STAT 650 Midterm 1 solutions.

(1) A particle on 3×3 squares moves to any of neighboring squares with equal probabailty. (Both horizontal, vertical and diagonal moves are allowed. Thus from a corner there are 3 possible moves, from the center there 8 possible moves and from a side square there are 5 possible moves). Find the stationary distribution.

Solution. Observe that the chain is irreducible since one can reach any site from any other site (if the target is not the center move to the center in the first step, if the target is the center do not move to the center in the first step). Let S be a symmetry of the square. Then if π_u is a stationary distribution then p_{Su} is also a stationary distribution. Since the stationary distribution is unique (due to irreducibility) we must have $\pi_u = \pi_{Su}$. Let π_c be the stationary probability of the center square , π_e be the stationary probability of the edge square and π_v be the stationary probability of the vertex square. The stationarity condition reads

$$\pi_v = \frac{2\pi_e}{5} + \frac{\pi_c}{8}, \quad \pi_e = \frac{2\pi_v}{3} + \frac{2\pi_e}{5} + \frac{\pi_c}{8}, \quad \pi_c = \frac{4\pi_v}{3} + \frac{4\pi_e}{5}.$$

Solving this system subject to the constraint $4\pi_v + 4\pi_e + \pi_c = 1$ we get

$$\pi_v = \frac{3}{40}, \quad \pi_e = \frac{5}{40}, \quad \pi_c = \frac{8}{40}.$$

(2) Consider Markov chain with states 1, 2, 3, 4 and the following transition matrix

(a) Classify the states of this chain.

(b) Find λ such that there exists

$$0 < \lim_{n \to \infty} p_{11}(n) / \lambda^n < \infty$$

and compute that limit.

(c) Compute $\lim_{\to\infty} p_{24}(n)$.

Solution. (a) For finite chains every site is either positively recurrent or transient. Site *i* is positively recurrent iff for any *j* with $i \rightarrow j$ we have $j \rightarrow i$. Hence 1 and 2 are transient and 3 and 4 are positively recurrent.

(b) We have

$$(p_{11}(n)p_{21}(n)) = (p_{11}(n-1)p_{21}(n-1)) \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} \end{pmatrix}$$

The eigenvalues of this matrix are $\frac{1+\sqrt{3}}{4}$ and $\frac{1-\sqrt{3}}{4}$. Hence

$$p_{11}(n) = A\left(\frac{1+\sqrt{3}}{4}\right)^n + B\left(\frac{1-\sqrt{3}}{4}\right)^n.$$

Using that $p_{11}(0) = 1, p_{11}(1) = \frac{1}{2}$. we find $A = B = \frac{1}{2}$. Hence

$$\lambda = \frac{1+\sqrt{3}}{4}$$
 and $\lim_{n \to \infty} \frac{p_{11}(n)}{\lambda^n} = \frac{1}{2}$.

(c) $p_{24}(n) = p_{21}(n/2)p_{14}(n/2) + p_{22}(n/2)p_{24}(n/2) + p_{23}(n/2)p_{34}(n/2) + p_{24}(n/2)p_{24}(n/2)$. By part (b) above as $n \to \infty$ $p_{21}(n/2) \to 0$ and $p_{21}(n/2) \to 0$. Hence $p_{23}(n/2) + p_{24}(n/2) \to 1$. On the other hand since the subchain $\{3, 4\}$ is recurrent

$$p_{34}(n/2) \to \pi_4, \quad \pi_{44}(n/2) \to \pi_4,$$

where π_4 is the stationary probability of site 4. Hence

$$\lim_{n \to \infty} p_{24}(n) = \pi_4$$

From the equation

$$\pi_4 = \frac{2\pi_3}{3} + \frac{\pi_4}{2}$$

and condition $\pi_3 + \pi_4 = 1$ we find $\pi_4 = 4/7$. Hence

$$\lim_{n \to \infty} p_{24} = 4/7$$

(3) Consider two chains on $\{0, 1, 2...\}$ with transition probabilities p_{ij} and q_{ij} such that $p_{ij} = q_{ij}$ for $i \neq 0$. Show that if both chains are irreducible and one of the chains is recurrent then the other is recurrent.

Solution. By irreducibility it is enough to show that 0 is recurrent for q-chain. If it was transient there would be a site i such that

$$P^{(q)}(X_n \text{ never visits } 0|X_0=i) > 0.$$

However

$$P^{(q)}(X_n \text{ never visits } 0|X_0=i) = P^{(p)}(X_n \text{ never visits } 0|X_0=i)$$

since two chains differ only at 0. Hence p-chain would also be transient giving a contradiction.

(4) Consider two irreducible chains on $\{0, 1, 2...\}$ with transition probabilities p_{ij} and q_{ij} . Consider a new chain whose space is the set

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of pairs (i, s) where $i \in \{0, 1, 2...\}$ and s = 1 or 2 and transition probabilities

$$\begin{aligned} r_{(i,1)(j,1)} &= p_{ij} \text{ if } i \neq 0, \quad r_{(i,2)(j,2)} = q_{ij} \text{ if } i \neq 0, \\ r_{(0,s),(j,1)} &= \frac{p_{0j}}{2}, \quad r_{(0,s),(j,2)} = \frac{q_{0j}}{2} \text{ for } s = 1, 2. \end{aligned}$$

In other words the state space of the new chain is the union of the state spaces of the two chains except that zero states of both chains are glued together and after coming to zero the particle choose either first or second chain with probability 1/2.

Which of the following statements are true:

(a) If both first and second chain are positively recurrent then the new chain is positively recurrent.

(b) If the first chain is positively recurrent and the second chain is null recurrent then the new chain is null recurrent?

Solution. If the chain starts from (0, 1) then the number of visits V to (0, 2) before returning to (0, 1) has geometric distribution with parameter 1/2. Thus

$$T_{(0,1)} = \sum_{j=1}^{V} X_j + Y$$

where X_j are iid having the same distribution as the return time to 0 of the second chain while Y has the same distribution as the return to 0 of the first chain. Hence

$$E(T_{(0,1)}) = EVEX + EY = EX + EY.$$

Thus $ET_{(0,1)} < \infty$ iff $EX < \infty$ and $EY < \infty$. Therefore both (a) and (b) are true.

(5) Consider the birth chain with intensities $\lambda_n = n + 1$. Suppose that $X_0 = 0$ and let T_2 be the time the chain enters state 2. Find the distribution of T_2 .

Solution. Holding time at 0 has Exp(1) distribution and holding time at 1 has Exp(2) distribution. Taking the convolution we obtain that T_2 has density

$$\int_0^x e^{-z} 2e^{2z-2x} dz = 2\left(e^{-x} - e^{-2x}\right).$$

(6) Let $T_1, T_2...T_N$ are points of the Poisson process with intensity 1 on [0,5] (thus N is Poisson(5)). Let $S = \sum_{j=1}^N \sin(T_j)$ (S = 0 if N = 0). Find the expected value and the variance of S.

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Solution. Applying Theorem 6.13.23 with $\lambda = 1$, t = 5, $r(x) = \sin(t-x)$, $W_x \equiv 1$, we obtain the following expression for the characteristic function of S

$$\phi_S(\theta) = E\left(e^{i\theta S}\right) = \exp\left(\int_0^5 e^{i\theta\sin s} ds\right).$$

Thus

$$\phi_{S}'(\theta) = i\phi_{S}(\theta) \int_{0}^{5} e^{i\theta\sin s} \sin s ds,$$

$$\phi_{S}''(\theta) = i^{2}\phi_{S}(\theta) \left[\left(\int_{0}^{5} e^{i\theta\sin s} \sin s ds \right)^{2} + \int_{0}^{5} e^{i\theta\sin s} \sin^{2} s ds \right].$$

ing $\theta = 0$ we get

Letting $\theta = 0$ we get

$$ES = \int_0^5 \sin s ds = 1 - \cos 5,$$

$$ES^2 = (ES)^2 + \int_0^5 \sin^2 s ds = (1 - \cos)^5 + \frac{5}{2} - \frac{\sin 10}{4}.$$

Thus

$$V(S) = ES^2 - (ES)^2 = \frac{5}{2} - \frac{\sin 10}{4}.$$