Final Topics.

1. Generating functions.

Local Limit Theorem.
$S_n = X_1 + X_2 + \dots X_n$ where X_j are iid, $X_j \sim X$, $EX = 0$, $VX = \sigma^2$. Let

$$\phi(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{z^2}{2\sigma^2}\right).$$

Theorem. (Lattice LLT) If X has lattice distribution with span h, so that $X \in b + h\mathbb{Z}$ then for any $x_n \in bn + h\mathbb{Z}$ such that $x_n/sqrtn \to z$ we have

$$\frac{\sqrt{n}P(S_n = x_n)}{h} \to \Phi(z).$$

Theorem. (Non-lattice LLT) If X has non-lattice distribution then for any $x_n \in bn + h\mathbb{Z}$ such that $x_n/sqrtn \to z$ for any a, b we have

$$\frac{\sqrt{n}P(S_n \in (x_n + a, x_n)}{h} \to (b - a)\Phi(z).$$

Recurrence of Random walks.
Therem. Let $S_n = X_1 + X_2 \dots X_n$ where X_j are iid and $X_j \sim X$ and $E|X| < \infty$.
(a) Suppose thar X is integer valued and considered as a Markov process on $\mathbb{Z}$, S_n is irreducible and aperiodic. Then S_n is recurrent if and only if $EX = 0$.
(b) Suppose that X is non lattice, then the following conditions are equivalent.
(i) $EX = 0$.
(ii) For any interval I, S_n visits I infinitely many times.
Branching processes.

$$Z_{n+1} = \sum_{j=1}^{Z_n} X_{jn},$$

where $Z_0 = 1$, X_j are iid, $X_j \sim X$, $\text{Var}X > 0$. Let p be the extinction probability

$$p = \lim_{n\to\infty} P(Z_n) = 0.$$

Theorem. If $EX \leq 1$ then $p = 1$. If $EX > 1$ then p is the smallest root of $p = G_X(p)$ where $G_X(s)$ is the generating function of X.

2. Markov chains.

n-step transition probabilities.
$P(n) = P^n$. $P(n+1) = PP(n) = P(n)P$. $\pi(n) = \pi(0)P^n$.
Classification of states.
Theorem. (a) j is transient iff $\sum_n p_{jj}(n) < \infty$.
(b) j is not positively recurrent iff $p_{jj}(n) \to 0$.
(c) If j is transient then $P(X \text{ returns } j\ n \text{ times}|X_0 = j) = p^n(1-p)$.
Classification of chains.
Theorem. (a) If $i \leftrightarrow j$ then i and j have the same period.
(b) If $i \leftrightarrow j$ then i and j are of the same type.
(c) If $i \to j$ and j is transient then i is transient.
(d) If $i \to j$ and i is recurrent then $j \to i$.
Decomposition Theorem. $S = T \bigcup C_1 \bigcup C_2 \cdots \bigcup C_n \dots$ where T consists of transient states and C_j are closed recurrent subchains.

$C_j = \bigcup_{l=1}^{n_j} C_{jl}$ where $X_0 \in C_{jl} \Rightarrow X_n \in C_{j;(l+n) \mod n_j}$ and $X_{n_j k}$ restricted to C_{jl} is irreducible and aperiodic.

STATIONARY DISTRIBUTION AND LIMIT THEOREMS.

Theorem An irreducible chain has a stationary distribution iff it is positively recurrent. In this case $\pi_j = \mu_j^{-1}$ where μ_j is the mean recurrence time.

Theorem (a) An irredicible chain is positively recurrent iff there exists positive summable solution to the equation $x = xP$.

(b) Let s be an element of an irreducible chain. The chain is transient iff there exists a non-zero solution y_j satisfying $|y_j| \leq 1$ to the equation

$$y_i = \sum_{j \neq s} p_{ij} y_j.$$

Theorem. (a) If the chain is irreducible an aperiodic then $p_{ij}(n) \to \mu_j^{-1}$ (where $\infty^{-1} = 0$).

(b) For an arbitrary irreducible chain

$$\frac{1}{N} \sum_{n=1}^{N} p_{ij}(n) \to \mu_j^{-1}.$$

CHAINS WITH FINITELY MANY STATES.

Theorem. (a) In a finite chain j every recurrent state is positively recurrent.

(b) j is positively recurrent iff $j \to i$ implies $i \to j$.

Theorem. In a finite irreducible aperiodic chain there exist constants $C > 0$ and $\theta < 1$ such that $|p_{ij}(n) - \pi_j| \leq C\theta^n$.

POISSON PROCESS.

Poisson process with intensity λ :

$S_n = \sum_{j=1}^{n} X_j$ where X_j are iid, $X_j \sim$Exp(λ). $N(t) = \max(n : S_n \leq t)$.

$N(t)$-Poisson(λ) iff $N(0) = 0$ N has independent increments and

$$P(N(t+h) - N(t) = k) = \begin{cases} 1 - \lambda h + o(h) & \text{if } k = 0 \\ \lambda h + o(h) & \text{if } k = 1 \\ o(h) & \text{if } k > 1 \end{cases}.$$

Let (X, μ) be a measure space. A point X-valued process is Poisson process with intensity measure μ if letting $N(A)$ to denote the number of points in A we have

(i) $N(A)$ has Poisson$(\mu(A))$ distribution.

(ii) If $A_1, A_2 \dots A_n$ are disjoint then $N(A_1), N(A_2) \dots N(A_n)$ are independent.

Theorem (a) If $\{x_j\}$ is a Poisson process on X with intensity μ and $f : X \to Y$ is a measurable map then $\{f(x_j)\}$ is a Poisson process on Y with intensity $\nu(B) = \mu(f^{-1}B)$.

(b) If N_1 and N_2 are independent Poisson processes with intensities μ_1 and μ_2 then $N = N_1 + N_2$ is a Poisson process with intensity $\mu = \mu_1 + \mu_2$.

(c) If N is a Poisson process with intensity μ and we discard the point x with probability $p(x)$ then the remaining points form Poisson process with intensity ν where

$$\nu(A) = \int_A p(x) d\mu(x).$$

Campbell-Hardy Theorem. Let $\{x_j\}$ be a Poisson process of $\mathbb{R}$ with intensity λ Let W_j be iid random variables independent of the Poisson process and r be a smooth function. Let

$$G(t) = \sum_{x_j > 0} r(t - x_j) W_j.$$

Then

$$E\left(\exp(i\theta G(t))\right) = \exp\left(\lambda \int_0^t e^{i\theta r(s) W} ds\right).$$

Birth and death chain.

Let $p_{n(n+1)} = p_n$, $p_{n(n-1)} = q_n$ and $p_{nj} = 0$ for $j \neq n \pm 1$.

Theorem. Let $\xi_n = \frac{p_0 p_1 \dots p_{n-1}}{q_1 q_2 \dots q_n}$. The process is positively recurrent iff $S = \sum_n \xi_n < \infty$. In this case $\pi_j = \xi_j / S$.

Theorem. Let $\eta_j = \frac{q_1 q_2 \dots q_j}{p_1 p_2 \dots p_j}$. The process is recurrent if $\sum_{j=0}^{\infty} \eta_j = \infty$.

Continuous time chains.

Forward equation $P' = PG$.

Backward equation $P' = GP$.

Transition probabilities and the generator $P(t) = \exp(tG)$.

Stationary distribution $\pi G = 0$.

Theorem. Suppose that $p_{ij}(t) \to \delta_{ij}$ as $t \to 0$ and that the chain is irreducible.
(a) If there exist stationary distribution then $p_{ij}(t) \to \pi_j$ as $t \to \infty$.
(b) If there is no stationary distribution then $p_{ij}(t) \to 0$ as $t \to \infty$.

Theorem. Suppose that either $|g_{ii}|$ are uniformly bounded or for each i there are only finitely many j such that $g_{ij} \neq 0$. Then
(a) The holding time at state i has $\mathrm{Exp}(-g_{ii})$ distribution.
(b) The probability that the first jump from state i is to state j equals $g_{ij}/|g_{ii}|$.

3. Stationary processes.

Linear prediction.

Theorem. Let X_n be a real stationary sequence with zero mean and covariance $c(m)$. Then the best linear predictor of X_{r+k} based on $X_r, X_{r-1} \dots X_{r-s}$ takes form

$$\hat{X}_{r+k} = \sum_{j=0}^{s} a_j X_{r-j}$$

where

$$\sum_{j=0}^{s} a_j c(|m - j|) = c(k + m) \text{ for } m = 0, 1 \dots s.$$

Spectral Density.

Continuous time Bochner Theorem. The following are equivalent
(a) $c(t)$ is a covariance function of a weakly stationary process;
(b) For any real $t_1, t_2 \dots t_n$ and $z_1, z_2 \dots z_n$

$$\sum_{jk} c(t_j - t_k) z_j \bar{z}_k \geq 0.$$

(c) There exists a measure μ on $\mathbb{R}$ such that

$$c(t) = \int_{-\infty}^{\infty} e^{it\lambda} d\mu(\lambda).$$

Discrete time Bochner Theorem. The following are equivalent
(a) $c(n)$ is a covariance function of a weakly stationary process;

(b) For any real $m_1, m_2 \dots m_n$ and $z_1, z_2 \dots z_n$

$$\sum_{jk} c(m_j - m_k) z_j \bar{z}_k \geq 0.$$

(c) There exists a measure μ on $(-\pi, \pi]$ such that

$$c(t) = \int_{-\pi}^{\pi} e^{it\lambda} d\mu(\lambda).$$

Ergodic Theorem.

Ergodic Theorem for stationary sequences. If X_n is a weakly stationary sequence then there exists a mean square limit

$$\bar{X} = \frac{1}{N} \lim_{n=1}^{N} X_n.$$

If X_n is strongly stationary then also $X_n \to \bar{X}$ with probability 1.

If the spectral measure satisfies $\mu(\{0\}) = 0$ then $\bar{X} = EX$.

Ergodic Theorem for measure preserving transformations. Let T be a transformation of space Ω preserving a probability measure μ. Let f be a square integrable observable. Then there exists a square integrable observable $\bar{f}(\omega)$ such that

$$\frac{1}{N} \sum_{n=1}^{N} f(T^n \omega)$$

converges to $\bar{f}(\omega)$ in mean squares and with probability 1.

If any T-invariant set A has measure either 0 or 1 then $\bar{f}(\omega) = Ef$.

4. Renewal Theory.

$$S_n = X_1 + X_2 \cdots + X_n$$

where X_n are positive iid with distribution F. Let $\mu = EX$, $N(t) = \max(n : S_n \leq t)$, $m(t) = E(N(t))$, $F_k = F * F\dot{F}$ (k times).

$$m(t) = \sum_{k=1}^{\infty} F_k(t).$$

Renewal equation.

$$\mu = H + \mu * F \Rightarrow \mu = H + H * m.$$

Limit Theorems for the renewal process.

Theorem. (a) $N(t)/t \to \frac{1}{\mu}$ almost surely.

(b) If $\sigma^2 = Var(X)$ satisfies $0 < \sigma^2 < \infty$ then

$$\frac{N(t) - t/\mu}{\sqrt{t\sigma^2/\mu^3}} \Rightarrow \mathcal{N}(0, 1).$$

Renewal Theorems.

Discrete Renewal Theorem. If X is integer valued and has span 1 then

$$P(\exists n : S_n = m) \to \frac{1}{\mu}$$

as $m \to \infty$.

Renewal Theorem. (a) $m(t)/t \to \frac{1}{\mu}$.
(b) If X is non-lattice then for any positive integrable non-increasing function g

$$g * m \to \frac{1}{\mu} \int_0^\infty g(s)ds.$$

(c) If $V(X) = \sigma^2 < \infty$ then

$$\lim_{t\to\infty} \left(m(t) - \frac{t}{\mu} \right) = \frac{\sigma^2 - \mu^2}{2\mu^2}.$$

5. Queues.

Let $\rho = ES/EX$ and $U_n = S_n - X_{n+1}$. Let W_n be the waiting time of n-th customer.

Theorem. (a) If $\rho \geq 1$ then $P(W_n < z) \to 0$ as $n \to \infty$.
(b) If $\rho < 1$ then as $n \to \infty$ W_n converges in distribution to

$$\max_{m\geq 0} \sum_{j=1}^m U_j.$$

6. Optimal Sampling.

Wald's equation. If X_j are iid and M is a stopping time then

$$E\left(\sum_{n=1}^M X_j \right) = EXEM.$$

Martingale Sampling. If (Y_n, F_n) is a martingale and T is a stopping time such that
(a) $P(T < \infty) = 1$;
(b) $P(|Y_T|) < \infty$;
(c) $E(Y_n 1_{n>T}) \to 0$ as $n \to \infty$ then
$E(Y_T|F_0) = Y_0$.

7. Diffusion processes.

Maximum of Brownian Motion. Let $W(t)$ be a Brownian Motion with zero drift and variance t. Let $M(t) = \sum_{s\in[0,t]} W(s)$, $\tau(x) = \min(t : W(t) = x)$. Then for $x > 0$

$$P(M(t) > x) = P(\tau(x) < t) = \frac{1}{2} P(W(t) > x).$$

Arc Sine Law.

$$P(W(t) \text{ has no zeros on } [t_0, t_1]) = \frac{2}{\pi} \sin^{-1} \left(\sqrt{t_0/t_1} \right).$$

Diffusions.
Let $X(t)$ be a diffusion process with drift $a(t, x)$ and diffusion coefficient $b(t, x)$. Let $p((s, x) \to (t, y))$ be transition density of $X(t)$.

Forward equation.

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial y} [a(t, y)p(t, y)] + +\frac{1}{2} \frac{\partial^2}{\partial y^2} [b(t, y)p(t, y)].$$

Backward equation.

$$\frac{\partial p}{\partial s} = -a(s,x)\frac{\partial p}{\partial x} - \frac{b(s,x)}{2}\frac{\partial^2 p}{\partial y^2}.$$

Suppose that a and b do not depend on time

Invariant density.

$$\frac{\partial}{\partial y}\left[a(t,y)p(t,y)\right] = \frac{1}{2}\frac{\partial^2}{\partial y^2}\left[b(t,y)p(t,y)\right].$$

Generator. If $f(t,x)$ is twice differentiable and the derivatives are bounded then

$$\lim_{h\to 0}\frac{E_{(t,x)}f(t+h,X_{t+h}) - f(t,x)}{h} = \frac{\partial f}{\partial t}(t,x) + a(x)\frac{\partial f}{\partial x}(t,x) + \frac{b(x)}{2}\frac{\partial^2 f}{Px^2}(t,x).$$

Martingales. $f(t,X_t)$ is a martingale iff

$$\frac{\partial f}{\partial t}(t,x) + a(x)\frac{\partial f}{\partial x}(t,x) + \frac{b(x)}{2}\frac{\partial^2 f}{Px^2}(t,x) = 0$$