STAT 650 Final.

(1) Consider the Markov chain on $\mathbb{Z}$ such that $X_{n+1} - X_n = \pm 1$ and if $X_n \neq 0$ then X_{n+1} moves towards 0 with probability 2/3 and away from 0 with probability 1/3. Suppose that $P(X_{n+1} = 1 | X_n = 0) = 3/4$ and $P(X_{n+1} = -1 | X_0 = 0) = 1/4$. Find the stationary distribution of this chain.

Solution. For $n \geq 2$ the stationarity equation gives

$$\pi_n = \frac{2\pi_{n+1}}{3} + \frac{\pi_{n-1}}{3}.$$

The general solution to this equation is

$$\pi_n = A_+ \left(\frac{1}{2}\right)^n + B_+, n \geq 1.$$

Since solution should be summable at $+\infty$ we have $B_+ = 0$. Likewise

$$\pi_{-n} = A_- \left(\frac{1}{2}\right)^n.$$

For $n = \pm 1$ we get

$$\frac{A_+}{2} = \frac{A_+}{6} + \frac{3\pi_0}{4} \Rightarrow A_+ = \frac{9\pi_0}{4},$$

$$\frac{A_-}{2} = \frac{A_-}{6} + \frac{\pi_0}{4} \Rightarrow A_- = \frac{3\pi_0}{4},$$

Next

$$1 = \sum_{n=1}^{\infty} \pi_n = \frac{3\pi_0}{4} + \pi_0 + \frac{9\pi_0}{4} = 4\pi_0 \Rightarrow \pi_0 = \frac{1}{4}.$$

Thus

$$\pi_n = \begin{cases} \frac{3}{16}\left(\frac{1}{2}\right)^{|n|} & \text{if } n < 0 \\ \frac{1}{4} & \text{if } n = 0 \\ \frac{9}{16}\left(\frac{1}{2}\right)^n & \text{if } n > 0 \end{cases}$$

(2) Let X_n be a weakly stationary sequence with $E(X_n) = 0, V(X_n) = 1$ and spectral density $f(\lambda)$. Let $Y_n = \sum_{j=0}^n X_j 2^j$. Find

$$\lim_{n\to\infty} \frac{\mathrm{Var}(Y_n)}{4^n.}$$

Solution.

$$\frac{\mathrm{Var}(Y_n)}{4^n} = E\left(\left(\frac{Y_n}{2^n}\right)^2\right) = E\left(\left(\sum_{j=1}^{n} X_j 2^{j-n}\right)^2\right) = E\left(\left(\sum_{k=0}^{n-1} X_{-k} 2^{-k}\right)^2\right)$$

where the last equation follows from stationarity. As $n \to \infty$ the last sum converges to $\sum_{k=0}^{\infty} X_{-k} 2^{-k}$. Therefore

$$\lim_{n\to\infty} \frac{\mathrm{Var}(Y_n)}{4^n} = E\left(\left(\sum_{k=0}^{\infty} X_{-k} 2^{-k}\right)^2\right)$$

$$= \int_{-\pi}^{\pi} \sum_{k,m=0}^{\infty} \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^m e^{i(m-k)\lambda} d\lambda = \int_{-\pi}^{\pi} \left[\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k e^{-ik\lambda}\right] \left[\sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^m e^{im\lambda}\right] d\lambda$$

$$= \int_{-\pi}^{\pi} \left|\frac{1}{1-\frac{1}{2}e^{-i\lambda}}\right|^2 f(\lambda) d\lambda = \int_{-\pi}^{\pi} \left|\frac{2}{2-e^{-i\lambda}}\right|^2 f(\lambda) d\lambda.$$

(3) Consider a renewal sequence $T_n = X_1 + X_2 + \dots X_n$ where X_j are iid having nonlattice distribution and such that $E(X^3) < \infty$. Let $N(t) = \max(n : T_n \leq t)$ and $D(t) = X_{N(t)+1}$. Compute

$$\lim_{t\to\infty} E(D^2(t)).$$

Solution. Let $r(t) = E(D^2(t))$. Then r satisfies

$$r(t) = H(t) + (r * F)(t)$$

where F is the distribution function of X and $H = E(X^2 1_{X>t})$.

Therefore $r = H * m$ and by Strong Renewal Theorem

$$\frac{1}{EX} \lim_{t\to\infty} r(t) = \int_0^{\infty} H(s) ds = \frac{1}{EX} \int_0^{\infty} E(X^2 1_{X>s}) ds = \frac{1}{EX} E\left(X^2 \int_0^X ds\right) = \frac{EX^3}{EX}.$$

(4) Consider a 3 dimensional Bessel propess X_t–the diffusion process on $[0, \infty]$ with drift $a(x) = 3$ and diffusion coefficient $b(x) = 4x$.

(a) Find functions $f(x)$ such that $f(X_t)$ is a martingale.

(b) For $0 < u < v < w$ find the probability that the process visits w before u given that $X_0 = v$.

Solution. (a) The condition on f reads

$$2xf'' + 3f' = 0.$$

Denoting $u = f'$ we get

$$\frac{u'}{u} = -\frac{3}{2x}.$$

Thus $u = Ax^{-3/2}$ and so $f(x) = Bx^{-1/2} + C$.

(b) Applying the optional sampling theorem to $M_t = (X_t)^{-1/2}$ we get

$$\frac{p}{\sqrt{w}} + \frac{1-p}{\sqrt{u}} = \frac{1}{\sqrt{v}} \Rightarrow p = \sqrt{\frac{w}{v}} \frac{\sqrt{u}-\sqrt{v}}{\sqrt{w}-\sqrt{u}}.$$