Solutions to midterm 1.

(1) Let $A_k = \{S_k = m, S_1 < m, S_2 < m \dots S_{k-1} < m\} >$ Then $\bigcup_k A_k = \{\max_{k \leq n} S_k \geq m\}$.

$$P(S_n > m) = \sum_{k=1}^{n-1} P((S_n > m) \bigcap \bigcup_k A_k) = \sum_k P(A_k \bigcap (S_n - S_k > 0))$$

$$= \sum_k P(A_k) P(S_n - S_k > 0).$$

But $P(S_n - S_k > 0) = 1/2$ by symmetry. Thus

$$P(S_n > m) = \sum_k \frac{1}{2} P(A_k) = \frac{1}{2} P(\max_{k \leq n} S_k \geq m).$$

(2) $$\prod_{j=1}^{n} \left(1 - \frac{U_j}{j}\right) = \exp\left[\sum_{j=1}^{n} \ln(1 - U_j/j)\right].$$

Hence it suffices to show that $\sum_j \ln(1 - U_j/j)$ converges. For $j \geq 2$ $\ln(1 - \frac{U_j}{j}) = -\frac{U_j}{j} + R_j$ where $R_j \leq \text{Const}/j^2$. Thus $\sum_j R_j$ converges and so it suffices to show that $\sum_j U_j/j$ converges. But $E(U_j/j) = 0$ $V(U_j/j) = 1/(3j^2)$ so the convergence of $\sum_j U_j/j$ follows from the Three Series Theorem.

(3)(a) $|\phi(t+h) - \phi(t)| = |\mathbb{E}(e^{i(t+h)X} - e^{itX})| = |\mathbb{E}(e^{itX}(e^{ihX} - 1))| \leq \mathbb{E}(|e^{ihX} - 1|)$.

(b) $\mathbb{E}(|e^{ihX_n} - 1|) = \mathbb{E}(|e^{ihX_n} - 1| 1_{|X_n| \leq M}) + \mathbb{E}(|e^{ihX_n} - 1| 1_{|X_n| > M}) = I + I\!I$.

If $|h| < \delta$ then

$$|I| \leq \text{Const}\mathbb{E}(\delta |X_n| 1_{|X_n| \leq M}) \leq \text{Const}\delta M.$$

On the other hand

$$|I\!I| \leq 2P(|X_n| > M).$$

Since X_n is tight given $\varepsilon > 0$ there exists M such that $P(|X_n| > M) \leq \varepsilon/4$ for all n. Choose δ such that $C\delta M < \varepsilon/2$ then

$$\mathbb{E}(|e^{ihX_n} - 1|) \leq \varepsilon.$$

(4) Let $\Delta_j = X_j - X_{j-1}$. Then Δ_j are iid

$$\mathbb{E}(\Delta_j) = 3\mathbb{E}(N_1) + 2\mathbb{E}(N_2) = \mathbb{E}(N_3) = 10.$$

$$V(\Delta_j) = 9V(N_1) + 4V(N_2) + V(N_3) = 20.$$

By the Law of Iterated Logarithm

$$\limsup_{\mathbb{N}\ni n\to\infty} \frac{X_n - 10n}{\sqrt{n\ln\ln n}} = \sqrt{2V(\Delta_1)} = 2\sqrt{10}.$$

Therefore

$$\limsup_{\mathbb{R}\ni t\to\infty} \frac{X_t - 10t}{\sqrt{t\ln\ln t}} \geq 2\sqrt{10}.$$

On the other hand if $n \leq t < n+1$ then

$$\frac{X_t - 10t}{\sqrt{t\ln\ln t}} \leq \frac{X_{n+1} - 10n}{\sqrt{(n+1)\ln\ln(n+1)}}$$

So

$$\limsup_{t\to\infty} \frac{X_t - 10t}{\sqrt{t\ln\ln t}} \leq \limsup_{n\to\infty} \frac{X_{n+1} - 10n}{\sqrt{(n+1)\ln\ln(n+1)}} = 2\sqrt{10}.$$

5(a) $\quad X = \sum_{j=1}^{n} A_j(N_{a_j} - N_{a_{j-1}}).$

The terms here are independent so

$$\phi_X(t) = \prod_{j=1}^{n} \phi_{A_j(N_{a_j}-N_{a_{j-1}})}(t) = \prod_{j=1}^{n} \phi_{N_{a_j}-N_{a_{j-1}}}(A_j t) = \prod_{j=1}^{n} \exp\left[(a_j - a_{j-1})(e^{itA_j} - 1)\right]$$

$$= \exp\sum_{j=1}^{n}\left[(a_j - a_{j-1})(e^{itA_j} - 1)\right].$$

(b) Let f_k be the functions of part (a) such that $f_k \to f$ pointwise. Then $X(f_k) \to X(f)$ so by continuity Theorem $\phi_{X(f)}(t) = \lim_{k\to\infty} \phi_{X(f_k)}(t)$. Observe that

$$\sum_{j=1}^{n}\left[(a_j - a_{j-1})(e^{itA_j} - 1)\right]$$

are Riemann sums of $e^{itf} - 1$ so

$$\phi_X(t) = \exp\left[\int_0^1 (e^{itf(x)} - 1)dx\right].$$

(6) Given $x \in \mathbb{R}$ let $Z_j = 1_{X_j > x/\sqrt{n}}$, $S_n = \sum_{j=1}^{n} Z_j$. Then

$$P(\sqrt{n}M_n > x) = P(M_n > x/\sqrt{n}) = P(S_n > n+1).$$

We have

$$\mathbb{E}(Z_j) = \frac{1}{2}\left(1 - \frac{x}{\sqrt{n}}\right) \quad V(Z_j) = \frac{1}{4}\left(1 - \frac{x^2}{n}\right) \to \frac{1}{4}$$

Hence by Central Limit Theorem

$$P(S_n > n{+}1) = P\left(\frac{S_n - \frac{2n+1}{2} - \frac{2n+1}{2}\frac{x}{\sqrt{n}}}{\sqrt{\frac{2n+1}{4}}} \geq \frac{n+1 - \frac{2n+1}{2} - \frac{2n+1}{2}\frac{x}{\sqrt{n}}}{\sqrt{\frac{2n+1}{4}}}\right) \to \Phi(\sqrt{2}x)$$

where Φ is the distribution function of the standard normal random variable. Hence $P(\sqrt{n}M_n > x/\sqrt{2}) \to \Phi(x)$ so $\sqrt{n}M_n$ is asymptotically normal with zero mean and variance $1/2$.