ON EQUIVALENCE OF QUENCHED AND ANNEALED STATISTICAL PROPERTIES FOR RANDOM DYNAMICAL SYSTEMS

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1. Introduction

1.1. **Quenched and annealed results.** Let M be a closed Riemannian manifold and consider the IID sequence of random diffeomorphisms distributed according to a measure μ on $\mathrm{Diff}^{\infty}_{\mathrm{vol}}(M)$ with compact support. The random dynamics is then driven by the product measure $\mu^{\mathbb{N}}$ on $\mathrm{Diff}^{\infty}_{\mathrm{vol}}(M)^{\mathbb{N}}$, and a particular realization of the random dynamics is given by a word $\omega \in (\mathrm{Diff}^{\infty}_{\mathrm{vol}}(M))^{\mathbb{N}}$. Let $F^N = F^N_{\omega} = f_{\sigma^{N-1}\omega} \circ \cdots \circ f_{\omega}$. We also consider the two point motion, which is the induced action F_2^N on $M \times M$ given by $F^N_{\omega} \times F^N_{\omega}$, i.e. $F_2^N(x, y) = (F^N(x), F^N(y))$. Below we will often suppress the dependence on ω. Denote by $\mathcal{H}^p_0(M)$ the space of zero mean functions that belong to the Sobolev space of index p.

For random systems, there are two basic versions of each limit theorem: quenched and annealed. In a quenched limit theorem, one shows that for a.e. realization ω of the random system that the limit theorem holds. In an annealed limit theorem, one additionally averages over the entire ensemble of possible realizations ω. Naturally, in the quenched case, one often wants an additional estimate on the set of ω where the limit theorem converges very slowly.

The goal of this note is to provide sufficient conditions for when quenched exponential mixing and the quenched central limit theorem follow from the annealed versions of these theorems, which are in principal easier to prove. The conditions that appear in this paper are not dynamical, and hence should be widely applicable. In particular, the results in this paper show that in the authors other work [DD25] that the central limit theorem obtained there is a quenched central limit theorem.

We follow an approach of [DKK04]. However, our results are stronger because [DKK04] considered a fixed observable while we obtain results which are valid for all sufficiently smooth functions. This extension requires additional work.

The structure of the paper is the following. After some preliminaries in Section 2, we show that annealed exponential mixing implies quenched exponential mixing in Section 3. Then in Section 5 we show how to deduce the quenched central limit theorem from the annealed one.

1.2. **Definitions.** The following definitions make sense in a wider context of skew products. Namely let σ be an automorphism of a probability space Ω preserving a

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probability measure $\boldsymbol{\mu}$ and consider the map $T\colon \Omega \times M \mapsto \Omega \times M$ given by the formula

$$T(\omega, x) = (\sigma\omega, f_\omega(x)), \tag{1.1}$$

where for each ω the map $f_\omega \in \mathrm{Diff}^\infty_{\mathrm{vol}}(M)$. Then the iterates of T satisfy $T^n(\omega, x) = (\sigma^n\omega, F^N_\omega(x))$. The two main concepts of this paper are the following.

Definition 1.1. We say that the random system T enjoys *annealed exponential mixing* if there exist $p \geq 0$ and $\alpha > 0$ such that for all $A, B \in \mathcal{H}^p_0(M)$ we have

$$\left|\mathbb{E}_{\boldsymbol{\mu}}\left(\int A(x)B(F^N_\omega x)dx\right)\right| \leq Ce^{-\alpha N}\|A\|_{\mathcal{H}^p_0}\|B\|_{\mathcal{H}^p_0}.$$

Definition 1.2. We say that the random system T enjoys *quenched exponential mixing* if there exist $p \geq 0$ and $\alpha > 0$ and a random variable $C(\omega)$ such that for all $A, B \in \mathcal{H}^p_0$ and almost every ω we have

$$\left|\int A(x)B(F^N_\omega x)dx\right| \leq C(\omega)e^{-\alpha N}\|A\|_{\mathcal{H}^p_0}\|B\|_{\mathcal{H}^p_0}. \tag{1.2}$$

Given a function A on M denote $S_N A(x, \omega) = \sum_{n=0}^{N-1} A(F^n_\omega x)$

Definition 1.3. We say that the random system enjoys the *annealed Central Limit Theorem* if there exists $p \geq 0$ such that there is a map $\mathcal{D}\colon \mathcal{H}^p_0(M) \to \mathbb{R}$, which is not identically equal to 0, such that for each $A \in \mathcal{H}^p_0$, if x is uniformly distributed on M and ω is distributed according to $\boldsymbol{\mu}$ then $\frac{S_N A(x,\omega)}{\sqrt{N}}$ converges as $N \to \infty$ to the normal distribution with zero mean and variance $\mathcal{D}(A)$.

Definition 1.4. We say that the random system enjoys the *quenched Central Limit Theorem* if there exists $p \geq 0$ such that there is a map $\mathsf{D}\colon \mathcal{H}^p_0(M) \to \mathbb{R}$, which is not identically equal to 0, such that for each $A \in \mathcal{H}^p_0(M)$, there are random variables $a_N(\omega), q_N(\omega)$, such that for almost every ω if x is uniformly distributed on M then $\frac{S_N A(x,\omega) - a_N(\omega)}{q_N(\omega)}$ converges as $N \to \infty$ to the normal distribution with zero mean and variance $\mathsf{D}(A)$.

1.3. **Counterexamples.** In general, quenched and annealed results are inequivalent even for IID random maps. Our first example shows that even if the annealed dynamics averages perfectly after a single iterate.

Example 1.5. Let $\omega_n \in (\mathbb{T}^d)^{\mathbb{N}}$ be uniformly distributed on $\mathbb{T}^d$ and let $f_\omega(x) = x + \omega_1$. Then $x_N = F^N_\omega x$ are IID uniformly distributed on $\mathbb{T}^d$ so the system enjoys annealed exponential mixing. However, in this case $x_N = x_0 + W_N$ where

$$W_N = \sum_{n=0}^{N-1} \omega_n, \tag{1.3}$$

so letting $A = e^{i\langle k,x\rangle}$, $B = e^{-i\langle k,x\rangle}$ for some $k \neq 0$ we see that $|\int A(x)B(F^N_\omega(x))dx| = 1$ for all N. Thus the system does not have quenched mixing. It is also possible to show that in that case quenched Central Limit Theorem does not hold.

Next we give two examples where quenched exponential mixing and the quenched central limit theorem hold, but the annealed result fails.

Example 1.6. (a) Let g be a linear Anosov map of $\mathbb{T}^d$. Let ω_n be IID integer valued random variables where $\mathbb{P}(\omega_n = -k) = \frac{0.001}{k^3}$ for $k < 0$ and $\mathbb{P}(\omega_n = 1) = \mathbb{P}(\omega_n = 2) = \dfrac{1 - 0.001\zeta(3)}{2}$. Let $f_\omega = g^{\omega_0}$. Then $F_\omega^N = g^{W_N}$ where W_N is given by (1.3). Using the .001 factor, it is easy to see that $\mathbb{E}(\omega_n) \in [1, 2]$, so by the Strong Law of Large Numbers for almost every ω we have that $W_N > N$ for large N. Then $\int A(x)B(g^{W_N}x)dx$ decays exponentially due to the exponential mixing of g. On the other hand letting A and B be as in the previous example we see that $\int A(x)B(F^{W_N}x)dx = \delta_{W_N,0}$ whence

$$\mathbb{E}\left(\int A(x)B(F^{W_N}x)dx\right) = \mathbb{P}(W_N = 0).$$

Since

$$\mathbb{P}(W_N = 0) \geq \sum_{k=N}^{2N} \mathbb{P}(W_{N-1} = k)\mathbb{P}(\omega_{N-1} = -k) \geq CN^{-2},$$

the annealed correlations for this system decay only polynomially.

(b) Now define f_ω as in part (a) but suppose that ω_n takes values ± 1 with probability $1/2$. Then the quenched Central Limit Theorem holds, but annealed one fails (see [DDKN23] for details).

We note that Examples 1.6(a) and (b) are special cases of so called *generalized* (T, T^{-1}) *transformations*. More information on limit theorems for these systems can be found in [DDKN22b, DDKN22a].

1.4. **Deriving annealed results from the quenched ones.** Here we recall the basic tools for deducing annealed results from the quenched ones. We will work in the general framework of skew products (1.1).

Proposition 1.7. *Suppose that the skew product* (1.1) *satisfies queched exponential mixing* (1.2) *and that corresponding prefactor $C(\omega)$ has a power tail:*

$$\mathbb{P}_\mu(C(\omega) > R) \leq K/R^\kappa,$$

for some $K, \kappa > 0$. Then annealed exponential mixing holds.

Proof. Note that $\left|\int A(x)B(F_\omega^N x)dx\right| \leq \min\left\{C(\omega)e^{-\alpha N}, 1\right\} \|A\|_{\mathcal{H}_0^p}\|B\|_{\mathcal{H}_0^p}$. The expectation of the first factor is bounded by $e^{-\alpha N/2} + \mathbb{P}_\mu\left(C(\omega) > e^{\alpha N/2}\right) \leq e^{-\alpha N/2} + Ke^{-\kappa\alpha N/2}$, completing the proof. □

The next result allows us to obtain the annealed Central Limit Theorem from the quenched one, see [DDKN22a, Lemma 5.6].

Proposition 1.8. *Suppose that the skew product* (1.1) *satisfies the quenched Central Limit Theorem and moreover that the quenched variance satisifes that $q_N(\omega)/\sqrt{N}$ converges in Law as $N \to \infty$ to a constant $q = q(A)$ while the quenched drift satisfies that $\frac{a_N(\omega)}{\sqrt{N}}$ converges as $N \to \infty$ to a normal distribution with zero mean and variance $\mathfrak{D}(A)$. Then the annealed Central Limit Theorem holds, that is, if ω is distributed according* *to $\boldsymbol{\mu}$ and x is uniformly distributed on M then $\frac{S_N A(\omega,x)}{\sqrt{N}}$ converges in law as $N\to\infty$ to a normal random variable with zero mean and variance $\mathcal{D}(A)=\mathsf{D}(A)q(A)+\mathfrak{D}(A)$.*

1.5. **Deriving quenched results from the annealed ones.** Our main results allow to obtain the quenched limit theorems from more easily accessible annealed results. We work in the setting of IID random systems. The following are the main results of this paper.

Theorem 1.9. *Suppose that μ is a measure supported on $\mathrm{Diff}^r_{\mathrm{vol}}(M)$, $r\geq 1$ and that the associated two point motion F_2^N enjoys annealed exponential mixing, i.e. there exists $p\geq 0$ such that for $A,B\in\mathcal{H}_0^p(M\times M)$,*

$$\left|\mathbb{E}\left(\iint A(x,y)B(F_2^N(x,y))dxdy\right)\right|\leq C\|A\|_{\mathcal{H}_0^p}\|B\|_{\mathcal{H}_0^p}e^{-\alpha N}. \tag{1.4}$$

Then F satisfies quenched exponential mixing, that is, for all $s\geq 0$ there exists β such that for almost every ω there exists $C=C(\omega)$ such that for all $A,B\in\mathcal{H}_0^s$

$$\left|\int A(F_\omega^N x)B(F_\omega^{N+k}x)dx\right|\leq CN\|A\|_{\mathcal{H}_0^s}\|B\|_{\mathcal{H}_0^s}e^{-\beta k}. \tag{1.5}$$

Next we discuss the quenched CLT. For this we will need to describe the variance of the resulting distribution. Given a function B on $M\times M$ let

$$S_B^N(x,y)=\sum_{n=1}^{n}B(F^n x,F^n y),$$

and let

$$\mathbb{D}(B)=\mathbb{E}\left(\int B^2dxdy\right)+2\sum_{k=1}^{\infty}\mathbb{E}\left(\int B(x,y)B(F_k x,F_k y)dxdy\right). \tag{1.6}$$

In the proof of Theorem 1.10 below, we will apply this formula where B has either the special form $B(x,y)=A(x)-A(y)$ or $B(x,y)=A(x)$. In these cases a simple calculation using that $\iint A(x)A(F^k(y))\,dx\,dy=0$ yields that

$$\mathbb{D}(A(x)-A(y))=2\mathcal{D}(A)\text{ and }\mathbb{D}(A(x))=\mathcal{D}(A). \tag{1.7}$$

Theorem 1.10. *Suppose that (1.4) holds and for each function $B(x,y)\in\mathcal{H}_0^s(M\times M)$ we have that S_B^N satisfies the Central Limit Theorem with polynomial convergence of characteristic functions, that is: there exists $\eta>0$ such that for each ξ*

$$\mathbb{E}\left(\iint e^{iS_B^N(x,y)\xi/\sqrt{N}}dxdy\right)=e^{-\mathbb{D}(B)\xi^2/2}+O\left(N^{-\eta}\right). \tag{1.8}$$

Then with probability 1, for each $A\in\mathcal{H}_0^s$ it holds that as $N\to\infty$ that $N^{-1/2}S_N(A)$ converges to a normal random variable with zero mean and variance $\mathcal{D}(A)$.

The same result holds with $\mathcal{H}_0^s$ replaced by C_0^s—the space of zero mean Hölder functions.

We note that the hypotheses of Theorem 1.10 hold for systems with spectral gap [Gou15, Thm. 3.7], see the discussion in the proof of [DD25, Thm. 7.13] for details. Hence we obtain:

Corollary 1.11. *If the generator of the two point motion has a spectral gap on* $\mathcal{H}_0^s$ *for some* $s \in \mathbb{R}$ *then quenched exponential mixing holds on* $\mathcal{H}_0^p$ *with* $p > 0$*, and the quenched Central Limit Theorem holds for* C^r *functions with* $r > |s|$*.*

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2. Preliminaries

In this section we recall some standard facts and introduce notation that will be used below.

2.1. **Harmonic Analysis.** Below we will make extensive use of the Sobolev spaces $\mathcal{H}_0^p$. In particular, in all the arguments below we will work with a fixed Fourier basis M of $L^2(M)$. These are the eigenfunctions φ_i of the Riemannian Laplacian Δ. We fix a basis φ_i of eigenfunctions of the Laplacian Δ that are normalized so that $\|\phi_i\|_{L^2} = 1$. For a zero integral function A write $A = \sum_i a_i \varphi_i$, where each $a_i \in \mathbb{R}$. Then we define the s-Sobolev norm by:

$$\|A\|_{\mathcal{H}_0^s} = \sum_{i\in\mathbb{N}} |a_i|^2 \lambda_i^s. \tag{2.1}$$

Below we will use some basic estimates on these functions, such as

$$\|\phi_i\|_{\mathcal{H}^p} = \lambda_i^{p/2} \text{ and } \|\phi_i\|_{C^0} \le C\lambda_i^{d/2}, \tag{2.2}$$

which follow from the Sobolev embedding theorem.

Below we will make use of the Weyl law for the eigenvalue of the Laplacian. One consequence is the following, which we state as a lemma as we will use it several times.

Lemma 2.1. *(Weyl Law) Suppose that* M *is a Riemannian manifold of dimension* d *and* $\{\lambda_i\}_{i\in\mathbb{N}}$ *are the eigenvalues of the Laplacian. Then there exists* C *such that the number of eigenvalues of norm at most* λ *is at most* $C\lambda^{d/2}$*. In particular,*

$$\sum_i \lambda_i^t$$

is finite as long as $t < -d/2$*.*

Proof. To begin, let b_n the number of eigenvalues of magnitude less than or equal to n. Then the sum in question is bounded above by $\sum_{n\in\mathbb{N}} n^{-\alpha}(b_n - b_{n-1})$ where $\alpha = -t$. Summation by parts shows that

$$\sum_{n=0}^{N} n^{-\alpha}(b_n - b_{n-1}) = n^{-\alpha}b_n - \sum_{n=0}^{N}(n^{-\alpha} - (n-1)^{-\alpha})b_n.$$

By the Weyl Law, $b_n \le n^{d/2}$. Thus the sum is convergent as long as $\alpha > d/2$. □

We will also use below the usual Sobolev embedding theorem:

Lemma 2.2. *(Sobolev embedding theorem) Let* M *be a closed, smooth Riemannian manifold. Then:* $C^s(M) \subset H^s(M) \subset C^{s-d/2}(M)$*.*

2.2. **Probability.** Throughout the rest of the paper, we will write $\mathbb{P}$ and $\mathbb{E}$ for the probability and expectation of a random variable; when we do this we are exclusively taking expectations over the random dynamics ω.

We will use a couple of different concentration inequalities. The first one is Azuma's inequality.

Proposition 2.3. [Ste97, Thm. 1.3.1] *(Azuma's inequality) Suppose that $X_1, \ldots, X_n$ is a martingale difference sequence. Then*

$$\mathbb{P}\left(\left|\sum_{i=1}^n X_i\right| \geq \lambda\right) \leq 2\exp\left(\frac{-\lambda^2}{2\sum_{i=1}^n \|X_i\|_{L^\infty}^2}\right).$$

The second concentration inequality applies to sums of Bernoulli random variables.

Proposition 2.4. [CL06, Thm. 4] *Let $X_1, \ldots, X_n$ be independent random variables with $\mathbb{P}(X_i = 1) = p_i$ and $\mathbb{P}(X_i = 0) = 1 - p_i$. Let $X = \sum_{i=1}^n X_i$. Then*

$$\mathbb{P}(X \geq \mathbb{E}(X) + \lambda) \leq \exp\left(-\frac{\lambda^2}{2(\mathbb{E}(X) + \lambda/3)}\right).$$

3. Exponential Mixing

Here we show that exponential mixing of the two point motion implies exponential mixing. The idea is that one can study the decay of correlations of for a basis of $\mathcal{H}_0^p$ comprised of eigenfunctions of the Laplacian, and deduce that most words exhibit good decay for all the low modes in this basis.

Proof of Theorem 1.9. By interpolation it is sufficient to prove the result for s sufficiently large, so we assume in the computations below that $s > p + (3d/2)$.

Let φ_j be an orthogonal basis consisting of eigenfunctions of Δ. Then $\Delta\varphi_j = \lambda_j^2\varphi_j$, $\|\varphi_j\|_{L^2} = 1$ and $\|\varphi_j\|_{\mathcal{H}_0^p} = \lambda_j^p$. Denote

$$\rho_{i,j,n,k} = \int (\varphi_i \circ F^n)(\varphi_j \circ F^{n+k})dx. \tag{3.1}$$

Then

$$\mathbb{E}(\rho_{i,j,n,k}) = \mathbb{E}(\rho_{i,j,0,k}) = O\left(\lambda_i^p \lambda_j^p e^{-\alpha k}\right),$$

$$\mathbb{E}(\rho_{i,j,n,k}^2) = \mathbb{E}(\rho_{i,j,0,k}^2) = \iint \varphi_i(x)\varphi_i(y)\varphi_j(F^k x)\varphi_j(F^k y)dxdy = O\left(\lambda_i^{2p+d}\lambda_j^{2p+d}e^{-\alpha k}\right)$$

where we have used 2-point mixing (1.4) for the function $\psi(x, y) = \varphi_i(x)\varphi_i(y)$, which satisfies

$$\|\psi\|_{\mathcal{H}^p} \leq \|\psi\|_{C^p} \leq C\|\varphi_i\|_{C^p}^2 \leq C\|\varphi_i\|_{\mathcal{H}_0^{p+d/2}}^2 = C\left(\lambda_i^{p+d/2}\right)^2.$$

See, e.g. [Hör76, Thm. A.7] for the second inequality, $\|\varphi_1\varphi_2\|_{C^k} \leq C\|\varphi_1\|_{C^k}\|\varphi_2\|_{C^k}$.

Hence by Chebyshev's inequality for any $\beta > 0$,

$$\mathbb{P}\left(|\rho_{i,j,n,k}| > n\lambda_i^{p+t}\lambda_j^{p+t}e^{-\beta k}\right) \leq Cn^{-2}\lambda_i^{d-2t}\lambda_j^{d-2t}e^{-(\alpha-2\beta)k}. \tag{3.2}$$

From Lemma 2.1, $\sum_i \lambda_i^{-\alpha}$ is finite for $\alpha > d/2$. By the Borel Cantelli Lemma the above events could happen only finitely many times if $t > d$. Hence for such t, and almost every ω, there exists N_ω such that for $n \geq N_\omega$ and all i, j, k:

$$|\rho_{i,j,n,k}| \leq n\lambda_i^{p+t}\lambda_j^{p+t}e^{-\beta k}. \tag{3.3}$$

Decomposing $A = \sum_i a_i\varphi_i$, $B = \sum_j b_j\varphi_j$, we obtain using (3.3) that

$$\left|\int A(F^N x)B(F^{N+k}x)dx\right| \leq Ne^{-\beta k}\sum_{i,j}|a_i|\,|b_j|\,\lambda_i^{p+t}\lambda_j^{p+t} = Ne^{-\beta k}\left[\sum_i |a_i|\,\lambda_i^{p+t}\right]\left[\sum_j |b_j|\,\lambda_j^{p+t}\right].$$

Note that

$$\sum_i |a_i|\lambda_i^{p+t} \leq \left(\sum_i |a_i|^2\lambda_i^{2s}\right)^{1/2}\left(\sum_i \lambda_i^{2p+2t-2s}\right)^{1/2} \leq \|A\|_{\mathcal{H}_0^s}\left(\sum_i \lambda_i^{2p+2t-2s}\right)^{1/2}. \tag{3.4}$$

As before, by the Weyl Law the last term is finite for $s > p + t + d/2$. If this condition on s holds, then we obtain the same estimate on $\sum_j |b_j|\,\lambda_j^{p+t}$ as well.

Note that if $s > p + (3d/2)$, we can choose t so that both $t > d$ and $s > p + t + (d/2)$, so the above estimate holds and we obtain the result. □

Corollary 3.1. *Under the assumptions of Theorem 1.9, for almost every ω we have that for each $A \in \mathcal{H}_0^s$ with $s > p + (3d)/2$ and for almost all x:*

$$\sum_{n=1}^N A(F_\omega^n x) = O\left(N^{1/2+\varepsilon}\right).$$

Proof. This follows from (1.5) and [DDKN22b, Lemma 8.1]. That lemma says that if X_n is a stationary sequence of random variables such that $\mathbb{E}[(\sum_{n=1}^N X_n)^2] \leq CN^{2\rho}$ then $S_N/N^{\max\{\rho,1/2\}+\epsilon}$ converges almost surely to zero. We apply this fact to the random variables $X_n = A\circ F_\omega^n(x)$ where ω is fixed and x is uniformly distributed on M. By(1.5) we have for $i \leq j$ that $|\mathbb{E}[X_iX_j]| \leq \min(C_\omega i e^{-|j-i|}\|A\|_{H_0^s}, \|A\|_{C^0}^2)$. Summing over i and j we conclude that there exists C_ω such that $\mathbb{E}[(\sum_{i=1}^n X_i)^2] \leq C_\omega(n\ln n)\|A\|_{H_0^s}^2$ and the conclusion follows. □

Remark 3.2. Note that if A is fixed first then the result follows from annealed mixing and Fubini Theorem, the novelty of this result is that the set of ωs of full measure could be taken independent of A.

4. Asymptotic Variance

To show the equivalence of the quenched and annealed versions of the central limit theorems we need to control the growth of quenched variances.

Let $S_N(A)(x) = \sum_{n=1}^N A(F^n x)$.

Lemma 4.1. *Under the hypotheses of Theorem 1.9, for any $s > p + 3d/2$ the following holds.*

(a) *For almost every ω and all $A \in \mathcal{H}_0^s$ there exists $K = K(\omega)$ such that*

$$V_N(A) := \int S_N^2(A)(x)dx \leq KN\|A\|^2_{\mathcal{H}_0^s}. \tag{4.1}$$

(b) *For almost every ω and all $A \in \mathcal{H}_0^s$*

$$\lim_{N\to\infty} \frac{V_N(A)}{N} = \mathcal{D}^2(A) := \int A^2 dx + 2\sum_{k=1}^{\infty} \mathbb{E}\left(\int A(x)A(F^k x)dx\right).$$

Proof. As in the proof of Theorem 1.9 let φ_i be a basis of eigenfunctions of Δ with $\|\phi_i\|_{L^2} = 1$. Define $\rho_{i,j,n,k}$ as in (3.1). Let

$$V_{N,i,j} := \int \left(\sum_{n=1}^{N} \varphi_i(F^n x)\right)\left(\sum_{m=1}^{N} \varphi_j(F^m x)\right) dx = \sum_{n,n+k\in\{1,\dots,N\}} \rho_{i,j,n,k}.$$

We claim that with probability 1, there exists $C(\omega)$ such that for all i and j and $N \in \mathbb{N}$:

$$|V_{N,i,j}| \leq C(\omega)N\lambda_i^{p+t}\lambda_i^{p+t}. \tag{4.2}$$

We will obtain this estimate by estimating the probability that each $\rho_{i,j,n,k}$ is large and then applying the Borel-Cantelli lemma. To this end, we define

$$V_{N,i,j,k} := \sum_{0\leq n\leq N-k} \rho_{i,j,n,k}.$$

First, we estimate $V_{N,i,j,k}$ where $|k| \geq N^{0.1}$, which are the easiest. Recall from (3.2), that for all t,

$$\mathbb{P}\left(|\rho_{i,j,n,k}| > \lambda_i^{p+t}\lambda_j^{p+t}e^{-\beta k}\right) \leq C_1\lambda_i^{d-2t}\lambda_j^{d-2t}e^{-(\alpha-2\beta)k}. \tag{4.3}$$

The sum of the probabilities that one of the events in the previous line occurs for some N, i, j and $k \geq N^{0.1}$ is

$$\sum_{i,j\in\mathbb{N}}\sum_{N\in\mathbb{N}}\sum_{0\leq n\leq N}\sum_{N^{0.1}\leq k\leq N-k} \lambda_i^{d-2t}\lambda_j^{d-2t}e^{-(\alpha-2\beta)k} \leq \sum_{i,j\in\mathbb{N}} \lambda_i^{d-2t}\lambda_j^{d-2t}. \tag{4.4}$$

So, for $t > 3d/4$ and $0 < \beta < \alpha/2$, it follows from Lemma 2.1 that the above sum is finite. By the Borel-Cantelli Lemma for any $0 < \eta < \alpha - 2\beta$ there are only finitely many i, j, n and $k > N^{0.1}$ such that $|\rho_{i,j,n,k}| > \lambda_i^{p+t}\lambda_j^{p+t}e^{-\beta k}$. Thus it follows that for almost every ω, there exists $C(\omega)$ such that for all $N, i, j,$ and $k \geq N^{0.1}$:

$$|V_{N,i,j,k}| = \left|\sum_{0\leq n\leq N-k} \rho_{i,j,n,k}\right| \leq C(\omega)\lambda_i^{p+t}\lambda_j^{p+t}e^{-\beta k}. \tag{4.5}$$

We now turn to the terms where $k < N^{0.1}$, for which the $\rho_{i,j,n,k}$ decay slower. We study $V_{N,i,j}$ by dividing into the sum of two terms, one that experiences a good correlation decay, $V'_{N,i,j}$, and another that does not, $V''_{N,i,j}$. Below, we will use repeatedly the observation that $\rho_{i,j,n,k}$ is independent of $\rho_{i,j,n+\ell,k}$ as long as $\ell \geq k$. To take advantage of this observation, for $|k| \leq N^{0.1}$, let

$$\rho'_{i,j,n,k} = \rho_{i,j,n,k} 1_{|\rho_{i,j,n,k}| \leq \lambda_i^{p+(t/2)} \lambda_j^{p+(t/2)} e^{-\beta|k|}}, \quad \rho''_{i,j,n,k} = \rho_{i,j,n,k} 1_{|\rho_{i,j,n,k}| > \lambda_i^{p+(t/2)} \lambda_j^{p+(t/2)} e^{-\beta|k|}},$$

$$V'_{N,i,j,k} = \sum_{0 \leq n \leq N-k} \rho'_{i,j,n,k}, \quad V'_{N,i,j,k,\ell} = \sum_{\substack{n \equiv \ell \bmod k \\ 0 \leq n \leq N-k}} \rho'_{i,j,n,k},$$

$$V''_{N,i,j,k} = \sum_{0 \leq n \leq N-k} \rho''_{i,j,n,k}, \quad V'_{N,i,j,k,\ell} = \sum_{\substack{n \equiv \ell \bmod k \\ 0 \leq n \leq N-k}} \rho''_{i,j,n,k},$$

$$V'_{N,i,j} = \sum_{0 \leq k \leq N} V'_{N,i,j,k}, \quad V''_{N,i,j} = \sum_{0 \leq k \leq N} V''_{N,i,j,k}.$$

We begin with the term $V'_{N,i,j,k}$ that experiences a fast decay of correlations. From the definition of $\rho'_{i,j,n,k}$, these terms are bounded by $\lambda_i^{p+(t/2)} \lambda_j^{p+(t/2)} e^{-\beta|k|}$. As these terms are also independent, they form a submartingale. From the bound on the $\rho'_{i,j,n,k}$, it then follows from Azuma's inequality, Proposition 2.3, that there exists $c > 0$ such that:

$$\mathbb{P}\left(V'_{N,i,j,k,\ell} \geq N^{.9} \lambda_i^{p+t} \lambda_j^{p+t} e^{-\beta|k|/4}\right) \leq 2 \exp\left[-cN^{.8} e^{\beta|k|/2} \lambda_i^t \lambda_j^t\right].$$

By bounding the left hand side of the following inequality by at least one of the events in the previous line happening, and summing over the at most $N^{0.1}$ residue classes mod k, we get

$$\mathbb{P}\left(V'_{N,i,j,k} \geq N \lambda_i^{p+t} \lambda_j^{p+t} e^{-\beta|k|/4}\right) \leq 2|k| \exp\left[-cN^{.8} e^{\beta|k|/2} \lambda_i^t \lambda_j^t\right]. \tag{4.6}$$

Next, we estimate the terms $V''_{N,i,j,k}$ with slow decay. Note that by the definition of $\rho''_{i,j,n,k}$ and (4.3)

$$\mathbb{P}(\rho''_{i,j,n,k} \neq 0) \leq C \lambda_i^{p-t} \lambda_j^{p-t} e^{-\eta|k|}.$$

Also, trivially from (2.2),

$$|\rho''_{i,j,n,k}| \leq \|\varphi_i\|_{C^0} \|\varphi_j\|_{C^0} \leq C \lambda_i^{d/2} \lambda_j^{d/2}. \tag{4.7}$$

First, we estimate $V''_{N,i,j,k,\ell}$. As before, the $\rho''_{i,j,n,k}$ are independent for different $n \equiv l \bmod k$. From (4.7), it follows that:

$$\mathbb{P}(V''_{N,i,j,k,\ell} \geq \lambda_i^{p+t} \lambda_j^{p+t} N\, |k|^{-5})$$

$$\leq \mathbb{P}\left(\sum_{\substack{n \equiv \ell \bmod k \\ 0 \leq n \leq N-k}} 1_{|\rho_{i,j,n,k}| > \lambda_i^{p+t/2} \lambda_j^{p+t/2} e^{-\beta|k|}} \geq \lambda_i^{p+t-d/2} \lambda_j^{p+t-d/2} N\, |k|^{-5}\right)$$

$$\leq \mathbb{P}(A(i,j,N,k,\ell) \geq 2Q(i,j,N,k))$$

Let us obtain an estimate on the expectation of $A(i,j,N,k,\ell)$.

$$\mathbb{E}[A(i,j,N,k,\ell)] \leq \sum_{\substack{n \equiv \ell \bmod k \\ 0 \leq n \leq N-k}} \mathbb{P}(\rho''_{i,j,n,k} \neq 0) \leq \sum_{\substack{n \equiv \ell \bmod k \\ 0 \leq n \leq N-k}} C \lambda_i^{p-t} \lambda_j^{p-t} e^{-\eta|k|} \leq N \lambda_i^{p-t} \lambda_j^{p-t} e^{-\eta|k|}$$

Note that for as long as t is sufficiently large, it follows that $\mathbb{E}[A(i,j,n,k)] \le Q$. Hence,

$$\mathbb{P}(A(i,j,N,k,\ell) \ge 2Q(i,j,n,k)) \le \mathbb{P}(A(i,j,N,k,\ell) \ge \mathbb{E}[A(i,j,N,k,\ell)] + Q(i,j,n,k))$$

As $A(i,j,N,k,\ell)$ is the sum of independent Bernoulli random variables, we can apply Theorem 2.4. In particular, as $\mathbb{E}[A(i,j,N,k,\ell)] + Q(i,j,N,k) \le 2Q$, we find that:

$$\mathbb{P}(A(i,j,N,k,\ell) \ge 2Q(i,j,n,k)) \le \exp\left(-\frac{Q^2}{4Q}\right) \tag{4.8}$$

But this gives that

$$\mathbb{P}(V''_{N,i,j,k,\ell} \ge \lambda_i^{p+t}\lambda_j^{p+t} N\left|k\right|^{-5}) \le \exp(-\lambda_i^{p+t-d/2}\lambda_j^{p+t-d/2}N\left|k\right|^{-5}/4). \tag{4.9}$$

Now summing over each possible residue class $\ell \bmod k$, it follows that

$$\mathbb{P}(V''_{N,i,j,k} \ge \lambda_i^{p+t}\lambda_j^{p+t} Nk^{-4}) \le k\exp(-\lambda_i^{p+t-d/2}\lambda_j^{p+t-d/2}N\left|k\right|^{-5}/4). \tag{4.10}$$

We now apply the Borel-Cantelli Lemma to $V'_{N,i,j,k}$ and $V''_{N,i,j,k}$. Observe from equations (4.6) and (4.10) that

$$\begin{aligned}&\mathbb{P}\left(V'_{N,i,j,k} \ge N\lambda_i^{p+t}\lambda_j^{p+t}e^{-\beta|k|/4}\right) + \mathbb{P}\left(V''_{N,i,j,k} \ge \lambda_i^{p+t}\lambda_j^{p+t}Nk^{-4}\right)\\ \le &\sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}}\sum_{N\in\mathbb{N}}\sum_{k\le N^{0.1}} k\left[\exp\left(-cN^{.8}e^{\beta|k|/2}\lambda_i^t\lambda_j^t\right) + \exp(-\lambda_i^{p+t-d/2}\lambda_j^{p+t-d/2}N\left|k\right|^{-5}/4)\right]\end{aligned}$$

This is finite as long as $t > d/2$. Thus for all but finitely many tuples (N,i,j,k), with $k \le N^{0.1}$, it follows that both

$$V'_{N,i,j,k} \le N\lambda_i^{p+t}\lambda_j^{p+t}e^{-\beta|k|/4} \text{ and } V''_{N,i,j,k} \le \lambda_i^{p+t}\lambda_j^{p+t}Nk^{-4}.$$

From the above line and (4.5), it follows that for $t > 3d/4$ and $0 < \beta < \alpha/2$ for a.e. ω, that there exists $C(\omega)$ such that that for all N, i, j, k:

$$\frac{V_{N,i,j,k}}{N\lambda_i^{p+t}\lambda_j^{p+t}} \le \begin{cases}\frac{C(\omega)}{k^4} & \text{if } k \le N^{0.1}\\ Ce^{-\beta|k|} & \text{if } k > N^{0.1}.\end{cases} \tag{4.11}$$

From the definitions it follows that $|V_{N,i,j}| \le 2\sum_{0\le k\le N}|V_{N,i,j,k}|$. Summing over k we conclude that there exists $t > 3d/4$ such that for a.e. ω there exists $D(\omega)$ such that for all N, i, and j,

$$|V_{N,i,j}| \le DN\lambda_i^{p+t}\lambda_j^{p+t}. \tag{4.12}$$

To conclude, we can expand an arbitrary $A \in \mathcal{H}_0^s$ as $A = \sum_j a_j\varphi_j$ and argue as in the proof of Theorem 1.9 to obtain part (a) of the lemma; this gives the constraint $s > p + (3d/2)$.

We now start the proof of (b). Define

$$V_N(A,B) := \int\left(\sum_{i=1}^N A\circ F^k\right)\left(\sum_{j=1}^N B\circ F^j\right)dx. \tag{4.13}$$

Due to bilinearity, we will write $V_N(A) = V_N(A,A)$ as

$$V_N(A,A) = \sum_{i,j} a_i a_j V_N(\varphi_i,\varphi_j).$$

To use this, we begin with the following claim.

Claim 4.2. *For a.e. ω and each i, j,*

$$\lim_{n\to\infty} \frac{V_N(\varphi_i,\varphi_j)}{N} = \int \varphi_i\varphi_j \, dx + \sum_{1\le k} \mathbb{E}\left[\int \varphi_i(x)\varphi_j(F^k(x))\, dx\right] + \mathbb{E}\left[\int \varphi_i(F^k(x))\varphi_j(x)\, dx\right]$$

Proof. To begin, observe that:

$$\begin{aligned} V_N(\phi_i,\phi_j) &= \int \sum_{n=1}^{N} \phi_i\phi_j + \sum_{k=1}^{N} \sum_{0\le n\le N-k} \phi_i(F^n x)\phi_j(F^{n+k}(x)) + \phi_i(F^{n+k}(x))\phi_j(F^n(x))\, dx \\ &= \int N\phi_i\phi_j \, dx + \sum_{k=1}^{N} V_{N,i,j,k} + V_{N,j,i,k} \end{aligned}$$

By the ergodic theorem, almost surely,

$$N^{-1}V_{N,i,j,k} \to \mathbb{E}\left[\int \phi_i\phi_j \circ F^k \, dx\right]$$

Thus we would like to show that the following line is dominated, so we can pass to the limit:

$$N^{-1}V_N(\phi_i,\phi_j) = \int \phi_i\phi_j \, dx + \sum_{k=0}^{N} \left[\frac{V_{N,i,j,k}}{N} + \frac{V_{N,j,i,k}}{N}\right].$$

By (4.11), the terms $V_{N,i,j,k}/N$ are dominated since $\left|V_{N,i,j,k}\right|/N$ is bounded independent of N by $C(\omega)k^{-4}$ which is summable in k. So we can pass to the limit and conclude that almost surely:

$$\lim_N N^{-1}V_N(\varphi_i,\varphi_j) = \int \varphi_i\varphi_j \, dx + \sum_k \mathbb{E}\left[\int \phi_i \circ F^k \phi_j \, dx\right] + \mathbb{E}\left[\int \phi_i\phi_j \circ F^k \, dx,\right]$$

as desired. ☐

Let us now conclude by using the claim. As before write $A = \sum_{i\in\mathbb{N}} a_i\varphi_i$. Then we do the following computation, which we will subsequently justify:

$$\lim_{N\to\infty} \frac{V_N(A,A)}{N} = \lim_{N\to\infty} \sum_{i,j\in\mathbb{N}} a_i a_j \frac{V_N(\varphi_i,\varphi_j)}{N} \tag{4.14}$$

(4.15)

$$= \sum_{i,j\in\mathbb{N}} a_i a_j \left(\int \varphi_i\varphi_j \, dx + \sum_{1\le k} \left[\int \varphi_i(x)\varphi_j(F^k(x))\, dx\right] + \mathbb{E}\left[\int \varphi_i(F^k(x))\varphi_j(x)\, dx.\right]\right)$$

$$= \int A^2 \, dx + 2\sum_{1\le k} \mathbb{E}\left[\int A(x)A(F^k(x))\, dx\right] \tag{4.16}$$

In order to pass from the first line to the second, we need to know that $\sum_{i,j} a_i a_j V_N(\phi_i, \phi_j)/N$ is dominated. This follows from the estimate on $V_N(\phi_i, \phi_j) = V_{N,i,j}$ in (4.11), which says that for $t > 3d/4$,

$$\sum_{i,j} \frac{|a_i|\,|a_j|\,|V_N(\phi_i, \phi_j)|}{N} \le \sum_{i,j} |a_i|\,|a_j|\, D(\omega)\lambda_i^{p+t}\lambda_j^{p+t} \le D(\omega)\left(\sum_i |a_i|\,\lambda_i^{p+t}\right)\left(\sum_j |a_j|\,\lambda_j^{p+t}\right)$$

But by (3.4), the terms in parentheses are finite as long as $s > p + t + d/2$, which we can ensure as we chose $s > p + 3d/2$. Thus we can pass from (4.14) to (4.15).

Next we need to check that the expression in (4.15) is absolutely summable in i, j, k so we can rearrange it to pass to line (4.16). Note that

$$\sum_{i,j,k} |a_i|\,|a_j| \left|\mathbb{E}\left[\int \phi_i \phi_j \circ F^k\,dx\right]\right| \le \sum_{i,j,k} |a_i|\,|a_j|\,\|\phi_i\|_{\mathcal{H}^p}\|\phi_j\|_{\mathcal{H}^p} e^{-\alpha k}$$

$$\le C \sum_{i,j} |a_i|\,|a_j|\,\lambda_i^{p/2}\lambda_j^{p/2} = C\left[\sum_i |a_i|\,\lambda_i^{p/2}\right]^2 \tag{4.17}$$

where we have used the exponential mixing hypothesis in the first inequality. For the right-hand expression in (4.17), consider:

$$(\sum_i |a_i|\,\lambda_i^{p/2}) \le (\sum_i |a_i|^2\,\lambda_i^{p+t})^{1/2}(\sum_i \lambda_i^{-t})^{1/2}.$$

The right term is finite as long as $t > d/2$ by Lemma 2.1. In order to take $t > d/2$ and have the left term be finite as well, we need that $A \in \mathcal{H}^s$ where $s > p + d/2$ by definition of the Sobolev norm. Thus as $A \in \mathcal{H}^s$ by assumption, (4.17) is summable, so we may pass to line (4.16) and we are done. □

We now record an additional estimate on $\mathcal{D}(A)$ that will be useful later.

Lemma 4.3. *Suppose as in Theorem 1.9 that we have annealed mixing on $\mathcal{H}_0^p$. Define*

$$\mathcal{D}(A, B) := \int AB\,dx + 2\sum_{k=1}^{\infty} \mathbb{E}\left[\int AB \circ F^k\,dx\right],$$

so that $\mathcal{D}(A, A) = \mathcal{D}(A)$. Then for any $A, B \in \mathcal{H}_0^p$, it follows that:

$$|\mathcal{D}(A) - \mathcal{D}(B)| = |\mathcal{D}(A - B, A + B)| \le \|A - B\|_{\mathcal{H}_0^s}\|A + B\|_{\mathcal{H}_0^s}.$$

Proof. There are two facts used above. The first is that $\mathcal{D}(A) - \mathcal{D}(B) = \mathcal{D}(A - B, B + A)$. This follows from exponential mixing in (1.4), which shows that the series defining these two quantities are absolutely convergent and hence can be rearranged. The second is the estimate $\mathcal{D}(A, B) \le C\|A\|_{\mathcal{H}_0^p}\|B\|_{\mathcal{H}_0^p}$ which holds because the norm of the kth term in the definition of $\mathcal{D}$ is $\|A\|_{\mathcal{H}_0^p}\|B\|_{\mathcal{H}_0^p} e^{-\alpha k}$ from annealed exponential mixing (1.4). □

5. Central Limit Theorem

We are now ready to prove Theorem 1.10.

Proof. To fix the notation, we prove the result for Sobolev spaces; the proof for smooth functions is identical.

We divide the proof into several steps, each of which simplifies what we must check until we have reduced to checking convergence of the characteristic function at rational frequencies.

Step 1. Let $N_m = m^a$ with $a > 1/\eta$ where η is the convergence rate in (1.8). It suffices to prove that for a.e. ω and each $A \in \mathcal{H}^s_0(M)$ that $(N_m)^{-1/2} S^{N_m}(A)$ converges to $\mathcal{N}(0, \mathcal{D}(A))$.

Indeed, suppose that we have convergence along this subsequence. Then, given an arbitrary N choose m so that $N_m \le N < N_{m+1}$. Then

$$\frac{S_N}{\sqrt{N}} = \frac{S_{N_m}}{\sqrt{N_m}} + \frac{S_N - S_{N_m}}{\sqrt{N}} + \frac{S_{N_m}}{\sqrt{N_m}} \left(\sqrt{\frac{N_m}{N}} - 1 \right).$$

By Theorem 1.9 for almost every ω there exists $C(\omega)$ such that

$$\left| \int A(F^n x) A(F^{n+k} x) dx \right| \le C(\omega) \|A\|_{\mathcal{H}^s_0} \min\left(1, n e^{-\beta k}\right).$$

Summing over $N \le n \le n+k \le N_m$ it follows that $\mathbb{E}[(S_N - S_{N_m})^2] \le C(N - N_m) \ln N$, so the second term converges to zero in probability due to the Chebyshev's inequality. Also, the third term converges to zero due to the Slutsky's theorem and our assumption that the CLT holds along N_m. Invoking again Slutsky's theorem we see that the central limit theorem holds for all N.

Step 2. It suffices to prove that the quenched Central Limit Theorem holds for a $\mathcal{H}^s_0$ dense set of functions.

Indeed let $\mathcal{A}$ be a dense collection of functions satisfying the quenched Central Limit Theorem. Take $A \in \mathcal{H}^s_0$ and let h be a compactly supported smooth test function on $\mathbb{R}$. Let J denote the support of h. We need to show that

$$\lim_{N\to\infty} \int h\left(\frac{S_N(A)(x)}{\sqrt{N}}\right) dx = \int_J h(u) \mathfrak{f}_{\mathcal{D}(A)}(u) du \tag{5.1}$$

where $\mathfrak{f}_D$ denotes the density of the normal random variable with zero mean and variance D. Fix $\varepsilon > 0$ and take $\tilde{A} \in \mathcal{A}$ such that $\|\tilde{A} - A\|_{\mathcal{H}^s_0} \le \varepsilon$. By Lemma 4.3, $|\mathcal{D}(A) - \mathcal{D}(\tilde{A})| \le C_A \varepsilon$. Now write

$$\int h\left(\frac{S_N(A)}{\sqrt{N}}\right) dx = \int h\left(\frac{S_N(\tilde{A})}{\sqrt{N}}\right) dx + \int \left[h\left(\frac{S_N(A)}{\sqrt{N}}\right) - h\left(\frac{S_N(\tilde{A})}{\sqrt{N}}\right)\right] dx.$$

Since $\tilde{A} \in \mathcal{A}$ the first term for large N is ε close to $\int_J h(u) \mathfrak{f}_{\mathcal{D}(\tilde{A})}(u) du$ and whence it is $C_A \varepsilon$ close to $\int_J h(u) \mathfrak{f}_{\mathcal{D}(A)}(u) du$. From the mean value theorem, the second term is smaller in absolute value than

$$\|h\|_{C^1}\varepsilon^{1/3}+\|h\|_{C^0}\mathrm{mes}\left(x:\left|\frac{S_N(A)(x)}{\sqrt{N}}-\frac{S_N(\tilde{A})(x)}{\sqrt{N}}\right|\geq\varepsilon^{1/3}\right).$$

By Lemma 4.1(a) and Chebyshev's inequality the above expression is $O(\varepsilon^{1/3})$. Since ε is arbitrary (5.1) holds for all h, and hence A satisfies the quenched CLT.

Step 3. Observe that since $\mathcal{H}_0^s$ contains a countable dense set, it is enough to show that the quenched CLT holds for a fixed function $A\in\mathcal{H}_0^s$.

Step 4. Almost surely, the functions $\Phi_{A,N}(\xi)=\displaystyle\int e^{iS_N(A)(x)\xi/\sqrt{N}}dx$ are equicontinuous with respect to N.

Indeed $\Phi_{A,N}(0)=1$, taking the first derivative gives:

$$\partial_\xi\Phi_{A,N}=\int\frac{iS_N(A)(x)}{\sqrt{N}}e^{iS_N(A)(x)\xi/\sqrt{N}}\,dx.$$

Note in particular that $\partial_\xi\Phi_{A,N}(0)=0$ (since A has zero mean), and by Cauchy-Schwarz,

$$|\partial_\xi\Phi_{A,N}(\xi)|\leq\left(\int\frac{S_N(A)^2(x)}{N}\,dx\right)^{1/2}$$

By Lemma 4.1(a), it follows that $\int N^{-1}S_N^2(A)(x)\,dx\leq K\|A\|^2_{\mathcal{H}_0^s}$ and hence $|\partial_\xi\Phi_{A,N}|$ is a bounded function. Thus $\Phi_{A,N}(\xi)$ is equicontinuous.

Combining Steps 1-4 above and we see that it suffices to show that for almost every ω that $\Phi_{A,N}(\xi)\to e^{-\mathcal{D}(A)\xi^2/2}$ for all rational ξ restricted to the sequence N_m from Step 1. Hence it suffices to show that the converges holds for a fixed (rational) ξ.

Next, from (1.7) if $B(x,y)=A(x)$ then $\mathbb{D}(B)=\mathcal{D}(A)$, while if $B(x,y)=A(x)-A(y)$ then $\mathbb{D}(B)=2\mathcal{D}(A)$. Next let

$$Z_N(\omega)=\int e^{i\xi S_N(A)(x)/\sqrt{N}}dx-e^{-\xi^2\mathcal{D}(A)/2}.$$

We claim that Z_{N_m} converges to zero almost surely. Indeed by (1.8) $\mathbb{E}(Z_N)=O(N^{-\eta})$ and in addition,

$$\begin{aligned}\mathbb{E}\left(Z_N\bar{Z}_N\right)=&\mathbb{E}\left(\left[\int e^{i\xi S_N(A)(x)/\sqrt{N}}dx-e^{-\xi^2\mathcal{D}(A)/2}\right]\left[\int e^{-i\xi S_N(A)(y)/\sqrt{N}}dy-e^{-\xi^2\mathcal{D}(A)/2}\right]\right)\\ =&\mathbb{E}\left(\iint e^{i\xi(S_N(A)(x)-S_N(A)(y))/\sqrt{N}}\,dxdy\right)+e^{-\xi^2\mathcal{D}(A)}\\ &-2e^{-\xi^2\mathcal{D}(A)/2}\mathbb{E}\left(\Re\int e^{i\xi S_N(A)(x)/\sqrt{N}}\,dx\right)\\ =&e^{-\xi^2\mathcal{D}(A)}+O(N^{-\eta})+e^{-\xi^2\mathcal{D}(a)}-2e^{-\xi^2\mathcal{D}(A)/2}(e^{-\xi^2\mathcal{D}(A)/2}+O(N^{-\eta}))=O\big(N^{-\eta}\big).\end{aligned}$$

Now from our choice of N_m and the term in the above line, Chebyshev's inequality gives that $\lim_{m\to\infty}\int e^{i\xi S_{N_m}(A)(x)\xi}dx = e^{-\xi^2\mathcal{D}(A)/2}$ for almost every ω completing the proof of the theorem. □

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