STAT 650 Midterm 2.

Show your work!!!

(1) Let X_n be a weakly stationary sequence with spectral density $f(\lambda)$. Find the spectral density of the sequence $Y_n = \frac{X_{n-1}+X_{n+1}}{2}$.

Solution. Let $\sigma^2 = E(X(0)^2)$, Then

$$E(Y_0\bar{Y_n}) = c(n+2)+2c(n)+c(n-2) = \sigma^2 \int_0^{2\pi} \left[e^{i\lambda(n+2)} + 2e^{i\lambda n} + e^{i\lambda(n-2)}\right] f(\lambda)d\lambda =$$

$$\sigma^2 \int_0^{2\pi} e^{i\lambda n} \left[2 + 2\cos(2\lambda n)\right] f(\lambda)d\lambda.$$

In particular

$$E(|Y_0|^2) = \sigma^2 \int_0^{2\pi} \left[2 + 2\cos(2\lambda)\right] f(\lambda)d\lambda.$$

Therefore the spectral density of Y_n is

$$g(\lambda) = \frac{\left[2 + 2\cos(2\lambda)\right] f(\lambda)}{\int_0^{2\pi} \left[2 + 2\cos(2\lambda)\right] f(\lambda)d\lambda}$$
$$= \frac{\left[1 + 1\cos(2\lambda)\right] f(\lambda)}{\int_0^{2\pi} \left[1 + 1\cos(2\lambda)\right] f(\lambda)d\lambda}.$$

(2) Consider a regression

$$X_n = \frac{X_{n-1} + X_{n-2} + Y_n}{3}$$

where Y_n is a stationary sequence with covariance $\rho(n) = \left(\frac{1}{2}\right)^{|n|}$. Find a distribution of X_0 which makes X_n stationary and compue its covariance function.

Solution. Iterating we get

$$X_0 = \frac{Y_0 + X_{-1} + X_{-2}}{3} = \frac{Y_0}{3} + \frac{Y_{-1}}{9} + \frac{4X_{-2}}{9} + \frac{X_{-3}}{9}$$
$$= \frac{Y_0}{3} + \frac{Y_{-1}}{9} + \frac{49_{-2}}{9} + \frac{7X_{-3}}{27} + \frac{4X_{-4}}{27} = \dots$$
$$= \frac{1}{3}(Y_0 + \frac{Y_{-1}}{3} + \dots a_n Y_{-n}) + a_{n+1}X_{-(n+1)} + \frac{a_n}{3}X_{-n+2}$$

where

$$a_n = \frac{a_{n-1} + a_{n-2}}{3}. \tag{1}$$

Observe that $a_n \to 0$ (for example $|a_n| < (2/3)^n$ and since X_n is stationary $E(X_n) = \text{Const}$ and so $a_n X_{-n} \to 0$ in mean square sense. Therefore letting $n \to \infty$ we obtain

$$X_0 = \sum_{j=0}^{\infty} a_n Y_{-n}.$$

The general solution to (1) takes form $Aa^n + Bb^n$ where A and B are the solutions of

$$\lambda^2 = \frac{\lambda + 1}{3}.$$

Thus

$$a = \frac{1+\sqrt{13}}{6} \quad b = \frac{1+\sqrt{13}}{6}.$$

Using the initial conditions $a_0 = 1$, $a_1 = \frac{1}{3}$ we find

$$A = \frac{\sqrt{13}+1}{2\sqrt{13}} \quad B = \frac{\sqrt{13}-1}{2\sqrt{13}}.$$

Thus

$$X_0 = \frac{1}{\sqrt{13}} \sum_{j=0}^{\infty} \left[\left(\frac{\sqrt{13}+1}{6} \right)^{j+1} - \left(\frac{\sqrt{13}-1}{6} \right)^{j+1} \right] Y_{-j}.$$

A similar argument gives

$$X_n = \frac{1}{\sqrt{13}} \sum_{j=0}^{\infty} \left[\left(\frac{\sqrt{13}+1}{6} \right)^{j+1} - \left(\frac{\sqrt{13}-1}{6} \right)^{j+1} \right] Y_{n-j}.$$

To compute the covariance one can use two methods.

Method 1. We compute the spectral desnity of Y

$$f_Y(\lambda) = \frac{1}{2\pi} \sum_j \rho(n) e^{-i\lambda j} = \sum_{j=0}^{\infty} \frac{e^{-i\lambda j}}{2^j} + \sum_{j=0}^{\infty} \frac{e^{i\lambda j}}{2^j} - 1 = \frac{1}{2\pi} R(e^{i\lambda})$$

where

$$R(z) = \frac{2}{2-z} + \frac{2}{2-1/z} - 1.$$

Next we use the following result

Claim. Let X_n satisfy

$$\sum_{k=0}^{p} c_k X_{n-k} = Y_n$$

when

$$E(X_n X_0) = \int_{-\pi}^{\pi} e^{i\lambda n} f_X(\lambda) d\lambda \text{ where}$$

$$f_X(\lambda) = \frac{f_Y(\lambda)}{A(z)A(1/z)} \text{ and} A(z) = \sum_{k=0}^{p} c_k z^k.$$

To prove this claim we write

$$E(Y_0 Y_n) = \int_{-\pi}^{\pi} f_Y(\lambda) d\lambda$$

$$= \sum_{k_1 k_2} c_{k_1} c_{k_2} E(X_{n-k_1} X_{-k_2}) = \int_{-\pi}^{\pi} f_X(\lambda) \sum_{k_1 k_2} c_{k_1} c_{k_2} e^{i\lambda n - k_1 + k_2} d\lambda$$

$$= \int_{-\pi}^{\pi} f_X(\lambda) e^{i\lambda_n} \left(\sum_{k_1} c_{k_1} e^{-i\lambda k_1} \right) \left(\sum_{k_2} c_{k_2} e^{i\lambda k_2} \right) d\lambda = \int_{-\pi}^{\pi} f_X(\lambda) e^{i\lambda_n} A(1/z) A(z).$$

Observe that in our case

$$A(z) = 1 - \frac{z + z^2}{3}.$$

It follows

$$E(X_0 X_n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f_X(\lambda) e^{i\lambda_n} \frac{R(e^{i\lambda})}{A(e^{-i\lambda})} d\lambda.$$

Making a change of varibles $z = e^{i\lambda}$ this integral is transformed to

$$E(X_0 X_n) = \frac{1}{2\pi i} \oint_{|z|=1} \frac{R(z) z^{n-1} dz}{A(z)A(1/z)} = \sum_{\zeta} \text{Res}_{z=\zeta} \left(\frac{R(z) z^{n-1}}{A(z)A(1/z)} \right)$$

where the sum is over the poles inside the unit circle. $R(z)$ has one pole $\zeta = 1/2$ with residue

$$\left(\frac{1}{2}\right)^n \frac{1}{A(2)A(1/2)}.$$

Next

$$A\left(\frac{1}{z}\right) = \frac{(z-a)(z-b)}{z^2}$$

so it contributes two poles a and b with residues

$$\frac{R(a) a^{n+1}}{(a-b)A(a)} \text{ and } \frac{R(b) b^{n+1}}{(b-a)A(b)}.$$

Finally $A(z)$ has roots $1/a$ and $1/b$ which are outside the unit circle so it contributes no residues. Hence

$$E(X_0 X_n) = \left(\frac{1}{2}\right)^n \frac{1}{A(2)A(1/2)} + \frac{R(a) a^{n+1}}{(a-b)A(a)} + \frac{R(b) b^{n+1}}{(b-a)A(b)}.$$

Method 2. Represent

$$X_n = \frac{1}{\sqrt{13}} (X_n^a - X_n^b)$$

where

$$X_n^a = \sum_{j=0}^{\infty} a^{j+1} Y_{n-j} \quad X_n^b = \sum_{j=0}^{\infty} b^{j+1} Y_{n-j}.$$

Next

$$E(Y_n X_0^a) = \sum_{j=0}^{\infty} \left(\frac{1}{2}\right)^{n+j} a^{j+1} = \frac{2a}{2-a}\left(\frac{1}{2}\right)^n.$$

$$\text{Likewise } E(Y_n X_0^b) = \frac{2b}{2-b}\left(\frac{1}{2}\right)^n.$$

Denote

$$\sigma_a^2 = E((X_0^a)^2) \quad \sigma_b^2 = E((X_0^b)^2) \quad \sigma_{ab} = E((X_0^a)(X_0^b)).$$

By stationarity

$$\sigma_a^2 = E(X_1^a X_1^a) = a^2 E(Y_1^2) + 2a^2 E(Y_1 X_0) + a^2\sigma_a^2 = \frac{(2+a)a^2}{2-a} + a^2\sigma_a^2.$$

Therefore

$$\sigma_a^2 = \frac{(2+a)a^2}{(2-a)(1-a^2)}.$$

Likewise

$$\sigma_b^2 = \frac{(2+b)b^2}{(2-b)(1-b^2)} \text{ and } \sigma_{ab} = \frac{ab(4-ab)}{(2-a)(2-b)(1-ab)}.$$

Therefore

$$E(X_n X_0) = \frac{1}{13}\left[\sum_{j=1}^{n} \left(a^{n+1-j} - b^{n+1-j}\right) E(Y_j(X_0^a - X_0^b)) + E\left(\left(a^n X_0^a - b^n X_0^b\right)\left(X_0^a - X_0^b\right)\right)\right].$$

The first sum equals

$$\sum_{j=1}^{n} \frac{a^{n+1-j} - b^{n+1-j}}{2^j}\left(\frac{2a}{2-a} - \frac{2b}{2-b}\right) = \left[\frac{(2^{n+1}a^n - 1)a}{(2a-1)2^n} - \frac{(2^{n+1}b^n - 1)b}{(2b-1)2^n}\right]\frac{4(a-b)}{(2-a)(2-b)}$$

while the second sum equals

$$a^n(\sigma_a^2 - \sigma_{ab}) - b^n(\sigma_{ab} - \sigma_b^2).$$

Summing up we obtain

$$13E(X_0 X_n) = \left[\frac{(2^{n+1}a^n - 1)a}{(2a-1)2^n} - \frac{(2^{n+1}b^n - 1)b}{(2b-1)2^n}\right]\frac{4(a-b)}{(2-a)(2-b)} + a^n(\sigma_a^2 - \sigma_{ab}) - b^n(\sigma_{ab} - \sigma_b^2).$$

(3) Consider Markov chain with states $1, 2, 3$ and the following generator

$$\begin{pmatrix} -1 & 1 & 0 \\ 2 & -6 & 4 \\ 0 & 3 & -3 \end{pmatrix}$$

Compute

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t p_{11}(s)ds.$$

Solution. From 2 the chain goes to 1 with probability $2/6 = 1/3$ and to 3 with probability $4/6 = 2/3$. Let V be the number of visits to 3 before the chain returns to 1. Then $P(V = k) = (2/3)^k 1/3$. If the chain visits 2 the average time before returning back to 2 is $\frac{1}{6} + \frac{1}{3} = \frac{1}{2}$ (since the chain stays at 2 is a sum $Exp(6) + Exp(3)$. Likewise the expected time to come from 1 to 2 and from 2 to one is $1 + \frac{1}{6} = \frac{7}{6}$. Hence the expected return time to 1 is

$$\mu_1 = \frac{7}{6} + \frac{1}{2} \sum_{k=1}^{\infty} k (\frac{2}{3})^k \frac{1}{3} = \frac{13}{6}$$

(where we have used the identity

$$\sum_{k=0}^{\infty} ks^k = s \sum_{k=0}^{\infty} ks^{k-1} = s\frac{d}{ds}\left(\sum_{k=0}^{\infty} s^k\right) = s\frac{d}{ds}\left(\frac{1}{1-s}\right) = \frac{s}{1-s^2}.$$

Hence the answer is

$$\frac{1}{\frac{13}{6} \times 1} = \frac{6}{13}.$$

(4) Consider a renewal sequence $T_n = X_1 + X_2 + \dots X_n$ where X_j are iid having nonlattice distribution and finite expectation. Let $N(t) = \min(n : T_n \le t)$. Compute

$$\lim_{t\to\infty} P(N(t) \text{ is odd}).$$

Solution. Let $p(t) = P(N(t) \text{ is odd})$. Denote by F_2 the distribution of T_2. Then

$$p(t) = P(T_1 \le t, T_2 > t) + \int_0^t p(t-s)dF_2(s).$$

Thus

$$p(t) = P(T_1 \le t, T_2 > t) * m \to \frac{1}{ET_2} \int_0^t P(T_1 \le t < T_2)dt.$$

However

$$\int_0^t P(T_1 \le t < T_2)dt = \int_0^t [P(T_2 \ge t) - P(T_1 \ge t)]dt = E(T_2) - E(T_1) = 2EX - EX = EX.$$

Thus $p(t) \to \frac{EX}{ET_2} = \frac{EX}{2EX} = \frac{1}{2}$.

(5) Consider a delayed renewal sequence $T_n = X_1 + X_2 + \ldots X_n$ where X_j are iid, with X_1 having distribution F^d and X_k having distribution F for $k \ge 2$. Let $N^d(t) = \min(n : T_n \le t)$, $m^d(t) = E(N^d(t))$. Assume that X_j are nonlattice and that $E(X^2) < \infty$. Compute

$$\lim_{t\to\infty} m^d(t) - \frac{t}{\mu}.$$

Solution.

$m^d = F^d + F^d * m = F^d + 1 * m - (1 - F^d) * m = F^d + m - (1 - F^d) * m.$

Therefore

$$\lim_{t\to\infty} m^d(t) - \frac{t}{\mu} = \lim_{t\to\infty} F^d(t) + \lim_{t\to\infty}\left[m(t) - \frac{t}{\mu}\right] - \lim_{t\to\infty}(1 - F^d) * m(t) =$$

$$1 + \left(\frac{EX^2}{2\mu} - 1\right) - \frac{1}{\mu}\int_0^\infty (1 - F^d)(s)ds = 1 + \left(\frac{EX^2}{2\mu^2} - 1\right) - \frac{EX_1}{\mu} = \frac{EX^2}{2\mu^2} - \frac{EX_1}{\mu}.$$