**MATH475 Midterm 1 solutions.**

(1) (a) How many ways are there to distribute 24 identical balls into 3 (different) boxes?

(b) How many ways are there to distribute 8 red, 8 blue and 8 green balls balls into 3(different) boxes? (The balls of the same color are identical)

(c) How many ways are there to distribute 24 different balls into 3 (different) boxes?

**Solution.** (a) Let $x_j$ be the number of balls in box $j$. This numbers completely determine the arrangement of balls so we have to count the number of solutions of

$$x_1 + x_2 + x_3 = 24$$

which equals to

$$\binom{24+3-1}{3-1} = \binom{26}{2}.$$

(b) Arguing as in part (a) we see that there are $\binom{10}{2}$ to distribute balls of each color. Since there are three colors the answer is

$$\left[\binom{10}{2}\right]^3.$$

(c) We have 3 possibilities for each ball. So the answer is $3^{24}$.

(2) (a) Find the number of permutations of the multiset $\{3A, 7B, 10C\}$.

(b) In how many permutations is distance between consecutive As at least 2?

(c) How many permutations from part (a) begin and end by the same letter?

**Solution.** (a) using the formula for the number of permutations of the multiset we obtain $\frac{20!}{3!7!10!}$.

(b) We first choose places for As. They divide the permutation into four groups of size $x_1, x_2, x_3$ and $x_4$. We have

$$x_1 + x_2 + x_3 + x_4 = 17, \quad x_2 \geq 2, x_3 \geq 2.$$

Let $y_2 = x_2 - 2, x_3 = y_3 - 2$. Then

$$x_1 + y_2 + y_3 + x_4 = 13$$

and all terms are nonnegative. Therefore there are

$$\binom{13+4-1}{4-1} = \binom{16}{3}$$

ways to choose places for As. After this is done we choose places for $B$. There are $\begin{pmatrix} 17 \\ 7 \end{pmatrix}$ possibilities. So the answer is

$$\begin{pmatrix} 16 \\ 3 \end{pmatrix} \begin{pmatrix} 17 \\ 7 \end{pmatrix}$$

(c) If the permutation starts and ends with A then there are $\frac{18!}{1!7!10!}$ ways to permute remaining 18 symbols. Likewise there are $\frac{18!}{3!5!10!}$ permutations what start and ends with B and $\frac{18!}{3!7!8!}$ permutations what start and ends with C. So the answer is

$$\frac{18!}{1!7!10!} + \frac{18!}{3!5!10!} + \frac{18!}{3!7!8!}.$$

(3) There are 10 people aged between 20 and 55 years. Show that there two groups $A$ and $B$ such that

(i) $A$ and $B$ are disjoint.

(ii) $size(A) = size(B) \leq 3$.

(iii) The sum of ages of people in $A$ equals the sum of ages of people in $B$.

**Solution.** There are $\begin{pmatrix} 10 \\ 3 \end{pmatrix} = 120$ groups of 3 people. The possible age sums of such groups vary from 60 to 165, so there are 106 possible age sums. By pigeonhole principal there are two groups with same age sum. Removing common people we obtain two disjoint group with the same age sums.

(4) Derive the formula for $\sum_{k=1}^{n} k^4$.

**Solution.** We want to represent

$$k^4 = a \begin{pmatrix} k \\ 4 \end{pmatrix} + b \begin{pmatrix} k \\ 3 \end{pmatrix} + c \begin{pmatrix} k \\ 2 \end{pmatrix} + d \begin{pmatrix} k \\ 1 \end{pmatrix}.$$

Solving linear equations for unknown coefficients we find

$$a = 4, \quad b = 36, \quad c = 14, \quad d = 1.$$

Now using the formula

$$\sum_{k=1}^{n} \begin{pmatrix} k \\ j \end{pmatrix} = \begin{pmatrix} n+1 \\ j+1 \end{pmatrix}$$

we find that

$$\sum_{k=1}^{n} k^4 = 4 \begin{pmatrix} n+1 \\ 5 \end{pmatrix} + 36 \begin{pmatrix} n+1 \\ 4 \end{pmatrix} + 14 \begin{pmatrix} n+1 \\ 3 \end{pmatrix} + \begin{pmatrix} n+1 \\ 2 \end{pmatrix}.$$

(5) Find $k$ which makes the value of $f(k) = \left( \begin{array}{c} 100 \\ k \end{array} \right) 2^k$ maximal.

We have

$$\frac{f(k)}{f(k+1)} = \frac{k+1}{2(100-k)} = \frac{k+1}{200-2k}.$$

This ratio is less than 1 if $k < \frac{199}{3}$ and it is greater that 1 if $k > \frac{199}{3}$. In particular

$$\dots f(66) < f(67) > f(68) \dots$$

so the maximal value of $f(k)$ is attained for $k = 67$.