(1) Three balls are drawn from three urns. The first urn contains 1 blue and 5 red balls, the second urn contains 2 blue and 4 red balls, and the third urn contains 3 red and 3 green balls.

(a) Find the probability that 2 red balls are chosen;

(b) Let X be the number of different colors chosen. Find the distribution of X.

Solution. (a) Considering three cases depending on which ball is not read we get

$$P(2 \text{ reds}) = \frac{1}{6}\frac{4}{6}\frac{3}{6} + \frac{5}{6}\frac{2}{6}\frac{3}{6} + \frac{5}{6}\frac{4}{6}\frac{3}{6} = \frac{102}{216}.$$

(b) If there is only one color then all balls should be red. Thus

$$P(X = 1) = \frac{5}{6}\frac{4}{6}\frac{3}{6} = \frac{60}{216}.$$

If there are three color then the last ball must be green. Considering whatever first or second ball are red we get

$$P(X = 3) = \frac{5}{6}\frac{2}{6}\frac{3}{6} + \frac{1}{6}\frac{4}{6}\frac{3}{6} = \frac{42}{216}.$$

Hence

$$P(X = 2) = 1 - P(X = 1) - P(X = 3) = \frac{114}{216}.$$

(2) 10000 bacteria are analyzed in the lab. It is known that the probability that a bacteria has gene A is $\frac{1}{2}$ and the probability that it has gene B is $\frac{1}{5000}$. Compute approximately the probability that

(a) 5010 or more bacteria will carry gene A;

(b) Exactly 3 bacteria care gene B.

Solution. Let X be the number of bacteria carrying gene A and Y be the number of bacteria carrying gene B.

(a) $X \approx N(5000, 2500)$, that is $X \approx 5000 + 50Z$ where Z is the standard normal. Accordingly

$$P(X \geq 5010) \approx P(Z \geq 0.2) = 1 - P(Z \leq 0.2) \approx 0.42.$$

(b) $Y \approx \text{Pois}(2)$. Accordingly $P(Y = 3) \approx \frac{2^3}{3!}e^{-2} \approx 0.18$.

(3) Jane finds a job which requires her to commute 5 days a week. On her way home Jane is in a harry so there is $\frac{1}{20}$ probability that she gets a speeding ticket

(a) Let X be the number of tickets Jane gets during first 6 weeks of work. Compute EX and VX.

(b) When Jane gets three tickets she needs to attend a driving school. Find the probability that Jane gets her third ticket on her 50th commute.

Solution. (a) $X \sim$Bin$(30, \frac{1}{20})$. Therefore

$$EX = 30 \times \frac{1}{20} = \frac{3}{2}, \quad VX = \frac{30\ 1}{\times\ 20} \times \frac{19}{20} = \frac{57}{40}.$$

(b) Using the formula for negative binomial distribution with parameters 3 and $\frac{1}{20}$ we see that the answer is

$$\binom{49}{2} \left(\frac{19}{20}\right)^{47} \left(\frac{1}{20}\right)^{3}.$$

(4) A class has 15 boys and 20 girls. 10 theater tickets are distributed at random.

(a) Find the probability that girls have exactly 6 tickets;

(b) Amanda's lunch mates are Barbara, Cindy, Dalia and Elena. Find the conditional probability that Amanda's table gets exactly 2 tickets given that girls got exactly 6 tickets.

Solution. (a) From the formula for hypergeometric distribution we see that the answer is

$$\frac{\binom{15}{4}\binom{20}{6}}{\binom{35}{10}}.$$

(b) Given that the girls get 6 tickets the number of tickets obtained by Amanda's table is hypergeometric with parameters $(20, 5, 6)$. So the answer is

$$\frac{\binom{5}{2}\binom{15}{4}}{\binom{20}{6}}.$$

(5) A number of misprints on a page has Poisson distribution with parameter $\frac{1}{2}$.

(a) Find the probability that exactly three of the next 10 pages will have at least two misprints.

(b) Let X be the first page which has a misprint. Find EX and VX.

Solution. (a) The probability that a page has no misprints is $\frac{1}{\sqrt{e}}$, the probability that a page has one misprint is $\frac{1}{2\sqrt{e}}$, therefore the probability that a page has two or more misprints is $1-\frac{3}{2\sqrt{e}}$. Using the formula for binomial distribution we see that the answer is

$$\binom{10}{3}\left(\frac{3}{2\sqrt{e}}\right)^7\left(1-\frac{3}{2\sqrt{e}}\right)^3.$$

(b) X has geometric distribution with parameter $1-\frac{1}{\sqrt{e}}$. Accordingly

$$EX=\sqrt{e},\quad VX=(1-\frac{1}{\sqrt{e}})(\sqrt{e})^2=e-\sqrt{e}.$$

(6) Let X_1 have density equal to c_1x^3 on $[0,1]$ and zero elsewhere and X_2 have density equal to c_2x^{10} on $[0,2]$ and zero elsewhere.
(a) Compute c_1 and c_2;
(b) Which of the two random variables above has a smaller variance?

Solution. (a) We have

$$c_1=\left(\int_0^1 x^3dx\right)^{-1}=4,\quad c_2=\left(\int_0^2 x^{10}dx\right)^{-1}=\frac{11}{2^{11}}.$$

(b) We have

$$EX_1=\int_0^1 4x^3dx=\frac{4}{5}, E(X_1^2)=\int_0^1 4x^5dx=\frac{2}{3}.$$

Hence

$$VX_1=\frac{2}{3}-\left(\frac{4}{5}\right)^2\approx 0.0267.$$

On the other hand

$$EX_2=\frac{11}{2^{11}}\int_0^2 x^{11}dx=\frac{2^{12}}{12}\frac{11}{2^{11}}=\frac{11}{6},\quad E(X_2^2)=\frac{11}{2^{11}}\int_0^2 x^{12}dx=\frac{2^{13}}{13}\frac{11}{2^{11}}=\frac{44}{13}.$$

Hence

$$VX_2=\frac{44}{13}-\left(\frac{11}{6}\right)^2=\frac{11}{468}\approx 0.0235.$$

Thus $VX_2<VX_1$.

(7) The lifetime of a light bulb (measured in days) has exponential distribution with parameter $1/100$.
(a) Find the distribution of the lifetime measured in hours;
(b) If the bulb is installed on a Wednesday at noon, find the probability that it will burn out on a Monday.

Solution. (a) Let X be the lifetime in days and Y be the lifetime in hours. Then $Y=24X$ hence $Y\sim$Exp$(\frac{1}{100}\frac{1}{24})$ =Exp$(\frac{1}{2400})$.

(b) Considering each Monday separately we get

$$P(\text{Burn out on Monday}) = \sum_{k=0}^{\infty} P(4.5 + 7k \le X \le 5.5 + 7k) =$$

$$\sum_{k=0}^{\infty} [e^{-0.045-0.07k} - e^{-0.045-0.07k}] = \frac{e^{-0.045} - e^{-0.055}}{1 - e^{-0.07}}.$$

(8) The amount of sales at a department store on a given day has normal distribution with mean 30000 and standard deviation 3000. Find the probability that the store sold

(a) more than 31000 worth of goods;

(b) between 28000 and 32000 worth of goods.

Solution. Let X be the amount of sales. Then $X = 30000 + 3000Z$ where Z is the standard normal random variable. Thus

$$(a) \quad P(X > 31000) = P(Z > \frac{1}{3}) = 1 - P(Z < \frac{1}{3}) \approx 0.37,$$

$$(b) \quad P(2800 < X < 3200) = P(-\frac{2}{3} < Z < \frac{2}{3}) =$$

$$P(Z < \frac{2}{3}) - P(Z < -\frac{2}{3}) = 2P(Z < \frac{2}{3}) - 1 \approx 0.50.$$

(9) Let X_1 and X_2 be independent each having density equial to $2x$ if $0 \le x \le 1$ and equal to 0 otherwise.

(a) Find $P(X_1 > 2X_2)$.

(b) Find the distribuition of $X_1 + X_2$.

Solution.

$$(a) \quad P(X_1 > 2X_2) = \int_0^1 2x_1 (\int_0^{x_1/2} 2x_2 dx_2) dx_1 = \int_0^1 2x_1 \frac{x_1^2}{4} dx_1 = \int_0^1 \frac{x_1^3}{2} = \frac{1}{8}.$$

(b) The possible values of Z are from 0 to 2. We consider two cases. (I) $0 \le z \le 1$:

$$f(z) = \int_0^z 2x2(z-x)dx = 4\int_0^z (xz - x^2)dx = 4\left(\frac{z^2}{2} - \frac{z^2}{3}\right) = \frac{2z^2}{3}.$$

(II) $1 \le z \le 2$.

$$f(z) = \int_{z-1}^1 2x2(z-x)dx = 4\int_{z-1}^1 (zx - x^2)dx = 4[\frac{z(1-(z-1)^2)}{2} - \frac{1-(z-1)^3}{3}]$$

$$4[\frac{2z^2 - z^3}{2} - \frac{2 - 3z + 3z^2 - z^3}{3}] = \frac{2}{3}[6z^2 - 3z^3 - 4 + 6z - 6z^2 + 2z^3]$$

$$= \frac{2}{3}[-4 + 6z - z^3] = \frac{-8 + 12z - 2z^3}{3}.$$

(10) Let $X_1, X_2 \dots X_n$ be independent each having denisity equial to $2x$ if $0 \le x \le 1$ and equal to 0 otherwise.

(a) Let N be the first time $X_N > \frac{2}{3}$. Find EX and VX.

(b) Let $n = 5$ and let $X_{(1)} > X_{(2)} > X_{(3)} > X_{(4)} > X_{(5)}$ be the corresponding order statistics. Find $P(X_{(3)} > X_{(4)} + 0.1)$.

Solution. (a) $P(X > \frac{2}{3}) = 1 - \frac{4}{9} = \frac{5}{9}$. Thus $X \sim$Geom$(\frac{5}{9})$. Accordingly

$$EX = \frac{9}{5}, \quad VX = \frac{\frac{4}{9}}{(\frac{5}{9})^2} = \frac{36}{25}.$$

(b) The joint density of $X_{(3)}$ and $X_{(4)}$ equals to $240(1-x_3^2)^2 x_3 x_4^3$. Therefore

$$P(X_{(3)} > X_{(4)} + 0.1) = 240 \int_{0.1}^{1} (x_3 - 2x_3^3 + x_3^5)(\int_0^{x_3 - 0.1} x_4^3 dx_4) dx_3$$

$$= 60 \int_{0.1}^{1} (x_3 - 2x_3^3 + x_3^5)(x_3 - 0.1)^4 dx_3$$

$$= 60 \int_{0.1}^{1} (x - 2x^3 + x^5)(0.0001 - 0.004x + 0.06x^2 - 0.4x^3 + x^4) dx$$

$$= 60 \int_{0.1}^{1} (0.0001x - 0.004x^2 + 0.0598x^3 - 0.392x^4 + 1.0001x^5 + 0.796x^6 - 1.94x^7 - 0.4x^8 + x^9) dx$$

$$= 60 \Bigg(\frac{0.0001[1-(0.1)^2]}{2} - \frac{0.004[1-(0.1)^3]}{3} + \frac{0.0598[1-(.01)^4]}{4} - \frac{0.392[1-(0.1)^5}{5}$$

$$+ \frac{1.0001[1-(0.1)^6]}{6} + \frac{0.796[1-(0.1)^7]}{7} - \frac{1.94[1-(0.1)^8]}{8} - \frac{0.4[1-(0.1)^9]}{9} + \frac{1-(0.1)^{10}}{10} \Bigg).$$

(11) Let (X, Y) have density $x + y$ if $0 \le x \le 1, 0 \le y \le 1$ and equal to zero otherwise.

(a) Find the marginal distribution of X.

(b) Are X and Y independent?

(c) Find the distribution of $Z = X/Y$. **Solution.**

$$(a) \quad f_X(x) = \int_0^1 (x + y) dy = x + \frac{1}{2}.$$

(b) Similarly to (a) we have $f_Y(y) = y + \frac{1}{2}$. Since $(x + \frac{1}{2})(y + \frac{1}{2}) \neq x + y$ X and Y are **NOT** independent. (c) We consider two cases. (I) $z < 1$. Then

$$P(Z > z) = \int_0^z (\int_{x/z}^1 (x + y) dy) dx = \frac{z}{3} + \frac{z^2}{6}.$$

Hence $P(Z \le z) = 1 - \frac{z}{3} - \frac{z^2}{6}$.

(II) $z > 1$. Then $P(Z \leq z) = P(Y/X > \frac{1}{z})$. By symmetry

$$P(Y/X > \frac{1}{z}) = \frac{1/z}{3} + \frac{(1/z)^2}{6} = \frac{1}{3z} + \frac{1}{6z^2}.$$