**Solutions to midterm 2.**

(1) Let us compute $P(X_{(n:n)}|X_{(n-1:n)} = y)$. By symmetry it is the same as

$$P(X_{(n:n)}|X_{(n-1:n)} = y), X_{(n:n)} = X_1, X_{(n-1:n)} = X_2)$$
$$= P(X_1|X_2 = y, X_j < y, j = 3 \dots n).$$

Thus

$$P(X_{(n:n)} > z|X_{(n-1:n)=y)} = \frac{1/z}{1/y}.$$

In particular

$$P(X_{(n:n)} > zy|X_{(n-1:n)=y)} = \frac{1}{z}.$$

Hence $X_{(n:n)}/X_{(n-1:n)}$ and $X_{(n-1:n)}$ are independent and

$$P(X_{(n:n)}/X_{(n-1:n)} > z) = \frac{1}{z}.$$

(2) We have $P(X^+ \geq x) = 1/(2x^{5/4})$ if $x > 1$. It follows that

$$\frac{\left(\sum_{j=1}^n X_j^+\right) - nE(X^+)}{(n/2)^{4/5}} \Rightarrow Y^+$$

where $Y^+$ has characteristic function

$$\phi^+(t) = \exp\left[\int_0^\infty \frac{e^{itx} - 1 - itx}{x^{9/4}} dx\right].$$

Likewise

$$\frac{\left(\sum_{j=1}^n X_j^-\right) - nE(X^-)}{(n/2)^{4/7}} \Rightarrow Y^-$$

In other words

$$\sum_{j=1}^n X_j^+ - nE(X_j^+) = \left(\frac{n}{2}\right)^{4/5} Y_n^+$$
$$\sum_{j=1}^n X_j^- - nE(X_j^-) = \left(\frac{n}{2}\right)^{4/7} Y_n^-$$

where $Y_n^\pm \to Y^\pm$. Subtracting and dividing by $n^{4/5}$ we get

$$\frac{S_n - nE(X)}{n^{4/5}} \to 2^{-4/5} Y^+.$$

(3)(a) $|\mu_1(A) - \mu_2(A)| \leq |\mu_1(A) - \mu_3(A)| + |\mu_2(A) - \mu_3(A)|$. Taking sup we obtain $d(\mu_1, \mu_2) \leq d(\mu_1, \mu_3) + d(\mu_2, \mu_3)$.

(b) Note that

$$\nu(A) - \mu(A) = \mu(A^c) - \nu(A^c)$$

so

$$d(\mu, \nu) = \sup_A \mu(A) - \nu(A) = \sup_A \left[ \int_A (1-f) d\mu - \nu_s(A) \right] \leq \int_\Omega (1-f)^+ d\mu.$$

But the last expression can be achieved for $A = \{f < 1\}$. Hence

$$d(\mu, \nu) = \sup_A \mu(A) - \nu(A) = \int_\Omega (1-f)^+ d\mu.$$

(4) Let $p_n(x)$ be the density of $\mu_n$. Then if $\nu(A) = \mu_4(A \bigcup [0,1])$ then $\frac{d\nu}{d\mu} = p_4$, so $\nu \ll \mu$ while $\mu([0,1]^c) = 0$. It follows that $\mu_4^{ac} = \nu$, so $\mu_4^{as}(\mathbb{R}^1) = \mu_4([0,1]$. Next for $x < 1$ we have $p_{n+1}(x) = \int_0^x p_n(s) ds$. Hence by induction $p_n(x) = \frac{x^{n-1}}{(n-1)!}$ (for $x < 1$). Hence

$$\mu_4([0,1]) = \int_0^1 \frac{x^3}{6} dx = \frac{1}{24}.$$

(5) $$\sqrt{S_n} - \sqrt{n} = \frac{S_n - n}{\sqrt{S_n} + \sqrt{n}} = \frac{1}{\sqrt{S_n/n} + 1} \frac{S_n - n}{\sqrt{n}}.$$

Since continuous functions preserve weak convergence the first term converses weakly to 1/2 while the second term converges weakly to $\mathcal{N}(0, \sigma^2)$. Thus the product converges to $\mathcal{N}\left(0, \frac{\sigma^2}{4}\right)$.

(6) Let $\bar{X}_j^{(K)} = X_j 1_{|X_j| > K}$, $\bar{S}_n^{(K)} = \sum_{j=1}^n \bar{X}_j^{(K)}$.
We have

$$P(S_n \geq a\sqrt{n}) \geq P(S_n^{(K)} \geq a\sqrt{n}), \bar{S}_n^{(K)} \geq 0).$$

By symmetry

$$P(S_n^{(K)} \geq a\sqrt{n}), \bar{S}_n^{(K)} \geq 0) \geq P(S_n^{(K)} \geq a\sqrt{n}), \bar{S}_n^{(K)} < 0)$$

(because for each $A \subset \{1, 2 \dots n\}$ we have

$$P(S_n^{(K)} \geq a\sqrt{n}), \bar{S}_n^{(K)} > 0, |X_j| > K \text{ for } j \in A, |X_j| \leq K \text{ for } j \notin A)$$

$$= P(S_n^{(K)} \geq a\sqrt{n}), \bar{S}_n^{(K)} < 0, |X_j| > K \text{ for } j \in A, |X_j| \leq K \text{ for } j \notin A)).$$

Since

$$P(S_n^{(K)} \geq a\sqrt{n}), \bar{S}_n^{(K)} \geq 0) + P(S_n^{(K)} \geq a\sqrt{n}), \bar{S}_n^{(K)} < 0) = P(S_n^{(K)} \geq a\sqrt{n})$$

part (a) follows.

(b) Suppose $E(X_j^2)=\infty$. We claim that $S_n/\sqrt{n}$ does not converge. Otherwise $S_n/\sqrt{n}$ would be tight. Thus there would exist $a$ such that

$$\limsup P(S_n \geq a\sqrt{n}) \leq 0.01$$

and by part (a) for any $K$

$$\limsup P(S_n^{(K)} \geq a\sqrt{n}) \leq 0.02$$

But

$$P(S_n^{(K)} \geq a\sqrt{n}) = P\left(\frac{S_n^{(K)}}{\sqrt{E\left(\left(X_1^{(K)}\right)^2\right)n}} \geq \frac{a}{\sqrt{E\left(\left(X_1^{(K)}\right)^2\right)}}\right).$$

Hence

$$\lim P(S_n^{(K)} \geq a\sqrt{n}) = \int_{a/\sqrt{E((X_1^{(K)})^2)}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt.$$

By the Monotone Convergence Theorem $E((X_1^{(K)})^2) \to \infty$ as $K \to \infty$. In particular we can choose $K$ such that

$$\lim P(S_n^{(K)} \geq a\sqrt{n}) \geq 0.4$$

a contradiction. Therefore $E(X_1^2) < \infty$ as claimed.

**Remark.** The assumption that $X_j$ are symmetric is redundant since if

$$\frac{\sum_{j=1}^n X_j}{\sqrt{n}} \Rightarrow Y$$

then

$$\frac{\sum_{j=1}^n (X_j - X_{j+n})}{\sqrt{n}} \Rightarrow Y_1 - Y_2$$

where $Y_1$ and $Y_2$ are independent and have the same distribution as $Y$. It follows from problem 6 that

$$E((X_1 - X_2)^2) = 2E((X_1)^2) < \infty.$$