(1) (a) How many ways are there to divide 5 different cakes and 5 identical cookies between 2 people so that the first person gets exactly 3 cakes.

(b) How many ways are there to divide 5 different cakes and 5 identical cookies between 2 people so that the each person gets exactly 5 items.

Solution. (a) To specify the cake distribution we need to describe which cakes the first person gets, so there are $\binom{5}{3}$ choices. To specify the cookie distribution we need to describe how many cookies the first person gets. There are 6 choices $(0, 1, \ldots 5)$. So the answer is

$$\binom{5}{3} \times 6.$$

(b) To specify the cake distribution we need to describe for each cakes who gets it. There are 2 choices for each cake, so alltogether there are 2^5 choices. Given the distribution of cakes there is unique way to divide the cookies so that each person gets 5 items. So the answer is 2^5.

(2) There are three urns. The first contains 2 red balls and 5 blue balls, the second contains 3 red balls and 4 blue balls and the third contains 4 red balls and 3 blue balls. An urn is chosen at random and then balls are drawn without replacement.

(a) Find the conditional probability that the third urn was chosen given that the first three draws resulted in 2 red and 1 blue ball.

(b) Find the conditional probability that the fourth ball will be blue given that the first three draws resulted in 2 red and 1 blue ball.

Solution. (a)

$$P(U_3|2R, 1B) = \frac{P(U_3)P(2R, 1B|U_3)}{P(U_1)P(2R, 1B|U_1) + P(U_2)P(2R, 1B|U_2) + P(U_3)P(2R, 1B|U_3)}$$

$$= \frac{\frac{1}{3}\binom{4}{2}\binom{3}{1}/\binom{7}{3}}{\frac{1}{3}\binom{2}{2}\binom{5}{1}/\binom{7}{3} + \frac{1}{3}\binom{3}{2}\binom{4}{1}/\binom{7}{3} + \frac{1}{3}\binom{4}{2}\binom{3}{1}/\binom{7}{3}}$$

$$= \frac{\binom{4}{2}\binom{3}{1}}{\binom{2}{2}\binom{5}{1} + \binom{3}{2}\binom{4}{1} + \binom{4}{2}\binom{3}{1}} = \frac{18}{18 + 12 + 5} = \frac{18}{35}.$$

Similarly

$$P(U_2|2R, 1B) = \frac{12}{35}, \quad P(U_1|2R, 1B) = \frac{5}{35}.$$

(b) Using the Law of Total Probability we find $P(B_4|2R, 1B) =$

$$P(U_1|2R, 1B)P(B_4|(U_1)(2R, 1B)) + P(U_2|2R, 1B)P(B_4|(U_2)(2R, 1B)) + P(U_3|2R, 1B)P(B_4|(U_3)(2R, 1B))$$

$$= \frac{5}{35}\frac{4}{4} + \frac{12}{35}\frac{3}{4} + \frac{18}{35}\frac{2}{4} = \frac{92}{140} = \frac{23}{35}.$$

(3) (a) 3 people get 3 cards each from the standard 52 card deck. Find the probability that at least one person has only red cards.

(b) 20 tables have 3 palyers and one standard 52 card deck per table. At each table each player is given 3 cards from the deck. Let X be the number of tables where at least one player has only red cards. Compute $P(X = 6)$.

Solution. Let A_j is the event that player j has only red cards. Since there are 26 red cards we have

$$P(A_j) = \frac{\binom{26}{3}}{\binom{52}{3}}, \quad P(A_jA_k) = \frac{\binom{26}{6}}{\binom{52}{6}}, \quad P(A_1A_2A_k) = \frac{\binom{26}{9}}{\binom{52}{9}}.$$

Therefore by inclusion exclusion formula the answer is

$$\mathbf{p} = P(A_1) + P(A_2) + P(A_3) - P(A_1A_2) - P(A_1A_3) - P(A_2A_3) + P(A_1A_2A_3)$$
$$= 3\frac{\binom{26}{3}}{\binom{52}{3}} - 3\frac{\binom{26}{6}}{\binom{52}{6}} + \frac{\binom{26}{9}}{\binom{52}{9}}.$$

(b) X has binomial distributions with parameters 20 and $\mathbf{p}$ where $\mathbf{p}$ is the answer to the part (a). Therefore

$$P(X = 6) = \binom{20}{6}\mathbf{p}^6(1-\mathbf{p})^{14}.$$

(4) Let X have denisty xe^{-x} if $x \geq 0$ and 0 otherwise.
(a) Compute $P(X > 2)$.
(b) Find the density of $Y = X^2$.

Solution. (a)

$$P(X > 2) = \int_2^\infty xe^{-x}dx = -\int_2^\infty xde^{-x} = 2e^{-2} + \int_2^\infty e^{-x}dx = 2e^{-2} + e^{-2} = 3/e^2.$$

(b) Using the formula for denisty of a function of random varaibles we get

$$f_Y(y) = f_X(x)/y' = \frac{xe^{-x}}{2x} = \frac{e^{-x}}{2} = \frac{e^{-\sqrt{y}}}{2}.$$

(5) Let (X, Y) have density equal to cxy if $x \geq 0$, $y \geq 0$ and $x + y \leq 1$ and equal to 0 otherwise.
(a) Compute $E(XY)$.
(b) Compute $P(X > 2Y)$.

Solution. We have

$$\int_0^1 x(\int_0^1 ydy)dx = \int_0^1 x\frac{(1-x)^2}{2}dx = \frac{1}{2}\int_0^1 (x - 2x^2 + x^3)dx = \frac{1}{2}\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) = \frac{1}{24}.$$

Accordingly $c = 24$.

$$(a) \quad E(XY) = 24\int_0^1 x^2(\int_0^{1-x} y^2dy)dx = 24\int_0^1 x^2\frac{(1-x)^3}{3}dx = 8\int_0^1 (x^2 - 3x^3 + 3x^4 - x^5)dx$$

$$= 8\left[\frac{1}{3} - \frac{3}{4} + \frac{3}{5} - \frac{1}{6}\right] = \frac{2}{15}.$$

(b) We have two constrains $Y < \frac{X}{2}$ and $Y < 1 - X$. Note that $min(\frac{X}{2}, 1 - X) = \frac{X}{2}$ if $X < \frac{2}{3}$ and $min(\frac{X}{2}, 1 - X) = 1 - X$ if $X > \frac{2}{3}$. Thus the probability in question equals to

$$24\int_0^{\frac{2}{3}} x(\int_0^{\frac{x}{2}} ydy)dx + 24\int_{\frac{2}{3}}^1 x(\int_0^{1-x} ydy)dx.$$

The first integral equals to

$$24\int_0^{\frac{2}{3}} \frac{x^3}{8}dx = \frac{3}{4}\left(\frac{2}{3}\right)^4 = \frac{4}{27}.$$

The second integral equals to

$$12\int_{\frac{2}{3}}^1 (x - 2x^2 + x^3)dx = 12\left(\frac{1}{2}\left[1 - \left(\frac{2}{3}\right)^2\right] - \frac{2}{3}\left[1 - \left(\frac{2}{3}\right)^3\right] + \frac{1}{4}\left[1 - \left(\frac{2}{3}\right)^4\right]\right) = \frac{3}{27}.$$

Therefore the answer is $\frac{4}{27} + \frac{3}{27} = \frac{7}{27}$.

(6) Let (X, Y) have density equal to cxy if $x \geq 0$, $y \geq 0$ and $x + y \leq 1$ and equal to 0 otherwise.

(a) Compute the marginal distribution of X.

(b) Find the moment generating function of X.

Solution.

$$(a) \quad f_X(x) = 24x\int_0^{1-x} ydy = 12(x - 2x^2 + x^3).$$

(b) We have

$$\int_0^1 xe^{xt}dx = \frac{1}{t}\int_0^1 xde^{xt} = \frac{e^t}{t} - \frac{1}{t}\int_0^1 e^{xt}dx = \frac{e^t}{t} - \frac{e^t}{t^2} + \frac{1}{t^2} = \frac{e^t}{t^2}\left[e^{-t} - 1 + t\right],$$

$$\int_0^1 x^2e^{xt}dx = \frac{1}{t}\int_0^1 x^2de^{xt} = \frac{e^t}{t} - \frac{2}{t}\int_0^1 xe^{xt}dx = \frac{2e^t}{t^3}\left[-e^{-t} + 1 - t + \frac{t^2}{2}\right]$$

where the last line uses the previous equation. Finally

$$\int_0^1 x^3e^{xt}dx = \frac{1}{t}\int_0^1 x^3de^{xt} = \frac{e^t}{t} - \frac{3}{t}\int_0^1 x^2e^{xt}dx = \frac{6e^t}{t^4}\left[e^{-t} - 1 + t - \frac{t^2}{2} + \frac{t^3}{6}\right]$$

where the last line uses the previous equation. Therefore

$$M_X(t) = \int_0^1 e^{tx}12(x - 2x^2 + x^3) =$$

$$12\times\frac{e^t}{t^2}\left[e^{-t} - 1 + t\right] - 24\times\frac{2e^t}{t^3}\left[-e^{-t} + 1 - t + \frac{t^2}{2}\right] + 12\times\frac{6e^t}{t^4}\left[e^{-t} - 1 + t - \frac{t^2}{2} + \frac{t^3}{6}\right]$$

$$= 12\times\frac{e^t}{t^2}\left[e^{-t} - 1 + t\right] - 48\times\frac{e^t}{t^3}\left[-e^{-t} + 1 - t + \frac{t^2}{2}\right] + 72\times\frac{e^t}{t^4}\left[e^{-t} - 1 + t - \frac{t^2}{2} + \frac{t^3}{6}\right].$$

(7) Let X and Y be independent, and X have uniform distribution on $(0, 1)$ Y have uniform distribution on $(0, 2)$.

(a) Find the density of $Z = X + Y$.

(b) Find $P(X > Y)$.

Solution. (a) We have

$$f_Z(z) = \int f_X(x) f_Y(z-x)dx.$$

$f_X(x)f_Y(z-x) \neq 0$ if

$$0 < x < 1 \text{ and } 0 < z - x < 2. \tag{1}$$

in which case $f_X(x)f_Y(z-x) = \frac{1}{2}$. Solving (1) we get

$$x \in \begin{cases} (0,z) & \text{if } z < 1 \\ (0,1) & \text{if } 1 < z < 2 \\ (z-2,1) & \text{if } 2 < z < 3 \end{cases}.$$

The integral of $f_X(x)f_Y(z-x)$ is equal to the half of the integration interval. Accordingly we obtain

$$f_Z(z) = \begin{cases} \frac{z}{2} & \text{if } z < 1 \\ \frac{1}{2} & \text{if } 1 < z < 2 \\ \frac{3-z}{2} & \text{if } 2 < z < 3 \end{cases}.$$

(b) The possible values of (X, Y) lie in the rectangle $[0,1]\times[0,2]$ and the density is constant $\frac{1}{2}$. In other words the pair (X, Y) has uniform distribution on this rectangle. The set $X > Y$ cuts a right equilateral triangle with sides 1. Thus $P(X > Y)$ is equl to the ratio the area of the triangle to the area of the recangle, i.e.

$$P(X > Y) = \frac{1/2}{2} = \frac{1}{4}.$$

(8) N men and N women are divided randomly into N pairs. Let X_N be the number of pairs consisting of two men.

(a) Compute $E(X_N)$ and $V(X_N)$.

(b) Prove a Weak Law of Large Numbers, that is, show that for each ε

$$P\left(\left|\frac{X_N}{N} - \frac{E(X_N)}{N}\right| > \varepsilon\right).$$

Solution. (a) Let $Y_j = 1$ if man j is paired with a man and $Y_j = 0$ if man j is paired with a woman. Then

$$P(Y_j = 1) = \frac{N-1}{2N-1}$$

where the numerator shows the number possible male partners and denominator is the total number of partners of man j. Note that $X = \frac{1}{2}\sum_j Y_j$. Hence

$$EX = \frac{1}{2}\sum_{j=1}^{N} E(Y_j) = \frac{N(N-1)}{2(2N-1)}.$$

Also $V(Y_j) = \frac{N(N-1)}{(2N-1)^2}$. Next Y_jY_k takes value 0 or 1. $Y_jY_k = 1$ if either man j is paired with man k or man j and man k have two different male partners. The probability of the first event is $\frac{1}{2N-1}$ the probability of the second event is

$$\frac{2N-2}{2N-1}\frac{\begin{pmatrix} N-2 \\ 2 \end{pmatrix}}{\begin{pmatrix} 2N-2 \\ 2 \end{pmatrix}} = \frac{(N-2)(N-3)}{(2N-1)(2N-3)}.$$

Hence

$$Cov(Y_j, Y_k) = \frac{1}{2N-1} + \frac{(N-2)(N-3)}{(2N-1)(2N-3)} - \frac{(N-1)^2}{(2N-1)^2}$$

and so

$$VX = \left(\frac{N^2(N-1)}{(2N-1)^2} + N(N-1)\left[\frac{1}{2N-1} + \frac{(N-2)(N-3)}{(2N-1)(2N-3)} - \frac{(N-1)^2}{(2N-1)^2}\right]\right).$$

(b) Note that

$$\frac{(N-2)(N-3)}{(2N-1)(2N-3)} < \frac{(N-1)^2}{(2N-1)^2}.$$

Accordingly

$$V(X_N) < \left[\frac{N^2(N-1)}{(2N-1)^2} + \frac{N(N-1)}{2N-1}\right].$$

Using the fact that

$$\frac{N-1}{2N-1} < \frac{1}{2}, \quad \frac{NN-1}{(2N-1)^2} < \frac{1}{4}$$

we get

$$V(X_N) < \left[\frac{N}{2} + \frac{N}{4}\right] = \frac{3N}{4}.$$

By Chebyshev inequality

$$P\left(\left|\frac{X_N - EX_N}{N}\right| > \varepsilon\right) \leq \frac{3}{4N\varepsilon^2} \to 0 \text{ as } N \to \infty.$$

(9) Let $X_1, X_2 \dots X_{240}$ be independent identically distributed random variables such that X_j has density $3x^2$ if $x \in [0,1]$ and 0 otherwise. Let

$$S = X_1 + X_2 + \cdots + X_{240}.$$

(a) Compute ES and VS.
(b) Find approximately $P(S > 183)$.

Solution.

$$(a) \quad EX_j = \int_0^1 3x^3 dx = \frac{3}{4}, \quad EX_j = \int_0^1 3x^4 dx = \frac{3}{5}, \quad VX_j = \frac{3}{5} - \left(\frac{3}{4}\right)^2 = \frac{3}{80}.$$

Therefore

$$ES = 240 \times \frac{3}{4} = 180, \quad VS = 240 \times \frac{3}{80} = 9.$$

(b) By the Central Limit Theorem S is well approximated by $180 + \sqrt{9}Z = 180 + 3Z$ where Z is the standard normal. Accordingly

$$P(S > 183) \approx P(180 + 3Z > 183) = P(Z > 1) = 1 - P(Z < 1) = 1 - 0.84 = 0.16$$

where we have used the normal table in the next to last equality.