Normalization of Thurston measures on the space of measured geodesic laminations

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Abstract. In the literature one can find two recurring definitions of Lebesgue class, mapping class group invariant measures on the space of measured geodesic laminations of a connected, oriented, finite type surface: one as the limit of integral multi-curve counting measures and the other one as the measure induced by a symplectic form in train track coordinates. By work of Masur, these measures agree up to a multiplicative factor. In this note we explicitely compute such multiplicative factor. Our result agrees with an independent computation of the same factor due to Monin and Telpukhovskiy.

1. Introduction

Let $g,n \geq 0$ be an arbitrary pair of integers satisfying $2-2g-n<0$. For the rest of this note we fix such a pair of integers and consider a fixed connected, oriented, smooth surface of genus g with n punctures (and negative Euler characteristic), which we denote by $S_{g,n}$.

The space of measured geodesic laminations on $S_{g,n}$, denoted by $\mathcal{ML}_{g,n}$, admits a $6g-6+2n$ dimensional piecewise integral linear structure induced by train track charts. The integral points of this structure are precisely the integral multi-curves on $S_{g,n}$; we denote them by $\mathcal{ML}_{g,n}(\mathbf{Z}) \subseteq \mathcal{ML}_{g,n}$. By work of Masur, see [**Mas85**], the mapping class group of $S_{g,n}$, denoted by $\mathrm{Mod}_{g,n}$, acts ergodically on $\mathcal{ML}_{g,n}$. In particular, there is a unique, up to scaling, Lebesgue class, mapping class group invariant measure on $\mathcal{ML}_{g,n}$. In the literature one can find two recurring definitions of such a measure. The first one, denoted by μ_{Thu}, is the limit of integral multi-curve counting measures on $\mathcal{ML}_{g,n}$. The other one, denoted by ν_{Thu}, is the measure induced by a symplectic form on $\mathcal{ML}_{g,n}$ coming from train track charts. See §2 for precise definitions of these measures. We explicitely compute the scaling factor relating these measures. Our result agrees with an independent computation due to Monin and Telpukhovskiy; see [**MT19**].

Theorem 1.1. *The measures μ_{Thu} and ν_{Thu} on $\mathcal{ML}_{g,n}$ satisfy*

$$\nu_{Thu} = 2^{2g-3+n} \cdot \mu_{Thu}.$$

2. Background material

Measured geodesic laminations. A geodesic lamination λ on a complete, finite volume hyperbolic surface X diffeomorphic to $S_{g,n}$ is a set of disjoint simple, complete geodesics whose union is a compact subset of X. A measured geodesic lamination is a geodesic lamination carrying an invariant transverse measure fully supported on the geodesic lamination. We can understand measured geodesic laminations by lifting them to a universal cover $\mathbf{H}^2 \to X$. A non-oriented geodesic on $\mathbf{H}^2$ is specified by a set of distinct points on the boundary at infinity $\partial^\infty \mathbf{H}^2 = S^1$. It follows that measured geodesic laminations on diffeomorphic hyperbolic surfaces may be compared by passing to the boundary at infinity of their universal covers. Thus, the space of measured geodesic laminations on X depends only on the underlying topological surface $S_{g,n}$. We denote by $\mathcal{ML}_{g,n}$ the space of measured geodesic laminations on $S_{g,n}$. By taking geodesic representatives, integral multi-curves on $S_{g,n}$ can be interpreted as elements of $\mathcal{ML}_{g,n}$; we denote them by $\mathcal{ML}_{g,n}(\mathbf{Z})$. For more details on the theory of measured geodesic laminations see §8.3 in [**Mar16**].

Train tracks. A train track τ on $S_{g,n}$ is an embedded 1-complex satisfying the following conditions:

(1) Each edge of τ is a smooth path with a well defined tangent vector at each endpoint. All edges at a given vertex are tangent.
(2) For each component R of $S \backslash \tau$, the double of R along the smooth part of ∂R has negative Euler characteristic.

The edges of τ are called branches. The vertices of τ where three or more smooth arcs meet are called switches. The inward pointing tangent vector of an edge divides the branches that are incident to a vertex into incoming and outgoing branches.

A train track τ on $S_{g,n}$ is said to be maximal if all the components of $S \backslash \tau$ are trigons, i.e. the interior of a disc with three non-smooth points on the boundary, or once punctured monogons, i.e. the interior of a punctured disc with one non-smooth points on the boundary . We say τ is generic if every switch is trivalent. Every train track can be turned into a generic train track by sliding its edges.

The PIL structure on the space of measured geodesic laminations. A geodesic lamination λ on $S_{g,n}$ is said to be carried by a train track τ if there is a differentiable map $f \colon S_{g,n} \to S_{g,n}$ homotopic to the identity, mapping λ into τ, and such that the restriction of f to every leaf of λ is nonsingular. Every geodesic lamination is carried by some train track τ. When λ has a transverse invariant measure μ, the carrying map f defines a positive counting measure u assigning a real positive number $u(e)$ to every edge e of τ. The invariance of μ implies that at each switch the sum of the counting measure of incoming edges equals the sum of the counting measure of outgoing edges. Such conditions are called *switch conditions.*

Given a train track τ on $S_{g,n}$, let $E(\tau)$ be the set of positive counting measures on τ. More precisely, $u \in E(\tau)$ is an assignment of positive real numbers on the edges of the train track τ satifying the switch conditions. We say τ is recurrent if $E(\tau) \neq \emptyset$. If τ is maximal and recurrent, then the $6g - 6 + 2n$ dimensional convex cone $E(\tau)$ gives rise to an open set $U(\tau) \subseteq \mathcal{ML}_{g,n}$ and an identification $\psi_\tau \colon U(\tau) \to E(\tau)$. The open set $U(\tau) \subseteq \mathcal{ML}_{g,n}$ is precisely the set of measured geodesic laminations on $S_{g,n}$ carried by τ that induce positive weights on the branches of τ. The integral points in $E(\tau)$ are in one-to-one correspondence with the integral multicurves in $U(\tau) \subseteq \mathcal{ML}_{g,n}$. The chart $\psi_\tau \colon U(\tau) \to E(\tau)$ is called the *train track chart* of $\mathcal{ML}_{g,n}$ associated to τ. Every measured geodesic lamination belongs to some open set $U(\tau)$ for some maximal, recurrent train track τ. The transition maps between train track charts are piecewise integral linear maps. It follows that these charts make $\mathcal{ML}_{g,n}$ into a $6g-6+2n$ dimensional piecewise integral linear (PIL) manifold whose integer points are precisely the integral multi-curves $\mathcal{ML}_{g,n}(\mathbf{Z})$. For more details on the definition of the PIL structure of $\mathcal{ML}_{g,n}$ induced by train track charts see §3.1 in [**PH92**].

The Thurston measure. For every $L > 0$ consider the counting measure μ^L on $\mathcal{ML}_{g,n}$ given by

$$\mu^L := \frac{1}{L^{6g-6+2n}} \sum_{\gamma \in \mathcal{ML}_{g,n}(\mathbf{Z})} \delta_{\frac{1}{L}\cdot\gamma}.$$

Using train track coordinates one can check that, as $L \to \infty$, this sequence of counting measures converges, in the weak-$\star$ topology, to a non-zero, locally finite, Lebesgue class measure μ_{Thu} on $\mathcal{ML}_{g,n}$. We refer to this measure as the *Thurston measure* on $\mathcal{ML}_{g,n}$. Directly from the definition one can check that μ_{Thu} is $\mathrm{Mod}_{g,n}$-invariant.

The Thurston symplectic form. Let τ be a maximal, recurrent train track on $S_{g,n}$ and $U(\tau) \subseteq \mathcal{ML}_{g,n}$ be the corresponding open subset of measured geodesic laminations. Let $W(\tau)$ be the set of all counting measures on τ. More precisely, $w \in W(\tau)$ is an assignment of (not necessarily positive) real numbers on the edges of the train track τ satifying the switch conditions. We identify the tangent space of $\mathcal{ML}_{g,n}$ at every point in $U(\tau)$ with $W(\tau)$.

Let τ be a generic, maximal, recurrent train track on $S_{g,n}$. Let V be the set of switches of τ. Given a switch $v \in V$, let a_v, b_v, c_v be the edges incident to v. Suppose that a_v is the incoming edge and b_v, c_v are the outgoing edges. Suppose moreover that, with respect to the orientation of $S_{g,n}$, b_v lies to the left and c_v lies to the right of v as we traverse a towards v; see Figure 1. For every pair of tangent vectors $\xi, \eta \in W(\tau)$ we define the pairing

$$\omega^\tau_{\mathrm{Thu}}(\xi,\eta) = \frac{1}{2} \cdot \sum_{v \in V} \det \begin{pmatrix} \xi(c_v) & \xi(b_v) \\ \eta(c_v) & \eta(b_v) \end{pmatrix}. \tag{2.1}$$

The pairing $\omega^\tau_{\mathrm{Thu}}$ on $W(\tau)$ is skew-symmetric, bilinear, and non-degenerate, i.e., it is a symplectic pairing. These pairings glue up to give a $\mathrm{Mod}_{g,n}$-invariant symplectic form ω_{Thu} on the PIL manifold $\mathcal{ML}_{g,n}$; see §3.2 in [**PH92**] for more details. We refer to this symplectic form as the *Thurston symplectic form* of $\mathcal{ML}_{g,n}$. We refer to the top exterior power $v_{\mathrm{Thu}} := \frac{1}{(3g-3+n)!} \bigwedge_{i=1}^{3g-3+n} \omega_{\mathrm{Thu}}$ as the *Thurston volume form* of $\mathcal{ML}_{g,n}$. We refer to the measure ν_{Thu} induced by v_{Thu} on $\mathcal{ML}_{g,n}$ as the *Thurston volume* of $\mathcal{ML}_{g,n}$. Notice that ν_{Thu} is $\mathrm{Mod}_{g,n}$-invariant and belongs to the Lebesgue measure class.

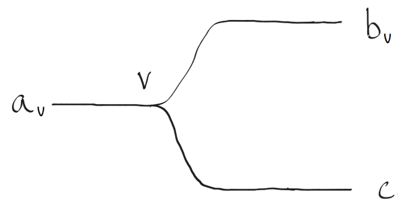


FIGURE 1. A trivalent switch v of a train track τ on $S_{g,n}$.

Comparing counting measures and volume forms. Let W be a finite dimensional real vector space of dimension $N \in \mathbf{N}$. Let $\Lambda \subseteq W$ be a lattice, i.e., a free $\mathbf{Z}$-submodule of W of dimension N. For every $L > 0$ the lattice Λ induces a natural counting measure μ_Λ^L on W given by

$$\mu_\Lambda^L := \frac{1}{L^N} \sum_{x \in \Lambda} \delta_{\frac{1}{L} \cdot x}.$$

As $L \to \infty$, μ_Λ^L converges in the weak-$\star$ topology to a non-zero, locally finite Lebesgue class measure μ_Λ on W.

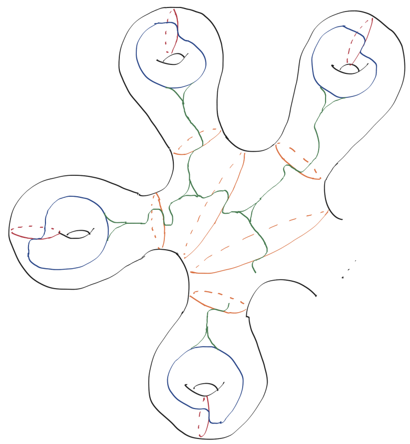
Let $\beta := (x_1, \dots, x_N)$ be an ordered basis of Γ as a free $\mathbf{Z}$-module. Consider the volume form $v_\Lambda := dx_1 \wedge \cdots \wedge dx_k$ on W. The measure induced by the volume form v_Λ on W is precisely μ_Λ. Let v be an arbitrary volume form on W and ν be the measure induced by v on W. As the top exterior power $\bigwedge^N W$ is one-dimensional, v and v_Λ are colinear. Moreover, $v = v(x_1, \dots, x_k) \cdot v_\Lambda$. In particular, $\nu = |v(x_1, \dots, x_k)| \cdot \mu_\Lambda$.

Assume W is even dimensional. Suppose v is the volume form induced by a symplectic form ω on W, i.e., $v = \frac{1}{(N/2)!} \bigwedge_{i=1}^{N/2} \omega$. Let $A := (a_{ij})_{i,j=1}^N$ be the $N \times N$ matrix with entries given by $a_{ij} := \omega(x_i, x_j)$. Then $|v(x_1, \dots, x_N)| = \sqrt{|\det(A)|}$.

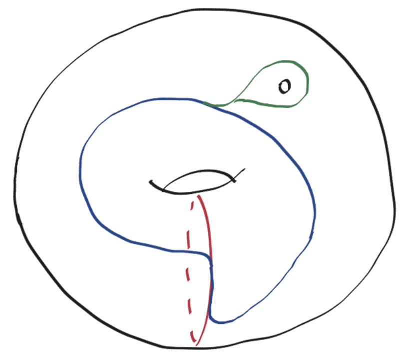
3. Proof of the main result

Normalization of Thurston measures. As explained in §1, μ_{Thu} and ν_{Thu} agree up to a multiplicative factor. To compute such factor it is enough to work on a particular train track chart. Let τ be the generic, maximal, recurrent train track on $S_{g,n}$ described in Figure 2. Let $W(\tau, \mathbf{Z}) \subseteq W(\tau)$ be the $6g - 6 + 2n$ dimensional free $\mathbf{Z}$-submodule (lattice) of integer solutions to the switch conditions of τ.

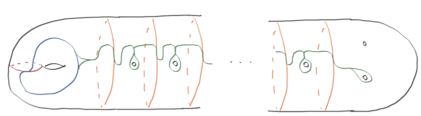
We begin by finding a basis of $W(\tau, \mathbf{Z})$. A natural candidate basis is described in Figure 3. Notice that the simple closed curves described by the red and orange basis elements form a pair of pants decomposition $\mathcal{P}$ of $S_{g,n}$. Exactly g curves in $\mathcal{P}$ are non-separating and $M := 2g - 3 + n$ are separating. Every red basis element r_i has a corresponding blue basis element b_i and every orange basis element o_j has



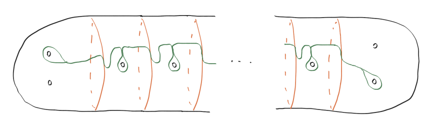
(a) Closed surface of genus ≥ 2.



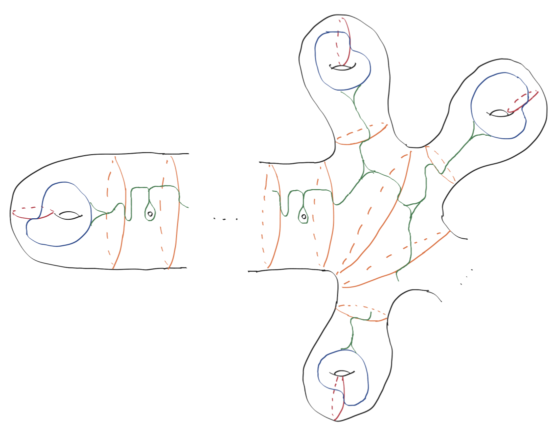
(b) Once punctured torus.



(c) Torus with multiple punctures.

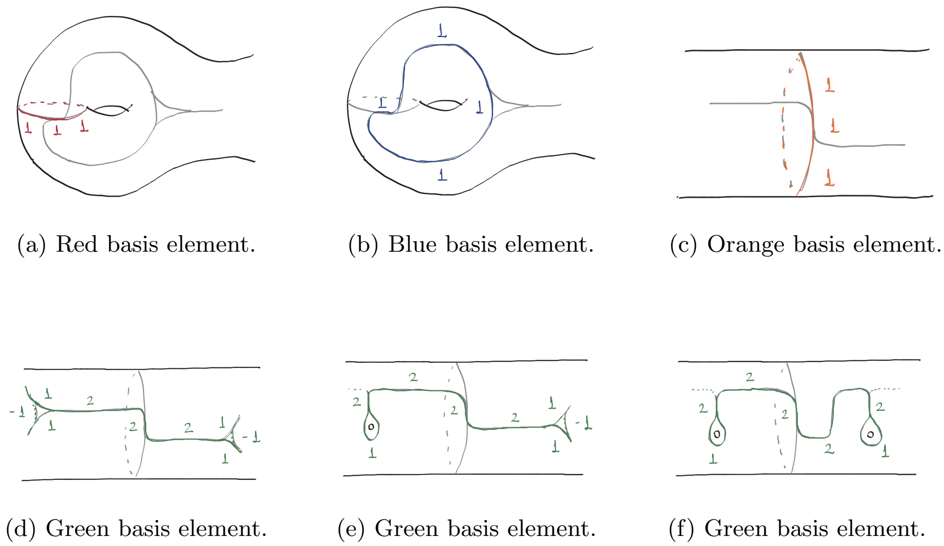


(d) Sphere with multiple punctures.



(e) Punctured surface of genus ≥ 2.

FIGURE 2. The train track τ on $S_{g,n}$ for different values of g and n. The branches of τ are colored to facilitate future references.



(a) Red basis element. (b) Blue basis element. (c) Orange basis element.

(d) Green basis element. (e) Green basis element. (f) Green basis element.

FIGURE 3. Elements of the candidate basis of $W(\tau, \mathbf{Z})$. The color of the branches describes the position of each local configuration in the global pictures presented in Figure 2. Red and blue basis elements as well as orange and green basis elements come in pairs. In total there are g pairs of red and blue basis elements and $2g - 3 + n$ pairs of orange and green basis elements.

a corresponding green basis element g_j. We consider the following ordering of the candidate basis elements:

$$\beta := (r_1, \ldots, r_g, o_1, \ldots, o_M, b_1, \ldots, b_g, g_1, \ldots, g_M). \tag{3.1}$$

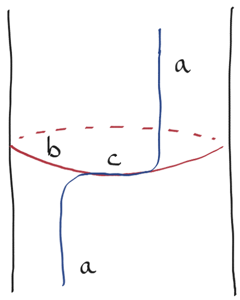
Elements in $W(\tau, \mathbf{Z})$ satisfy certain rigidity properties described Figure 4. Using these properties one can check that the candidate basis β is an actual basis of the free $\mathbf{Z}$-module $W(\tau, \mathbf{Z})$.

Let $N := 6g - 6 + 2n$. Let $A := (a_{ij})_{i,j=1}^N$ be the $N \times N$ matrix with entries given by $a_{ij} := \omega_{\text{Thu}}(\beta_i, \beta_j)$, where β_i denotes the i-th vector of the basis β with respect to the ordering in (3.1). Straightforward computations using (2.1) show

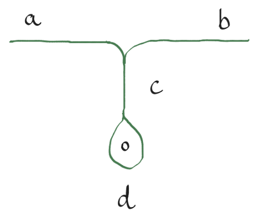
$$A = \left(\begin{array}{cc:cc} 0 & 0 & I_g & 0 \\ 0 & 0 & 0 & 2I_M \\ \hdashline -I_g & 0 & 0 & 0 \\ 0 & -2I_M & 0 & * \end{array} \right).$$

It follows that $|v_{\text{Thu}}(\beta)| = \sqrt{|\det(A)|} = 2^M$. We conclude

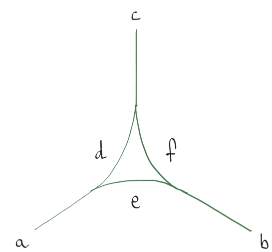
$$\nu_{\text{Thu}} = 2^M \cdot \mu_{\text{Thu}}.$$



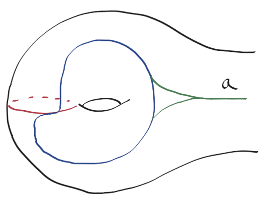
(a) Rigidity property 1.



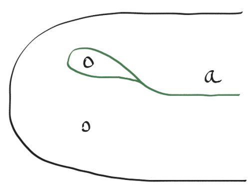
(b) Rigidity property 2.



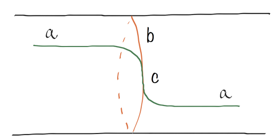
(c) Rigidity property 3.



(d) Rigidity property 4.



(e) Rigidity property 5.



(f) Rigidity property 6.

FIGURE 4. Rigidity properties of elements in $W(\tau, \mathbf{Z})$. The color of the branches describes the position of each local configuration in the global pictures presented in Figure 2. In (a) the parameters a, b can take any value in $\mathbf{Z}$ and completely determine the value of $c \in \mathbf{Z}$. In (b) the parameters a, b can take any value in $\mathbf{Z}$ and completely determine the values of $c \in 2\mathbf{Z}$ and $d \in \mathbf{Z}$. In (c) the parameters a, b, c can take any value in $\mathbf{Z}$ and completely determine the values of $d, e, f \in \mathbf{Z}$. In (d) the parameter a takes values in $2\mathbf{Z}$. In (e) the parameter a takes values in $2\mathbf{Z}$. By induction, the parameter a in (f) takes values in $2\mathbf{Z}$; the parameter b can take any value in $\mathbf{Z}$ and the value of $c \in \mathbf{Z}$ is completely determined by the values of a, b.

References

[Mar16] Bruno Martelli, *An Introduction to Geometric Topology*, ArXiv e-prints (2016).

[Mas85] Howard Masur, *Ergodic actions of the mapping class group*, Proc. Amer. Math. Soc. **94** (1985), no. 3, 455–459. MR 787893

[MT19] Leonid Monin and Vanya Telpukhovskiy, *On normalizations of Thurston measure on the space of measured laminations*, arXiv e-prints (2019), arXiv:1902.04533.

[PH92] R. C. Penner and J. L. Harer, *Combinatorics of train tracks*, Annals of Mathematics Studies, vol. 125, Princeton University Press, Princeton, NJ, 1992. MR 1144770