**Sample final problems.**

(1) Consider a random walk $S_n = X_1 + X_2 + \dots X_n$ where $X_j$ are idd taking values $-1$ and $2$ each with probability $1/2$.

(a) Find $\theta$ such that $\theta^{S_n}$ is a martingale.

(b) Find expected number to visits to $-1$.

(2) Consider a Markov chain with states $\{1, 2, 3, 4\}$ and transition matrix

$$\begin{pmatrix} 0 & 1/2 & 0 & 1/2 \\ 1/3 & 0 & 2/3 & 0 \\ 0 & 2/3 & 0 & 1/3 \\ 1/2 & 0 & 1/3 & 0 \end{pmatrix}.$$

Find the expected time till the first visit to 3 given that the state starts at 1.

(3) Let $W(t)$ be a standard Brownian Motion. Find the best linear prediction of $W(2)$ based on $W(1)$ and $W(3)$.

(4) Let $X_n$ be a weakly stationary sequence having twice differentiable spectral density. Show that the series

$$\sum_{n=1}^{\infty} \frac{X_n}{n}$$

converges in the mean square sense.

(5) Alice and Bob arrive to the bus stop at 6PM. Alice waits for bus A while Bob waits for bus B. Both buses start running at 6AM. Interarrival times for bus A have uniform distribution on $[0, 10]$ min and Interarrival times for bus B have exponential distribution on with mean 10 min. Suppose that interarrival times are iid independent from each other. Compute (approximately) the probability that Alice will leave before Bob.

(6) A coin is tossed $n$ times. By a *run of heads* we mean a maximal uninterrupted sequence of heads. Let $R_n$ be the number of runs of heads up to time $n$. Find $a_n$ and $b_n$ such that $\frac{R_n - a_n}{b_n}$ has a non-trivial limiting distribution and compute that distribution.

(7) Consider $M(1-\varepsilon)/M(1)/1$-queue. Let $L_\varepsilon$ be the queue length at equilibrium. Find $a_\varepsilon$ and $b_\varepsilon$ such that $\frac{L_\varepsilon - a_\varepsilon}{b_\varepsilon}$ has a non-trivial limiting distribution and compute that distribution.

(8) Consider a queue with interarrival times $X_n$ being Uniform(0,1) and service times being exponential with parameter 1. Find the conditional distribution of $X_n$ given that no customer was served during that time.

(9) Let $W(t)$ be a standard Brownian Motion. Let $M(t) = \max_{[0,t]} W(s)$.

(a) Find $P(M(t) > s | W(t) = u)$.

(b) Find the distribution of $M(t) - W(t)$.

(10) Consider a diffusion process $X_t$ with drift $a(x) = |x|$ and diffusion coefficient $b(x) = x^2 + 1$.

(a) Find a non-constant function $\phi$ such that $\phi(X_t)$ is a martingale.

(b) Let $M = \min_{[0,\infty)} X_t$. Find the distribution of $M$.