STAT410 Midterm 2.

(1) (a) An urn contains 6 red and 4 blue balls. 4 balls are chosen without replacement. Find the probability that 3 balls are red and one is blue.

(b) 20 students each pick 4 balls without replacement from an urn having 6 red and 4 blue balls (the urn is refilled for each new student). Let X be the number of students which draw 3 red and 1 blue ball. Compute EX and VX.

Solution. (a) Let R be the number of red balls chosen. Then R has hypergeometric distribution with parameters $10, 6, 4$. Hence

$$P(R=3) = \frac{\binom{6}{3}\binom{4}{1}}{\binom{10}{4}} = \frac{\frac{6\times5\times4}{3!}\times 4}{\frac{10\times9\times8\times7}{4!}} = \frac{8}{21}.$$

(b) X has binomial distribution with parameters $(20, \frac{8}{21})$. Hence

$$EX = 20 \times \frac{8}{21}, \quad VX = 20 \times \frac{8}{21} \times \frac{13}{21}.$$

(2) Let X and Y be independent random variables such that X has exponential distribution with parameter 1 and Y has exponential distribution with parameter 2.

(a) Let $Z = X + Y$. Find the density of Z.

(b) Find the joint density of $U = X^2$ and $V = XY$.

Solution. Note that X has density e^{-x} and Y has density $2e^{-y}$. Hence (a)

$$f_Z(z) = \int_0^z 2e^{-x}e^{2x-2z}dx = 2e^{-2z}\int_0^z e^x dx = 2e^{-2z}(e^z - 1) = 2(e^{-z} - e^{-2z}).$$

(b) Note that the Jacobian equals to

$$J = \det\begin{pmatrix} 2x & 0 \\ y & x \end{pmatrix} = 2x^2 = 2u.$$

Also $X = \sqrt{U}$ and hence $Y = \frac{V}{X} = \frac{V}{\sqrt{U}}$. Since X and Y are independent their joint density equals to $f_{X,Y}(x,y) = 2e^{-x}e^{-2y}$. Hence

$$f_{U,V}(u,v) = \frac{f_{X,Y}(x,y)}{J} = \frac{2e^{-x}e^{-2y}}{2u} = \frac{e^{-\sqrt{u}}e^{-2v/\sqrt{u}}}{u}.$$

(3) Let X_1, X_2, X_3, X_4, X_5 be independent random variables each having uniform distribution on $[0,1]$. Let M be their median (the third largest value).
(a) Find the cumulative distribution function of M.
(b) Compute EM and VM.

Solution. Recall that X has density 1 and cdf x on $[0,1]$. Hence using the formula for the distribution of the third largest variable we have

$$f_M(m) = \frac{5!}{2!2!} m^2(1-m)^2 = 30(m^2 - 2m^3 + m^4).$$

Integrating we find

$$F_M(m) = 30 \int_0^m 30(x^2 - 2x^3 + x^4) = 30\left(\frac{m^3}{3} - \frac{m^4}{2} + \frac{m^5}{5}\right) = 10m^3 - 15m^4 + 6m^5.$$

Next

$$EM = \int_0^1 30(x^2 - 2x^3 + x^4)xdx = 30(x^3 - 2x^4 + x^5)dx = 30 \times \left(\frac{1}{4} - \frac{2}{5} + \frac{1}{6}\right) = \frac{1}{2}$$

and

$$EM^2 = \int_0^1 30(x^2 - 2x^3 + x^4)x^2dx = 30(x^4 - 2x^5 + x^6)dx = 30 \times \left(\frac{1}{5} - \frac{2}{6} + \frac{1}{7}\right) = \frac{2}{7}.$$

Thus

$$VM = E(M^2) - (EM)^2 = \frac{2}{7} - \frac{1}{4} = \frac{1}{28}.$$

(4) Let (X,Y) have density equal to $\frac{3x+y}{2}$ if $0 \le x \le 1, 0 \le y \le 1$ and equal to 0 otherwise. Let $W = \max(X,Y)$.
(a) Compute the distribution of W.
(b) Find $P(W = X)$.

Solution Note that $W < w$ iff $X < w$ and $Y < w$ hence

$$P(W < w) = \int_0^w \int_0^w \frac{3x+y}{2} dxdy = \frac{3w}{2} \int_0^w xdx + \frac{w}{2} \int_0^w ydy = \left(\frac{3}{4} + \frac{1}{4}\right) w^3 = w^3.$$

Also $W = X$ iff $X > Y$ and

$$P(X > Y) = \int_0^1 \left(\int_0^x \frac{3x+y}{2} dy\right) dx = \int_0^1 \left(\frac{3x^2}{2} + \frac{x^2}{4}\right) dx = \frac{7}{4} \int_0^1 x^2 dx = \frac{7}{4} \times \frac{1}{3} = \frac{7}{12}.$$