) = \int_0^x \exp - \left(\int_0^y \frac{2a}{b}(u)du \right) dy.$$

Let $\tau_{x',x''}$ be the first time X hits either x' or x''. Then the optional stopping theorem implies that

$$\mathbb{P}(X(\tau_{x',x''}) = x') = \frac{s(x'') - s(x)}{s(x'') - s(x')}.$$

Let ϕ be a monotone function, then $\phi(X(t))$ is also a diffusion process. Note that

$$\mathcal{L}(f \circ \phi) = a\phi' f' + \frac{b}{2}(f'\phi')' = \frac{b}{2}(\phi')^2 f'' + \left(a\phi' + \frac{b}{2}\phi'' \right) f'.$$

Thus

$$\phi(X) \sim \mathbf{D}(a\phi' + \frac{b}{2}\phi'', b(\phi')^2). \tag{7.9}$$

Next we consider time changes.

THEOREM 7.15. *Let $X(t) \sim \mathbf{D}(a,b)$. Consider a time change $s(t,X)$ defined by $ds = k(x)dt$. Then $X(s) \sim \mathbf{D}(\frac{a}{k}, \frac{b}{k})$.*

Intuitively, $\Delta s = h$ means that $\Delta t = \frac{h}{k}$ so that

$$\mathbb{E}_x(\Delta X) = a\frac{h}{k} + o(h), \quad \mathbb{E}_x((\Delta X)^2 = b\frac{h}{k} + o(h).$$

In particular if $X(t)$ is on a natural scale then $X(t) \sim \mathbf{D}(0,b)$ and so the time change $ds = b(X)dt$ makes $X(s) \sim \mathbf{D}(0,1)$, that is $X(s)$ is a standard Brownian motion. In other words any one dimensional diffusion process can be obtained from the Brownian motion by the change of space and time variables.

In our study we shall need diffusion processes defined on a subinterval $(\alpha, \beta) \subset \mathbb{R}$. Let a and $b > 0$ be smooth functions on (α, β). Let $\tau = \tau^X$ be the first time the process X reaches either α or β.

THEOREM 7.16. *There exists unique Markov process $X(t)$ which is stopped at time τ and such that $M^f(\min(t, \tau))$ is a martingale for each $f \in \mathcal{D}$.*

To continue the process beyond τ we need to classify the boundary points. Let τ_y denote the first time the process hits y.

A boundary point α is called *accessible* if $\mathbb{P}(\tau_\alpha < \tau_\beta, \tau_\alpha < \infty) > 0$ and it is called *inaccessible* otherwise. If α is inaccessible then it is called *entrance boundary* if for all t $\lim_{x\to\alpha} \mathbb{P}_x(\tau_y < t) > 0$ and it is called *natural boundary* otherwise. To proceed further it is convenient to put process on a natural scale

LEMMA 7.17. *If X is on a natural scale and $\alpha = -\infty$ then it is inaccessible.*

PROOF. X visits $x+\varepsilon$ infinitely many times before coming to α. $\square$

Suppose that α is accessible and X is on a natural scale. Then α is finite. By the foregoing discussion X is obtained from a Brownian Motion $W(s)$ by a time change $dt = k(W)ds$. We can extend W beyond the first hit of α as a reflected Brownian Motion. The intristic time passed before W reaches y is $T = \int_0^{\tau_y^W} k(W(s))ds$. We say that α is an *exit boundary* if $\mathbb{P}_x(T < t) = 0$ and that it is *a regular boundary* otherwise.

The following fact is helpful in our studies.

LEMMA 7.18. *Let $X \sim \mathbf{D}(0, cx^{2\gamma})$.*

(a) ∞ is a natural boundary if $\gamma \leq 1$ and an entrance boundary otherwise.

(b) 0 is a natural boundary if $\gamma \geq 1$, an exit boundary if $\frac{1}{2} \leq \gamma < 1$ and a regular boundary if $\gamma < \frac{1}{2}$ and

PROOF. Let T_0 be the first time when $X(T) = 2^m$ for some m. If $X(T_j) = 2^{m_j}$ let T_{j+1} be the first time after T_j when $X(T) = 2^{m_j \pm 1}$. Let $\sigma_j = T_{j+1} - T_j$. Note that the random variables $\tilde{\sigma}_j = \frac{\sigma_j}{2^{(1-\gamma)m_j}}$ are iid random variables. Let J be the first time when $m_J = 0$. Let $m_j(t) = k$ if $T_j \leq t \leq T_{j+1}$ and $m_j = k$. Let $V_k = \text{mes}(t : m_j(t) = k$. Suppose that $\gamma > 1$. Observe that

$$\mathbb{P}(m_{j+1} = m_j - 1) = \frac{2}{3}.$$

Thus m_j is transient to the left and so there exits $\rho > 0$ such that

$$\mathbb{P}(m_j(t+s) \neq k \text{ for } s \geq 2^{2(1-\gamma)k} | m_j(t) = k) > \rho.$$

Accordingly

$$\mathbb{P}(V_k \geq l 2^{2(1-\gamma)k}) \leq (1-\rho)^l. \tag{7.10}$$

Next there is a constant C such that if $\tau_J \geq t$ then there is k such that

$$V_k \geq \frac{t}{C} 2^{2(1-\gamma)k}$$

so

$$\mathbb{P}(\tau_J \geq t) \leq \sum_k (1-\rho)^{tk/C} \leq \text{Const}(1-\rho)^{t/C}.$$

Hence ∞ is an entrance boundary.

On the other hand if $\gamma \leq 1$ then $\sigma_j \geq 2^{2(1-\gamma)m_j} \tilde{\sigma}_j \geq \tilde{\sigma}_j$ so $\tau_J \geq \sum_{j=1}^{j} \tilde{\sigma}_j$. Since $J \to \infty$ as $X(0) \to \infty$ we have

$$\mathbb{P}(\tau_J \leq t) \leq \mathbb{P}(\sum_{j=1}^{J} \tilde{\sigma}_j \leq t) \to 0.$$

This completes the proof of part (a).

(b) Consider first the case where $\gamma \geq 1$. Then using the notation of part (a) we see that $\sigma_j = \tilde{\sigma}_j$ and since the walk m_j should take a growing number of steps to pass either from 0 to $-N$ or to $-N$ to 0 we see that both

$$\mathbb{P}_1(\tau_{2^{-N}} \leq t) \text{ and } \mathbb{P}_{2^{-N}}(\tau_1 \leq t)$$

tend to 0. Hence 0 is a natural boundary.

Next consider the case $\gamma < 1$. Since the walk is transient to the left

$$\mathbb{P}(m_j < 0 \text{ for all } j) > 0.$$

On theother hand (7.10) implies that for each $x \in (2^{-N}, 1)$

$$\mathbb{P}_x(\tau_{2^{-N},1} \geq l) \leq (1-\rho)^{t/C}.$$

It follows that there exist numbers ζ, t

$$\mathbb{P}_{1/2}(\tau_{2^{-N}} < \tau_1, \tau_N \leq t) > \zeta.$$

However

$$\cap_N \{\tau_{2^{-N}} < \tau_1, \tau_{2^{-N}} \leq t\} \supset \{\tau_0 \leq t\}.$$

Therefore 0 is accessible.

Next $\mathbb{E}(\text{Card}(j \leq J : m_j = -k) = \text{Const} 2^k$ because the number of excursions from 2^{-k} to 0 before reaching 1 has geometric distribution with parameter 2^{-k}. Accordingly

$$\mathbb{E}(V_{-k}) = \text{Const} 2^k 2^{-2(1-\gamma)k} = \text{Const} 2^{(2\gamma-1)k}. \tag{7.11}$$

Summing over k we see that

$$\mathbb{E}_x(\tau_1) = \infty. \tag{7.12}$$

if $x \in (0,1)$ and $\gamma \geq \frac{1}{2}$. Assume that

$$\mathbb{P}_0(\tau_1 \leq t) > \zeta \tag{7.13}$$

for some t, ζ and $\frac{1}{2} \leq \gamma < 1$. Since x has continuous trajectories it passes through each point in $(0,1)$ before reaching 1 and so $\mathbb{P}_x(\tau_1 \leq t) > \zeta$ for all $x \in [0,1)$. By induction

$$\mathbb{P}_x(\tau_1 \geq lt) \leq (1-\zeta)^l \tag{7.14}$$

contradicting (7.12). Hence (7.13) is false, that is 0 is an exit boundary.

Finally if $\gamma < \frac{1}{2}$ then summing (7.11) over k we see that

$$\mathbb{E}_0(\tau_1 < \infty.$$

It follows that (7.13) and hence (7.14) are true. Hence 0 is a regular boundary. □

EXERCISE 7.2. Let $X \sim \mathbf{D}(0, cx^{2\gamma})$ where $\gamma \geq 1$. Show that $X(t) \to 0$ as $t \to \infty$.

EXERCISE 7.3. Suppose that a and b grow not faster than linear functions. Prove that $\pm\infty$ are inaccessible.

THEOREM 7.19. *Suppose that β is inaccessible boundary.*

(a) If α is inaccessible then there is unique process with continuous paths such that $M^f(t)$ is a martingale for all $f \in \mathcal{D}$.

(b) If α is a regular boundary then there is unique process Markov process $X(t)$ such that X spends zero measure proportion of time at α and $X \sim \mathbf{D}(a,b)$ between the visits to α. If X is on a natural scale then this process is characterized as a process with continuos paths such that $M^f(t)$ is a martingale for all $f \in \mathcal{D}$ such that $\frac{\partial f}{\partial s}(\alpha) = 0$. We call this process the diffusion process with drift a and diffusion coefficient b reflected at α.

(c) If α is an exit boundary then the only Markov process with continuos paths such that $X \sim \mathbf{D}(a,b)$ between visits to α has $X(t) \equiv \alpha$ for $t \geq \tau_\alpha$. This process is characterized as a process with continuos paths such that $M^f(t)$ is a martingale for all $f \in \mathcal{D}$ such that $f(\alpha) = 0$.

If β is accessible then X has to be supplemented by boundary conditions at β which are analogous to boundary conditions at α.

THEOREM 7.20. *Let $X_x(t)$ denote the diffusion process with drift a and diffusion coefficient b started from x and stopped at reaching β. If α is an entrance then $X_x(t)$ converges weakly to a limiting process $X_\alpha(t)$.*

We will call $X_\alpha(t)$ the diffusion process with drift a and diffusion coefficient b started from α.

5.2. Bessel processes. Let $X(t)$ be a diffusion process on $(0,\infty)$ which is invariant by rescalings $x \to cx, t \to ct$. That is we assume that $Y(t) = cX(t/c)$ has the same distribution as X. By the foregoing discussion

$$a_Y(y) = a\left(\frac{y}{c}\right), \quad b_Y(y) = cb\left(\frac{y}{c}\right).$$

Choosing $c = y$ we get

$$a_Y(y) = a(1), \quad b_Y(y) = yb(1).$$

That is X is invariant by rescalings iff $X \sim \mathbf{D}(A, BX)$. Changing time $s = \lambda t$ replaces (A, B) by $(A/\lambda, B/\lambda)$ so the processes differ only by the choice of time scale if $A_1/B_1 = A_2/B_2$. It is customary to chose time scale so that $B = 4$. Thus we call $\mathbf{D}(A, Bx)$ *square Bessel process of dimension* $4A/B$. We will denote square Bessel process of dimension d by SqBess_d. For example if W is a standard Brownian Motion and $X = W^2$ then by (7.9) $X \sim \mathbf{D}(1, 4x)$ that is $W^2 \sim \mathrm{SqBess}_1$.

Next let $X_1 \sim \mathbf{D}(A_1, Bx)$ and $X_2 \sim \mathbf{D}(A_2, Bx)$ be independent Bessel processes, $X = X_1 + X_2$. Then by approximating any function by sum of products we see that

$$\phi(X_1(t), X_2(t)) - \phi(X_1(0), X_2(0)) -$$
$$\int_0^t \left(A_1\frac{\partial\phi}{\partial x_1} + A_2\frac{\partial\phi}{\partial x_2} + Bx_1\frac{\partial^2\phi}{\partial x_1^2} + \frac{B}{2}x_2\frac{\partial^2\phi}{\partial x_2^2}\right)(X_1(s), X_2(s))ds$$

is a martingale. If $\phi(x_1, x_2) = f(x_1+x_2)$ then the integrand reduces to $(A_1 + A_2)f' + \frac{B}{2}(x_1+x_2)f''$ so that $X \sim \mathbf{D}(A_1 + A_2, Bx)$. In particular if $W_1, W_2 \dots W_j$ are independent Brownian Motions then

$$W_1^2 + W_2^2 + \dots W_d^2 \sim \mathrm{SqBess}_d.$$

Next all diffusion processes invariant by rescalings $x \to c^\rho x, t \to ct$ can be obtained by are of the form X^ρ where $X \sim \mathrm{SqBess}_d$. For example if $X \sin \mathrm{SqBess}_\delta$ then $Y = \sqrt{X}$ is called *Bessel process of dimension* δ. According to (7.9) if $X \sim \mathbf{D}(A, Bx)$ then

$$Y \sim \mathbf{D}\left(\frac{A - \frac{B}{4}}{2y}, \frac{B}{4}\right).$$

We will denote Bessel processes by Bess_δ.

Let X be a standard square Bessel process of dimension d. The equation for natural scale takes form $2xs'' + \delta s' = 0$ so if we define for

$\delta \neq 2$

$$Y = X^{1-\frac{\delta}{2}} \tag{7.15}$$

then $Y \sim \mathbf{D}(0, (2-\delta)^2 y^{2\gamma})$ where $\gamma = \frac{1-\delta}{2-\delta}$. (Recall that for $\delta = 2$ we just have an absolute value of two dimensional Brownian Motion.) Now Lemma 7.18 gives the following

COROLLARY 7.21. *0 is an entrance boundary if $\delta \geq 2$, a regular boundary if $0 < \delta < 2$ and an exit boundary if $\delta < 0$.*

PROOF. If $\delta > 0$ then Y maps 0 to ∞. Also $\gamma = \frac{\delta-1}{\delta-2} > 1$ and so ∞ is an entrance boundary for Y. If $\delta < 2$ then Y maps 0 to 0. $\gamma = 1 - \frac{1}{2-\delta}$ so $\gamma \in (-\infty, \frac{1}{2})$ if $0 < \delta < 2$ and $\gamma \in [\frac{1}{2}, 1)$ if $\delta < 0$. In the former case 0 is a regular boundary for Y, in the later case 0 is an exit boundary for Y. Finally the case $\delta = 2$ follows since X is the absolute value of 2 dimensional Brownian Motion. □

EXERCISE 7.4. Show that SqBess_δ is recurrent if $0 < \delta < 2$ and transient otherwise.

In particular 0 is reached almost surely if $\delta < 2$. We need the asymptotics of the probability that 0 is not reached for long time. Let $\boldsymbol{\nu} = 1 - \frac{\delta}{2}$.

LEMMA 7.22.

$$\mathbb{P}_1(\tau_0 > t) \leq Ct^{-\boldsymbol{\nu}}.$$

PROOF. Let k_0 be the largest number such that $\tau_{2^k} < \tau_0$. Denote $N = \log_2 t$ and let R, l_0 be large numbers to be specified later. Consider the sets

$$A_k = \{\mathrm{mes}(u \leq \tau_0 : 2^{k-1} < X(u) \leq 2^k) \geq R(N-k)2^k\}, k > 1$$

$$A_0 = \{\mathrm{mes}(u \leq \tau_0 : X(u) \leq 1) \geq \frac{t}{10}\}, k > 1$$

Note that

$$\sum_{k=1}^{N-l_0} R(N-k)2^k = tR \sum_{l=l_0}^{\infty} l2^{-l}$$

so if l_0 is so large that $\sum_{l=l_0}^{\infty} l2^{-l} < \frac{9}{10R}$ then $\tau_0 > t$ implies that either $k > N - l_0$ or A_k happens for some $k \geq 0$. The probability of the first alternative is less than $Ct^{-\boldsymbol{\nu}}$. On the other hand for $k > 1$

$$\mathbb{P}(A_k) \leq \mathbb{P}_1(\tau_k < \tau_0)\mathbb{P}_{2^k}(\mathrm{mes}(u \leq \tau_0 : 2^{k-1} < X(u) \leq 2^k) \geq R(N-k)2^k).$$

The first factor equals to $2^{-\boldsymbol{\nu}k}$ while the second is $O(\theta^{R(N-k)})$ for some $\theta < 1$ because there exists $\rho > 0$ such that

$$\mathbb{P}(X(s) \notin [2^{k-1}, 2^k] \text{ for all } s \in [t+2^k, \tau_0] | X(t) \in [2^{k-1}, 2^k]) \geq \rho.$$

Choose R so large that $2^{\boldsymbol{\nu}}\theta^R < 1$ in which case

$$\sum_{k=1}^{N-l_0} \mathbb{P}(A_k) \leq \sum_k \left(2^{\boldsymbol{\nu}}\theta^R\right)^{-k} \theta^{NR} \leq C2^{-\boldsymbol{\nu}(N-l_0)} \leq Ct^{-\boldsymbol{\nu}}.$$

Finally a similar argument shows that $\mathbb{P}(A_0) \leq e^{-\gamma t}$ completing the proof of the lemma. □

LEMMA 7.23. *The limit* $\lim_{t\to\infty} t^{\boldsymbol{\nu}}\mathbb{P}_1(\tau_0 > t)$ *exists and is positive.*

PROOF. If $s > t$ then

$$t^{\boldsymbol{\nu}}\mathbb{P}_1(\tau_0 > t) \geq (t^{\boldsymbol{\nu}}\mathbb{P}(\tau_0 > \tau_{t/s}))\mathbb{P}_1(\tau_0 > t|\tau_0 > \tau_{t/s}) = s^{\boldsymbol{\nu}}\mathbb{P}_{t/s}(\tau_0 > t) = s^{\boldsymbol{\nu}}\mathbb{P}_1(\tau_0 > s).$$

Thus $t^{\boldsymbol{\nu}}\mathbb{P}_1(\tau_0 > t)$ is an increasing function of t. Since it is bounded by Lemma 7.22 the result follows. □

LEMMA 7.24. *The limit* $\lim_{t\to\infty} t^{\boldsymbol{\nu}}\mathbb{P}_1(\int_0^{\tau_0} X(u)du > t^2)$ *exists and is positive.*

PROOF. If $s > t$ then

$$\begin{aligned} t^{\boldsymbol{\nu}}\mathbb{P}_1(\int_0^{\tau_0} X(u)du > t^2) &\geq (t^{\boldsymbol{\nu}}\mathbb{P}(\tau_0 > \tau_{t/s}))\mathbb{P}_1(\int_0^{\tau_0} X(u)du > t^2|\tau_0 > \tau_{t/s}) \\ &= s^{\boldsymbol{\nu}}\mathbb{P}_{t/s}(\int_0^{\tau_0} X(u)du > t^2) = s^{\boldsymbol{\nu}}\mathbb{P}_1(\int_0^{\tau_0} X(u)du > s^2) \end{aligned}$$

Thus $t^{\boldsymbol{\nu}}\mathbb{P}_1(\int_0^{\tau_0} X(u)du > t^2)$ is an increasing function of t. It remains to show that it is bounded. We have

$$t^{\boldsymbol{\nu}}\mathbb{P}_1(\int_0^{\tau_0} X(u)du > t^2) \leq t^{\boldsymbol{\nu}}\mathbb{P}(\tau_0 > \tau_t) + t^{\boldsymbol{\nu}}\mathbb{P}_1(\int_0^{\tau_0} X(u)du > t^2, \tau_0 < \tau_t).$$

The first term equals to 1. Next observe that if $\int_0^{\tau_0} X(u)du > t^2$ and $X(u) < t$ for $t \leq \tau_0$ then $\tau_0 > t$ so the second term is bounded by Lemma 7.22. The result follows. □

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