

## Differential topology notes

Following is what we did each day with references to the bibliography at the end. If I skipped anything, please let me know.

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We defined  $n$  dimensional topological manifold, a topological space  $X$  which is locally homeomorphic to  $\mathbb{R}^n$  and is Hausdorff and has a countable dense set. This implies a number of topological properties,  $X$  is normal, metrizable, paracompact and second countable for example. Actually I'm not so sure of this now, so to be safe I will change our official definition of a topological manifold to be a topological space  $X$  which is locally homeomorphic to  $\mathbb{R}^n$  and is Hausdorff and second countable. The properties we will use most are

- There is a sequence of compact subsets  $K_i \subset X$  so that  $K_i \subset \text{Int}K_{i+1}$  for all  $i$  and  $\bigcup_i K_i = X$ .
- Given any open cover  $U_\alpha$  of  $X$  there are continuous  $\psi_\alpha: X \rightarrow [0, 1]$  so that  $\text{Cl}(\psi_\alpha^{-1}((0, 1])) \subset U_\alpha$  for all  $\alpha$  and so that every point of  $X$  has a neighborhood  $O$  so that only finitely many  $\alpha$  are nonzero at any point of  $O$ , and so that  $\sum_\alpha \psi_\alpha(x) = 1$  for all  $x \in X$ . We call  $\{\psi_\alpha\}$  a partition of unity for the cover  $\{U_\alpha\}$ . (Note this is a little different from the version given in [B, 36], which is gratuitously more complicated.)

We gave two definitions of  $n$  dimensional smooth manifold, an  $n$  dimensional topological manifold  $X$  together with some extra data which allows us to do calculus. We call this extra data a differentiable structure on  $X$ . Two (equivalent) forms this extra data could take were:

- A list of differentiable real valued functions (which we left vague and did not develop).
- A compatible atlas of charts.

A chart is a homeomorphism  $\psi: V \rightarrow U$  where  $V \subset X$  is open and  $U \subset \mathbb{R}^n$  is open. A parameterization is the inverse of a chart, a homeomorphism  $\phi: U \rightarrow V$ . A compatible atlas of charts is a family of charts  $\psi_\alpha: V_\alpha \rightarrow U_\alpha$  so that  $\bigcup_\alpha V_\alpha = X$  and the restriction of each  $\psi_\beta \psi_\alpha^{-1}$  to  $\psi_\alpha(V_\alpha \cap V_\beta)$  is smooth. (Here and in the rest of this course, smooth means infinitely differentiable.)

We ended with a list of examples. A zero dimensional smooth manifold is just a countable set of points with the discrete topology. A one dimensional manifold is a countable disjoint union of circles and lines. We gave some examples of two dimensional manifolds, any open subset of  $\mathbb{R}^2$ , and the sphere  $S^2 = \{x \in \mathbb{R}^3 \mid \|x\| = 1\}$ . An atlas consisting of two charts is given by stereographic projection from the north and south poles. In fact if  $S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$  there is a compatible atlas of two charts  $\psi_0: S^n - (0, 0, \dots, 0, 1) \rightarrow \mathbb{R}^n$  and  $\psi_1: S^n - (0, 0, \dots, 0, -1) \rightarrow \mathbb{R}^n$  where  $\psi_0(x_1, \dots, x_{n+1}) = (x_1, \dots, x_n)/(1 - x_{n+1})$  and  $\psi_1(x_1, \dots, x_{n+1}) = (x_1, \dots, x_n)/(1 + x_{n+1})$ . You can check that  $\psi_i \psi_j^{-1}$  is always smooth where it is defined so this is a compatible atlas. Note that the graph of a continuous nowhere differentiable function from  $\mathbb{R}$  to  $\mathbb{R}$  has the structure of a smooth manifold (you only need one chart so it must be compatible). However when we discuss submanifolds we will see that it is not a submanifold of  $\mathbb{R}^2$ . References: [B, 68-72], [H, 7-14], [BJ, 1-10], [Mu, 225-227]. I also looked at some of these topics on Wikipedia and it appeared to be reliable.

For the record here is how we could alternatively develop a differentiable structure on  $X$  by giving a list of differentiable functions. For each open  $U \subset X$  we give a list  $\Gamma(U)$  of the real valued functions on  $U$  which we declare to be differentiable.  $\Gamma(U)$  should be closed under addition and multiplication and should contain all constant functions. If  $V \subset U$  and  $f \in \Gamma(U)$  then the restriction  $f|_V \in \Gamma(V)$ . If  $f \in \Gamma(U)$  and  $g \in \Gamma(W)$  and  $f|_{U \cap W} = g|_{U \cap W}$  then the combined function  $f \cup g: U \cup W \rightarrow \mathbb{R}$  is in  $\Gamma(U \cup W)$ . Finally, every point in  $X$  has a neighborhood  $V$  and a chart  $\psi: V \rightarrow U \subset \mathbb{R}^n$  so that for every open  $W \subset V$ , we have  $\Gamma(W) = \{f: W \rightarrow \mathbb{R} \mid f \psi^{-1} \text{ is smooth at every point of } \psi(W)\}$ . In fancy language,  $\Gamma$  is a subsheaf of the sheaf of continuous functions which is locally isomorphic to the sheaf of smooth functions on  $\mathbb{R}^n$ . To go back and forth between the two notions of differentiable structure, note that as an exercise you can prove that the charts referred to above form a compatible atlas. Conversely, if you have a compatible atlas  $\{\psi_\alpha: V_\alpha \rightarrow U_\alpha\}$  you can define  $\Gamma(W) = \{f: W \rightarrow \mathbb{R} \mid f \psi_\alpha^{-1} \text{ is smooth at each point of } \psi_\alpha(W \cap V_\alpha) \text{ for all } \alpha\}$ .

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From now on, unless otherwise stated, all manifolds will be smooth and if we take a chart in a manifold  $X$  it will be assumed without stating explicitly that the chart is a compatible chart in the given smooth structure on  $X$ .

A map  $f: X \rightarrow Y$  between manifolds is smooth if it is smooth in local coordinates, in other words, given any  $x \in X$  there is a chart  $\psi: U \rightarrow V \subset \mathbb{R}^n$  with  $x \in U$  and a chart  $\psi': U' \rightarrow V' \subset \mathbb{R}^m$  so that  $\psi' f \psi^{-1}$  is smooth at  $x$ . Note that it does not matter which charts we pick to check smoothness. We say  $f$  is a diffeomorphism if  $f$  is smooth,  $f^{-1}$  exists, and  $f^{-1}$  is smooth. We say  $X$  and  $Y$  are diffeomorphic if there is a diffeomorphism from  $X$  to  $Y$ . In general we will consider diffeomorphic manifolds to be the same, although of course they are technically different. We showed that diffeomorphic manifolds have the same dimension, which ended up being a linear algebra statement. This will be clearer once we have the notion of tangent space. The idea of the proof is to look in local coordinates (i.e. compose with charts). If  $f: U \rightarrow V$  is a diffeomorphism between an open subset  $U \subset \mathbb{R}^n$  and an open subset  $V \subset \mathbb{R}^m$  and if  $L$  is the matrix of partial derivatives of  $f$  and  $K$  is the matrix of partial derivatives of  $f^{-1}$  then the chain rule says  $KL = \text{identity}$  and  $LK = \text{identity}$ . So  $n = m$  and  $K = L^{-1}$ .

Finally, we looked at some ways of generating manifolds:

- If  $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$  is a smooth map, then for almost all points  $c \in \mathbb{R}^m$ , the inverse image  $f^{-1}(c)$  is a manifold of dimension  $n - m$ . I.e., the solutions to a random set of smooth equations in  $\mathbb{R}^n$  is a submanifold. This is not obvious but we'll see why later.
- Any open subset of a smooth manifold is naturally a smooth manifold. (just restrict charts)
- The cartesian product of two manifolds is a manifold. (take products of charts)
- If you glue two manifolds along diffeomorphic open subsets and the result is Hausdorff, then you have a smooth manifold. (cover by charts in one or the other manifold)
- The connected sum of two manifolds is a manifold. Note: we will show later that there are at most two possibilities for the connected sum of two connected manifolds, despite the choices involved in the construction. We did not quite finish defining the connected sum.

Some references [H, 15-16], [BJ, 6-9], [B, 70-72].

**Exercise 1:** Fill in the details of why the cartesian product of two smooth manifolds  $X$  and  $Y$  has a natural smooth structure. Show that the projections  $X \times Y \rightarrow X$  and  $X \times Y \rightarrow Y$  are smooth. Suppose that  $X$  and  $M$  are diffeomorphic and  $Y$  and  $N$  are diffeomorphic. Show that  $X \times Y$  is diffeomorphic to  $M \times N$ .

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We finished talking about the connected sum. In the process we gave most of a proof that if  $M$  is a connected manifold and  $p$  and  $q$  are any two points of  $M$ , then there is a diffeomorphism  $h: M \rightarrow M$  so that  $h(p) = q$ . Thus there are no "special" points in a connected manifold, any two points look the same. A proof is in [H1, 22-24] and I also give a proof below. We also defined submanifold (not as in [Br, 79], but as in [BJ, 9], [H, 13], c.f [Br, 82-83]).

**No distinguished points in a manifold:** Just to complete the discussion on the existence of the diffeomorphism  $h$  above, Let  $g: \mathbb{R} \rightarrow \mathbb{R}$  be a smooth function so that  $g(t) = 0$  for all  $t \geq 1$ ,  $g(0) = 1$ , and  $g([0, 1]) = [0, 1]$ . For example  $g(t) = e^{t/(t-1)}$  for  $t < 1$ . For  $u \in \mathbb{R}^n$  define  $f_u: \mathbb{R}^n \rightarrow \mathbb{R}^n$  by  $f_u(x) = x + g(|x|^2)u$ . Then if  $u$  is small enough,  $f_u$  will be a diffeomorphism, and furthermore  $f_u(x) = x$  if  $|x| \geq 1$ . (To see that  $f_u$  is one to one, note that if  $f_u(x) = f_u(y)$  then  $y = x + tu$  for  $t = g(|x|^2) - g(|x|^2 + t^2|u|^2 + 2tx \cdot u)$  so  $|t| = |g(|x|^2) - g(|x|^2 + t^2|u|^2 + 2tx \cdot u)| \leq K|t||u|(2 + |t||u|) \leq 4K|t||u|$  if  $K = \max_{0 \leq s \leq 1} g'(s)$  and say  $|u| < .5$ . So if  $|u| < 1/(4K)$  then  $f_u$  is one to one. The Jacobian matrix of  $f_u$  is  $I + 2g'ux^T$ . This is nonsingular with inverse  $I - 2g'ux^T/(1 + 2g'x \cdot u)$  if  $|u| < 1/(2K)$ . So by the inverse function theorem,  $f_u$  is a local diffeomorphism. So  $f_u$  is a diffeomorphism. (Missing: a proof that  $f_u$  is onto which I leave to you.) Note that  $f_u(0) = u$ . So if  $M$  is a manifold and  $p \in M$  and  $q$  is close enough to  $p$ , then we can find a diffeomorphism  $h: M \rightarrow M$  so that  $h(p) = q$ . Just take a chart  $\phi: U \rightarrow \mathbb{R}^n$  and after perhaps translating and scaling, assume that  $\phi(p) = 0$  and  $\phi(U)$  contains the disc of radius 2. Then for  $q$  close enough to  $p$ ,  $\phi(q)$  is small enough that  $f_{\phi(q)}$  is a diffeomorphism. so define  $h$  by  $h(x) = x$  if  $x \notin U$  and  $h(x) = \phi^{-1} f_{\phi(q)} \phi(x)$  if  $x \in U$ . Note that  $h$  and  $h^{-1}$  are smooth and  $h(p) = q$ . Define an equivalence relation on  $M$  where  $p \sim q$  if there is a diffeomorphism  $h: M \rightarrow M$  so that  $h(p) = q$ . Then we showed above that equivalence classes are open sets. But the compliment of an equivalence class is the union of all the other equivalence classes which is open. So equivalence classes are both open and closed. Thus if  $M$  is connected there is exactly one equivalence class so for any  $p$  and  $q$  there is a diffeomorphism  $h$  with  $h(p) = q$ .

**Exercise 1:** Prove that the map  $f_u$  above is onto.

**Submanifolds:** If  $X$  is a smooth manifold and  $Y \subset X$  we say  $Y$  is a smooth submanifold of  $X$  if for each point  $p \in Y$  there is a chart  $\phi: U \rightarrow \mathbb{R}^n$  and a linear subspace  $L \subset \mathbb{R}^n$  so that  $U$  is a neighborhood of  $p$  in  $X$  and  $\phi^{-1}(L) = U \cap Y$ . For example  $S^n = \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}$  is a submanifold of  $\mathbb{R}^{n+1}$ . If  $g: \mathbb{R}^n \rightarrow \mathbb{R}^m$  is smooth then the graph of  $g$  is a smooth submanifold of  $\mathbb{R}^{n+m}$ . The implicit function theorem [Br, 65-66] implies that if the  $k \times n$  matrix of partial derivatives of  $f: \mathbb{R}^n \rightarrow \mathbb{R}^k$  has rank  $k$  at  $x$  for every  $x \in f^{-1}(c)$ , then  $f^{-1}(c)$  is a  $n - k$  dimensional submanifold of  $\mathbb{R}^n$ .

**Exercise 2:** Suppose  $g: U \rightarrow \mathbb{R}^m$  is smooth where  $U \subset \mathbb{R}^n$  is open. Show that its graph  $\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m \mid x \in U \text{ and } y = g(x)\}$  is a submanifold of  $\mathbb{R}^n \times \mathbb{R}^m$ .

**Exercise 3:** Explicitly describe a submanifold  $X$  of  $\mathbb{R}^2$  such that  $X$  is a closed subset of  $\mathbb{R}^2$ ,  $X \cap \{(x, y) \mid x^2 + y^2 \geq 1\} = \{(x, 0) \mid |x| \geq 1\}$ , but  $X$  is not the  $x$  axis.

### 1/31

We started with more examples of smooth manifolds:

- If  $X$  is a manifold and  $G$  is a group of self-diffeomorphisms of  $X$ , then sometimes  $X/G$  is a smooth manifold. For example this is true if each point of  $X$  has a neighborhood  $U$  so  $U \cap g(U)$  is empty for each non-identity  $g \in G$ . Examples,  $\mathbb{R}P^n = S^n/G$  where  $G$  is the 2 group generated by the antipodal map.  $L(p, q)$  the lens space  $S^3/G$  where  $G$  is the cyclic group of order  $p$  generated by  $(z, w) \mapsto (ze^{2\pi i/p}, we^{2q\pi i/p})$ , where  $p$  and  $q$  are relatively prime and  $z, w$  are complex with  $|z|^2 + |w|^2 = 1$ . Another example is  $\mathbb{C}^{n+1} - 0/G = \mathbb{C}P^n$  where  $G$  is the diffeomorphisms of the form  $(z_0, z_1, \dots, z_n) \mapsto (\omega z_0, \omega z_1, \dots, \omega z_n)$  where  $\omega \in \mathbb{C} - 0$ . A nonexample is  $S^3/G$  where  $G$  is the group of order 2 generated by  $(x, y, z, w) \mapsto (-x, -y, -z, w)$ . In this case a nbhd of the north or south pole  $(0, 0, 0, \pm 1)$  is homeomorphic to the cone on  $\mathbb{R}P^2$  and hence is not a manifold. A useful generalization of the notion of manifold is an orbifold, where such singularities are allowed.
- Many natural spaces in mathematics end up being manifolds. For example,  $O(n)$  the space of orthogonal  $n \times n$  matrices is a smooth submanifold of  $\mathbb{R}^{n^2}$  with dimension  $n(n-1)/2$ . The Grassmanian  $G(n, k)$  of  $k$  planes in  $n$  space is a smooth manifold. One way to describe  $G(n, k)$  is as the set of  $n \times n$  matrices  $P$  so  $P = P^T$ ,  $P^2 = P$ , and  $\text{trace} P = k$ , here  $P$  is the matrix of orthogonal projection to a  $k$  dimensional linear subspace. Another way is the quotient.  $O(n)/O(k) \times O(n-k)$  where we quotient out by matrices of the form  $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$  with  $A \in O(k)$  and  $B \in O(n-k)$ .

**Tangent space of  $U \subset \mathbb{R}^n$ :** We started looking at the tangent space to a smooth manifold, first just looking at an open subset  $U$  of  $\mathbb{R}^n$ . We had three different points of view of tangent vectors to  $U$ . I'll give them names bestowed in [BJ, 14-20].

- (Physicist, POV1) A tangent vector is a point in  $U \times \mathbb{R}^n$ . A tangent vector at  $p$  is a point in  $p \times \mathbb{R}^n$ . We give each  $p \times \mathbb{R}^n$  a vector space structure, the usual structure on  $\mathbb{R}^n$ .
- (Geometer, POV2) A tangent vector is the velocity vector of some parameterized curve  $\alpha: \mathbb{R} \rightarrow U$ . If  $\alpha(t_0) = p$  this corresponds to  $(p, \alpha'(t_0))$  in POV1. Technically we should define an equivalence relation on such curves to distinguish when two curves give the same tangent vector, but we refrain from doing so at this point. An advantage of POV2 is that it extends immediately to a general manifold. A disadvantage is that the vector space structure is not clear.
- (Algebraist, POV3) Let  $C^\infty(U, \mathbb{R})$  denote the vector space of real valued smooth functions on  $U$ . Let  $C^\infty(U, \mathbb{R})^*$  denote its dual, the vector space of linear transformations of  $C^\infty(U, \mathbb{R})$  to  $\mathbb{R}$ . We say a derivation at a point  $p \in U$  is a  $D \in C^\infty(U, \mathbb{R})^*$  which obeys the product rule  $D(fg) = f(p)Dg + g(p)Df$  for all  $f, g \in C^\infty(U, \mathbb{R})$ . The derivations are a subspace of  $C^\infty(U, \mathbb{R})^*$ . It turns out a basis for this subspace is the partial derivative  $\partial/\partial x_i$ . (I'll show this below). Then we may write any derivation  $D$  at  $p$  as  $D(f) = \sum_{i=1}^n c_i \partial f / \partial x_i|_{x=p}$ . We identify  $D$  with the POV1 point  $(p, (c_1, c_2, \dots, c_n))$ . This POV3 has the advantage that it immediately extends to a general manifold. The vector space structure is also apparent. But it is a more complicated object than POV2.

We already showed how to go from POV2 to POV1. We go from POV1 to POV2 by for example identifying  $(p, x)$  with the velocity vector of the parameterized curve  $\alpha(t) = p + tx$  at  $t = 0$ . We go from POV2 to POV3 by identifying the tangent vector of the curve  $\alpha$  at  $t_0$  with the derivation  $D_\alpha(f) = d(f(\alpha(t)))/dt|_{t=t_0}$ . We can now say what the equivalence relation in POV2 is, two curves give the same tangent vector if they correspond to the same derivation.

We delayed showing that the partial derivatives give a basis for the derivations. First note that if  $D$  is a derivation then  $D(1) = D(1 \cdot 1) = 1 \cdot D(1) + 1 \cdot D(1) = 2D(1)$  so  $D(1) = 0$  and hence by linearity,  $D(c) = 0$  for any constant function  $c$ . Next note that if  $f: U \rightarrow \mathbb{R}$  is smooth and  $p \in U$  then there are smooth  $f_i: U \rightarrow \mathbb{R}$  so that  $f(x) = f(p) + \sum_{i=1}^n (x_i - p_i) f_i(x)$  and  $f_i(p) = \partial f / \partial x_i |_{x=p}$ . (Note: Although this result is true, the following argument requires  $U$  to be star shaped from  $p$ , i.e., for each  $x \in U$  the line segment from  $p$  to  $x$  is contained in  $U$ .) Just let  $f_i(x) = \int_0^1 \partial f / \partial x_i |_{p+t(x-p)} dt$ . By the chain rule and fundamental thm of calculus,

$$f(x) - f(p) = \int_0^1 df(p + t(x-p))/dt dt = \int_0^1 \sum_{i=1}^n (x_i - p_i) \partial f / \partial x_i |_{p+t(x-p)} dt = \sum_{i=1}^n (x_i - p_i) f_i(x)$$

So if  $D$  is a derivation we have

$$\begin{aligned} D(f) &= D(f(p) + \sum_{i=1}^n (x_i - p_i) f_i(x)) = D(f(p)) + \sum_{i=1}^n D((x_i - p_i) f_i(x)) \\ &= 0 + \sum_{i=1}^n (p_i - p_i) Df_i + f_i(p) D(x_i - p_i) = \sum_{i=1}^n f_i(p) D(x_i) = \sum_{i=1}^n D(x_i) \partial f / \partial x_i |_{x=p} \end{aligned}$$

Thus  $D = \sum_{i=1}^n c_i \partial / \partial x_i$  where  $c_i = D(x_i)$ .

**Exercise 1:** We showed correspondences  $\text{POV1} \rightarrow \text{POV2} \rightarrow \text{POV3} \rightarrow \text{POV1}$ . Show that the composition is the identity.

**Exercise 2:** Show that  $\mathbb{R}P^n$  can be covered by  $n+1$  compatible charts  $\phi_i: U_i \rightarrow \mathbb{R}^n$  and explicitly compute  $\phi_i \phi_j^{-1}$ . Maybe start with  $n=2$  to get the idea. These charts are useful in algebraic geometry.

**Exercise 3:** Let  $\mathbb{Z}^2$  act on  $\mathbb{R}^2$  by letting  $(m, n)$  take  $(x, y)$  to  $(x+m, y+n)$ . Show that  $\mathbb{R}^2/\mathbb{Z}^2$  has the structure of a smooth manifold so that the quotient map  $\mathbb{R}^2 \rightarrow \mathbb{R}^2/\mathbb{Z}^2$  is a local diffeomorphism, i.e., each point of  $\mathbb{R}^2$  has a neighborhood which is mapped diffeomorphically to an open subset of  $\mathbb{R}^2/\mathbb{Z}^2$ . What familiar manifold is  $\mathbb{R}^2/\mathbb{Z}^2$ ?

## 2/2

**Tangent space of a manifold:** Much of what I did was written up above in 1/31. Then we looked at how POV1 can be adapted to a general smooth manifold  $X$ . If  $\psi_i: V_i \rightarrow U_i \subset \mathbb{R}^n$ ,  $i=0,1$  are two charts, we have that the vectors based at  $V_i$  are identified with  $U_i \times \mathbb{R}^n \approx V_i \times \mathbb{R}^n$ . So we have two ways of identifying the vectors based at  $V_0 \cap V_1$  as  $(V_0 \cap V_1) \times \mathbb{R}^n$  and we need to reconcile the two. Using POV2, we take a parameterized curve  $\alpha: \mathbb{R} \rightarrow V_0 \cap V_1$  and let  $p = \alpha(0)$ . Then we want to identify  $(p, (\psi_0 \alpha)'(0))$  in  $V_0 \times \mathbb{R}^n$  with  $(p, (\psi_1 \alpha)'(0))$  in  $V_1 \times \mathbb{R}^n$ . But  $\psi_1 \alpha = (\psi_1 \psi_0^{-1}) \circ \psi_0 \alpha$  so by the chain rule we have  $(\psi_1 \alpha)'(0) = J_p(\psi_0 \alpha)'(0)$  where  $J_p$  is the matrix of partial derivatives of  $\psi_1 \psi_0^{-1}$ , evaluated at  $\psi_0(p)$ . Let me do the example of the  $n$  sphere  $S^n$ . On 1/24 we described the usual smooth structure on  $S^n$  by two charts given by stereographic projection. Solving, we get  $\psi_0^{-1}(y) = (2y, |y|^2 - 1)/(|y|^2 + 1)$  and then  $\psi_1 \psi_0^{-1}(y) = y/|y|^2$ . We calculate  $J = |y|^{-2} I - 2|y|^{-4} y y^T$  (where we think of  $y$  as a column vector). Let us now specialize to  $S^2$ . Rather than think of gluing  $\mathbb{R}^2$  to  $\mathbb{R}^2$  to get  $S^2$  we will just think of gluing the upper and lower hemispheres. So the intersection where we do the gluing has  $|y| = 1$ . We then have  $J = I - 2y y^T$ . If we let  $y = (\cos \theta, \sin \theta)^T$  then

$$J = \begin{bmatrix} 1 - 2 \cos^2 \theta & -2 \sin \theta \cos \theta \\ -2 \sin \theta \cos \theta & 1 - 2 \sin^2 \theta \end{bmatrix} = \begin{bmatrix} \cos(2\theta) & -\sin(2\theta) \\ \sin(2\theta) & \cos(2\theta) \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

which means  $J$  is rotation by  $2\theta$  times a fixed matrix. So I was not quite right in class. I should have known there was a 2 involved since  $S^2$  has Euler characteristic 2. We also showed that the tangent bundle of  $S^1$  is trivial,  $S^1 \times \mathbb{R}$ . We could look at  $S^1$  as a submanifold of  $\mathbb{R}^2$ ,  $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ . Then  $TS^1 = \{((x, y), (s, t)) \mid x^2 + y^2 = 1, xs + ty = 0\}$ . We have an isomorphism  $S^1 \times \mathbb{R} \rightarrow TS^1$  given by  $((x, y), u) \mapsto ((x, y), (uy, -ux))$ . We made the notation  $TX$  mean the tangent bundle to  $X$ , the union of tangent vectors at all points of  $X$ . We have a map  $\pi: TX \rightarrow X$  which takes a tangent vector to the point it is based at. We let  $T_p X = \pi^{-1}(p)$  = the vector space of all tangent vectors at  $p$ . Your assignment for next time is to think of a natural map  $df: TX \rightarrow TY$  if  $f: X \rightarrow Y$  is smooth.

**Vector bundles:** A real rank  $n$  vector bundle is a map  $\pi: E \rightarrow B$  so that

- For each  $x \in B$ , the fibre  $\pi^{-1}(x)$  is given the structure of an  $n$  dimensional real vector space.
- Each  $x \in B$  has a neighborhood  $U$  in  $B$  and a bundle chart  $f: \pi^{-1}(U) \rightarrow U \times \mathbb{R}^n$ , that is, a homeomorphism  $f$  so that if  $\pi_1: U \times \mathbb{R}^n \rightarrow U$  and  $\pi_2: U \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  are projections, then  $\pi_1 f = \pi$  and for each  $y \in U$ ,  $\pi_2 f: \pi^{-1}(y) \rightarrow \mathbb{R}^n$  is an isomorphism of vector spaces.

We call  $E$  the total space,  $B$  the base space, and  $\pi$  the bundle projection.

Note that if  $f: \pi^{-1}(U) \rightarrow U \times \mathbb{R}^n$  and  $g: \pi^{-1}(V) \rightarrow V \times \mathbb{R}^n$  are two bundle charts, then for each  $y \in U \cap V$ , the map  $\mathbb{R}^n \rightarrow \mathbb{R}^n$  given by  $w \mapsto \pi_2 f g^{-1}(y, w)$  is a linear isomorphism, i.e., multiplication by a nonsingular matrix  $\phi(y)$ . Thus we get a (continuous) transition map  $\phi: U \cap V \rightarrow GL_n$  where  $GL_n$  is the space of  $n \times n$  nonsingular matrices. If  $B$  is a smooth manifold and these maps  $\phi$  are all smooth then we say the vector bundle is smooth.

We then looked at examples of vector bundles with base  $S^1$ , and stated without proof that these are all examples. If  $A \in GL_n$  let  $E_A = [0, 1] \times \mathbb{R}^n / (0, y) \sim (1, Ay)$ . The bundle projection  $\pi_A: E_A \rightarrow S^1$  is given by  $\pi_A(t, y) = e^{2\pi i t}$  where we think of  $S^1$  as the unit complex numbers. If  $\phi: [0, 1] \rightarrow GL_n$  is a continuous map with  $\phi(0) = I$  and  $\phi(1) = AB^{-1}$  then we get a map  $h: E_B \rightarrow E_A$  defined by  $h(t, y) = (t, \phi(t)y)$  with inverse  $h^{-1}(t, y) = (t, \phi(t)^{-1}y)$ . (This map is well defined and continuous since  $h(1, By) = (1, \phi(1)By) = (1, Ay) = (0, y) = h(0, y)$ .)

Finally we showed that  $GL_n$  has just two components, the matrices with positive determinant and the matrices with negative determinant. To see this, first of all we have a deformation retraction  $GL_n \rightarrow O(n) =$  the group of orthogonal matrices given as follows. Any  $A \in GL_n$  has a unique decomposition  $A = QR$  where  $Q \in O(n)$  and  $R$  is upper triangular with positive diagonal entries. ( $Q$  is obtained by the Gram-Schmidt process on the columns of  $A$ .) Then the map  $(QR, t) \mapsto Q(tI + (1-t)R)$  gives a deformation retraction of  $GL_n$  to  $O(n)$ . Now for any  $Q$  in  $O(n)$  with positive determinant there is another  $P \in O(n)$  so that  $PQP^{-1}$  is the matrix taking  $(x_1, \dots, x_n)^T$  to  $(R_{\theta_1}(x_1, x_2), R_{\theta_2}(x_3, x_4), \dots, R_{\theta_k}(x_{2k-1}, x_{2k}), x_n)$  where  $n = 2k$  or  $2k+1$ ,  $R_\theta$  is rotation in  $\mathbb{R}^2$  by an angle  $\theta$ , and the last entry is there only if  $n$  is odd. Then a path from  $I$  to  $Q$  is  $P^{-1}(R_{t\theta_1}(x_1, x_2), R_{t\theta_2}(x_3, x_4), \dots, R_{t\theta_k}(x_{2k-1}, x_{2k}), x_n)P$  for  $t \in [0, 1]$ . There are other ways to show this connectedness, I'm not sure of references. Anyway, the upshot is that for each  $n$  we only get at most two different bundles this way,  $E_I = S^1 \times \mathbb{R}^n$  and  $E_A$  for any  $n \times n$  matrix with negative determinant. In fact for each  $n$  there are exactly two different rank  $n$  bundles over  $S^1$ , given by the two bundles above. We'll see this later.

**Exercise 1:** Show that the total space of a smooth vector bundle is a smooth manifold whose charts are given by some bundle charts. What is its dimension?

**Exercise 2:** Show that  $\pi_A: E_A \rightarrow S^1$  is a smooth vector bundle. The only problem of course is of course when  $t = 0, 1$ .

**Exercise 3:** If  $\det(A) < 0$ , show that  $E_A = M \times \mathbb{R}^{n-1}$  where  $M$  is the Möbius band  $M = [0, 1] \times \mathbb{R} / (0, y) = (1, -y)$ .

It is a common shorthand to identify a bundle by its total space. Thus a bundle  $\pi: E \rightarrow B$  will often be referred to as “the bundle  $E$ ”. Of course there is more to a vector bundle than just the topological space  $E$ . There is also the decomposition of  $E$  into a union of vector spaces. But this is all the extra stuff there is, since  $B$  is homeomorphic to the subspace of  $E$  consisting of all the 0 vectors in these subspaces. In fact it is sometimes useful to identify  $B$  with this set of 0 vectors.

**Random fact:** Any vector bundle over a contractible paracompact space is trivial [H, 97].

**Bundle morphisms:** If  $\pi: E \rightarrow B$  and  $\rho: F \rightarrow C$  are vector bundles then then a bundle morphism is a pair of maps  $f: C \rightarrow B$  and  $g: F \rightarrow E$  so that  $\pi g = f \rho$  and so that for each  $x \in C$ , the restriction  $h|: \rho^{-1}(x) \rightarrow \pi^{-1}(f(x))$  is a linear transformation. We might also say that  $g$  is a bundle morphism, omitting the reference to  $f$  since  $f$  is determined by  $g$  anyway. There is no general agreement on the name “bundle morphism”, it is used in [H, 88], but is called a linear map in [BJ, 27]. Unfortunately, in my view, the name bundle map has come to mean a bundle morphism which is an isomorphism on each fibre. We must live with

it. If  $f$  and  $g$  are homeomorphisms, then the restriction of  $g$  to fibers is necessarily a linear isomorphism so  $g^{-1}$  is also a bundle morphism and we say  $g$  is a bundle isomorphism.

A special case of a bundle morphism is a subbundle. Suppose  $\pi: E \rightarrow B$  is a bundle and  $F \subset E$ . then we say that  $F$  is a (rank  $k$ ) subbundle if:

- a) Each  $F \cap \pi^{-1}(x)$  is a  $k$  dimensional linear subspace of  $\pi^{-1}(x)$ .
- b) These subspace vary continuously. That is, for each  $x \in B$  there is a neighborhood  $U$  of  $x$  and a bundle chart  $g: \pi^{-1}(U) \rightarrow U \times \mathbb{R}^n$  so that  $g(F \cap \pi^{-1}(U)) = U \times L$  for some  $k$  dimensional subspace  $L$  of  $\mathbb{R}^n$ . Equivalently, for every bundle chart  $g$  of  $E$  the map  $U \rightarrow G(n, k)$  the Grassmanian of  $k$  planes in  $n$  space is continuous, (the map being  $x \mapsto \pi_2 g(F \cap \pi^{-1}(x))$ ).

As an example, if  $S^2$  is the unit sphere in  $\mathbb{R}^3$ , then  $T(S^2) = \{(x, y) \in S^2 \times \mathbb{R}^3 \mid x \cdot y = 0\}$  is a subbundle of  $S^2 \times \mathbb{R}^3$ . More generally, let  $X$  be a smooth submanifold of  $\mathbb{R}^m$ . Then  $T(X)$  is a subbundle of the trivial bundle  $X \times \mathbb{R}^m$ ,  $T(X) = \{(x, y) \in X \times \mathbb{R}^m \mid y \text{ is tangent to } X \text{ at } x\}$ .

**The morphism  $df$  between tangent bundles:** If  $f: X \rightarrow Y$  is a smooth map we get a bundle morphism  $df: T(X) \rightarrow T(Y)$  defined in a natural way. We also denote  $f_* = Tf = df$ . The easiest point of view is POV2 where a vector  $v$  in  $T(X)$  is the velocity vector of a parameterized curve  $\alpha(t)$  at  $t_0$ . Then  $df(v)$  is the velocity of  $f\alpha(t)$  at  $t_0$ . In POV3,  $v$  is represented by a derivation  $D$  at a point  $p \in X$ . Then  $df(v)$  is represented by the derivation  $E$  at  $f(p)$  with  $E(g) = D(g \circ f)$ . In POV1 we have charts  $\phi: U \rightarrow \mathbb{R}^n$  and  $\psi: V \rightarrow \mathbb{R}^k$  where  $U$  is a neighborhood of  $p$  and  $V$  is a neighborhood of  $f(p)$ . Then  $v$  is represented by a vector  $w \in \mathbb{R}^n$  and  $df(v)$  is represented by  $Jw \in \mathbb{R}^k$  where  $J$  is the Jacobian matrix of first partials of  $\psi f \phi^{-1}$  evaluated at  $\phi(p)$ .

**The induced bundle:** If  $\pi: E \rightarrow B$  is a vector bundle and  $f: C \rightarrow B$  is continuous, there is a natural induced vector bundle  $\rho: f^*(E) \rightarrow C$  where  $f^*(E) = \{(x, y) \in C \times E \mid f(x) = \pi(y)\}$ . The projection  $\rho$  is  $\rho(x, y) = x$ . Note that for each  $c \in C$ , the fibre  $\rho^{-1}(c) = c \times \pi^{-1}(f(c))$  has the structure of a vector space. If  $g: \pi^{-1}(U) \rightarrow U \times \mathbb{R}^n$  is a bundle chart for  $E$ , we get a bundle chart for  $f^*(E)$  given by  $h: \rho^{-1}(f^{-1}(U)) \rightarrow f^{-1}(U) \times \mathbb{R}^n$  with  $h(x, y) = (x, \pi_2 g(y))$  and  $h^{-1}(x, z) = (x, g^{-1}(f(x), z))$ . Note that if  $B$  and  $C$  are smooth manifolds,  $f$  is smooth, and  $E$  is a smooth vector bundle, then  $f^*(E) \rightarrow C$  will also be a smooth vector bundle.

A special case is where  $C \subset B$ . Then the inclusion induces a bundle which we denote  $E|_C$ , the total space is  $\pi^{-1}(C)$  and the bundle map is the restriction of  $\pi$ .

**Direct sum of vector bundles:** In general if there is a mathematical operation we can perform on vector spaces then there is a corresponding operation on vector bundles. The operations we will use are: direct sum, quotient, the dual space, and the space of alternating  $k$  forms. Right now we will talk about the direct sum and the quotient and leave the others until later.

If  $V$  and  $W$  are vector spaces, then we may form a vector space  $V \oplus W$  called the direct sum as follows. A vector in  $V \oplus W$  is an ordered pair  $(v, w)$  where  $v \in V$  and  $w \in W$ . The vector space operations are  $(v, w) + (v', w') = (v + v', w + w')$  and  $c(v, w) = (cv, cw)$ . We have  $\dim(V \oplus W) = \dim V + \dim W$ .

This leads to a corresponding direct sum operation on vector bundles (also called the Whitney sum). Suppose  $\pi: E \rightarrow B$  and  $\rho: F \rightarrow B$  are two bundles over  $B$ . The direct sum  $E \oplus F$  has total space  $\{(y, z) \in E \times F \mid \pi(y) = \rho(z)\}$ . The bundle projection  $\mu: E \oplus F \rightarrow B$  is  $\mu(x, y) = \pi(x)$ . If  $b \in B$  there are neighborhoods  $U$  and  $U'$  of  $b$  and bundle charts  $g: \pi^{-1}(U) \rightarrow U \times \mathbb{R}^n$  and  $g': \rho^{-1}(U') \rightarrow U' \times \mathbb{R}^k$ . Let  $U'' = U \cap U'$ . We get a bundle chart  $h: \mu^{-1}(U'') \rightarrow U'' \times (\mathbb{R}^n \oplus \mathbb{R}^k)$  where  $h(y, z) = (\pi(y), (\pi_2 g(y), \pi_4 g'(z)))$  where  $\pi_4: U' \times \mathbb{R}^k \rightarrow \mathbb{R}^k$  is projection.

Let us do an example. Consider  $S^2$  the unit sphere in  $\mathbb{R}^3$ . One bundle will be the tangent bundle  $T(S^2)$ . The second will be the trivial rank 1 bundle over  $S^2$ ,  $F = S^2 \times \mathbb{R}$ . We can write  $T(S^2) = \{(x, y) \in S^2 \times \mathbb{R}^3 \mid x \cdot y = 0\}$ . Then the total space of  $T(S^2) \oplus F$  is  $\{(x, y, t) \in S^2 \times \mathbb{R}^3 \times S^2 \times \mathbb{R} \mid x \cdot y = 0\}$ , but we may as well drop the third coordinate and get

$$T(S^2) \oplus F = \{(x, y, t) \in S^2 \times \mathbb{R}^3 \times \mathbb{R} \mid x \cdot y = 0\}$$

Consider the map  $h: T(S^2) \oplus F \rightarrow S^2 \times \mathbb{R}^3$  given by  $h(x, y, t) = (x, y + tx)$ . Note that  $h$  is a bundle morphism since it restricts to a linear map on each fibre, and in fact it is a bundle isomorphism since it restricts to a linear isomorphism on each fibre, i.e., for each fixed  $x$  the map  $T_x(S^2) \oplus \mathbb{R} \rightarrow \mathbb{R}^3$  given by  $(y, t) \mapsto y + tx$  is

an isomorphism. Note one feature of this example. I have told you (admittedly without proof) that  $T(S^2)$  is not a trivial bundle, and yet when we add a trivial bundle to it, it becomes trivial.

**Quotient vector bundle:** If  $V$  is a vector space and  $W \subset V$  is a subspace then there is a quotient vector space  $V/W$  defined as follows. We define an equivalence relation  $\sim$  on  $V$  by saying  $v \sim v'$  if  $v' - v \in W$ . The vectors in  $V/W$  are equivalence classes of this relation. Let  $[v]$  denote the equivalence class of  $v$ . The vector space operations are defined by  $[v] + [v'] = [v + v']$  and  $c[v] = [cv]$ . In the finite dimensional case,  $\dim(V/W) = \dim V - \dim W$ .

The quotient space may be new to you or in any case unfamiliar. (Although if you have seen the quotient of a group by a subgroup you have seen something very similar.) As a simple example, let  $W = \{(x, y) \in \mathbb{R}^2 \mid x = 2y\}$ . Then  $(x, y) \sim (x', y')$  if  $x - x' = 2(y - y')$ . Note that every equivalence class contains a unique vector on the  $x$  axis, since  $(x, y) \sim (x - 2y, 0)$ . So we have an isomorphism  $h: \mathbb{R}^2/W \rightarrow$  the  $x$  axis given by  $h(x, y) = (x - 2y, 0)$ . In a similar manner, if  $L$  is any line through the origin not equal to  $W$  we get an isomorphism  $\mathbb{R}^2/W \rightarrow L$  by the map which sends  $(x, y)$  to the unique point in  $L$  equivalent to  $(x, y)$ . In general, if  $W'$  is a complementary subspace to  $W$  then  $V/W$  is naturally isomorphic to  $W'$ . The advantage of  $V/W$  over  $W'$  is that it is not necessary to choose such a  $W'$  (and in infinite dimensions a complimentary subspace may not exist). The disadvantage is that it is more abstract. If you need to do calculations you might want to pick such a  $W'$ , for example the orthogonal complement in a finite dimensional inner product space.

If  $F$  is a subbundle of  $E$  then we get a quotient bundle  $E/F$ . Just put an equivalence relation on  $E$  where  $x \sim y$  if  $\pi(x) = \pi(y)$  and  $x - y \in F$  and let  $E/F$  be the quotient space  $E/\sim$ . Suppose  $g: \pi^{-1}(U) \rightarrow U \times \mathbb{R}^n$  is a bundle chart so that  $g(F \cap \pi^{-1}(U)) = U \times L$  for some subspace  $L$  of  $\mathbb{R}^n$ . Consider the map  $h: \pi^{-1}(U)/\sim \rightarrow U \times \mathbb{R}^n/L$  induced by  $g$ , which is continuous by definition of the quotient topology. Note  $h$  has a continuous inverse induced by  $g^{-1}$  and is linear on each fibre. Hence by composing  $h$  with an isomorphism of  $\mathbb{R}^n/L$  with  $\mathbb{R}^{n-k}$  we get bundle charts for  $E/F$ .

As an example, consider our by now well-known friend  $S^2$ . Let  $E = T(\mathbb{R}^3)|_{S^2} = S^2 \times \mathbb{R}^3$ . We will show that  $E/T(S^2)$  is a trivial rank 1 bundle. We have  $E/T(S^2) = S^2 \times \mathbb{R}^3/\sim$  where  $(x, y) \sim (x, y')$  if  $x \cdot (y - y') = 0$ . The morphism  $E \rightarrow S^2 \times \mathbb{R}$  given by  $(x, y) \mapsto (x, x \cdot y)$  induces an isomorphism  $E/T(S^2) \rightarrow S^2 \times \mathbb{R}$ .

For another example, if  $E$  and  $F$  are two vector bundles over  $B$  then  $(E \oplus F)/F$  is isomorphic to  $E$ . See if you can prove this as an exercise.

**The normal bundle:** If  $X \subset Y$  is a smooth submanifold we define the normal bundle of  $X$  in  $Y$  to be  $T(Y)|_X/T(X)$ . We will eventually see (in the tubular neighborhood theorem) that this normal bundle is diffeomorphic to a neighborhood of  $X$  in  $Y$ .

If  $X$  is a submanifold of  $\mathbb{R}^n$  we could alternatively define the normal bundle of  $X$  in  $\mathbb{R}^n$  to be the bundle of vectors perpendicular to  $X$ .

We saw above that the normal bundle of  $S^2$  in  $\mathbb{R}^3$  is the trivial rank 1 bundle.

As another example, consider  $S^1 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1, z = 0\}$ . Then the normal bundle of  $S^1$  in  $\mathbb{R}^3$  is trivial. To see this and introduce the notion of section, let  $\sigma_1: S^1 \rightarrow T(\mathbb{R}^3) = S^1 \times \mathbb{R}^3$  and  $\sigma_2: S^1 \rightarrow T(\mathbb{R}^3)$  be the maps  $\sigma_1(x, y, 0) = ((x, y, 0), (x, y, 0))$  and  $\sigma_2(x, y, 0) = ((x, y, 0), (0, 0, 1))$ . Note that  $\pi \sigma_i$  is the identity (where  $\pi: T(\mathbb{R}^3) \rightarrow \mathbb{R}^3$  is the bundle map). For this reason we call the  $\sigma_i$  sections of the bundle. Also, for each  $p \in S^1$ ,  $\sigma_i(p) \perp T_p(S^1)$  so in fact  $\sigma_i$  are sections of the normal bundle of  $S^1$ . Also note that for each  $p \in S^1$ ,  $\sigma_1(p), \sigma_2(p)$  forms a basis of  $T_p(S^1)^\perp$ . So we get an isomorphism  $S^1 \times \mathbb{R}^2 \rightarrow$  the normal bundle of  $S^1$  in  $\mathbb{R}^3$  given by  $(p, u, v) \mapsto u\sigma_1(p) + v\sigma_2(p)$ .

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Let  $f: X \rightarrow Y$  be a smooth map with  $\dim X = m$  and  $\dim Y = n$ .

- We say  $f$  is a submersion if the rank of  $df_p$  is  $n$  for all  $p \in X$ .
- We say  $f$  is an immersion if the rank of  $df_p$  is  $m$  for all  $p \in X$ .

By the inverse function theorem [Br, 67-68], [BJ, 44], [H, 214], [Mi1, 4], we know that if  $f$  is both an immersion and a submersion, then  $f$  is a local diffeomorphism.

Suppose  $f$  is a submersion. We claim that for every  $p \in X$  and every chart  $\phi: V \rightarrow \mathbb{R}^n$  where  $V$  is a neighborhood of  $f(p)$ , there is a chart  $\psi: U \rightarrow \mathbb{R}^m$  where  $U$  is a neighborhood of  $p$  so that  $\phi f \psi^{-1}(x_1, \dots, x_m) = (x_1, \dots, x_n)$ , i.e., in local coordinates,  $f$  is just projection to the first  $n$  coordinates.

To see this, pick any chart  $\mu: U' \rightarrow \mathbb{R}^m$  around  $p$ . We then let  $\psi(q) = (\phi f(q), L\mu(q)) \in \mathbb{R}^n \times \mathbb{R}^{m-n}$  where  $L: \mathbb{R}^m \rightarrow \mathbb{R}^{m-n}$  is any linear map which restricts to an isomorphism of the null space of  $d(\phi f\mu^{-1})$  at  $p$ . Then  $d(\psi\mu^{-1}) = \begin{bmatrix} d(\phi f\mu^{-1}) \\ L \end{bmatrix}$  has full rank  $m$  at  $p$  so by the inverse function theorem we know  $\psi$  restricted to some neighborhood of  $p$  is a chart. But  $\phi f\psi^{-1}(x_1, \dots, x_m) = (x_1, \dots, x_n)$ .

Now suppose that  $f$  is an immersion. We claim that for every  $p \in X$  and every chart  $\psi: U \rightarrow \mathbb{R}^m$  with  $U$  a neighborhood of  $p$  there is a chart  $\phi: V \rightarrow \mathbb{R}^n$  with  $V$  a neighborhood of  $f(p)$  so that  $\phi f\psi^{-1}(x_1, \dots, x_m) = (x_1, \dots, x_m, 0, \dots, 0)$ , i.e., in local coordinates,  $f$  is just inclusion. To see this, pick any chart  $\eta: V' \rightarrow \mathbb{R}^n$ . Let  $U' = \psi(U)$  and let  $L: \mathbb{R}^{n-m} \rightarrow \mathbb{R}^n$  be a one to one linear map whose image  $S$  is a subspace of  $\mathbb{R}^n$  complementary to  $d\eta df(T_p X)$ . Define  $h: U' \times \mathbb{R}^{n-m} \rightarrow \mathbb{R}^n$  by  $h(x, y) = \eta f\psi^{-1}(x) + Ly$ . Note  $dh$  is onto at  $\psi(p)$  so by the inverse function theorem there is a neighborhood  $V''$  of  $\eta f(p)$  and a smooth  $g: V'' \rightarrow U' \times \mathbb{R}^{n-m}$  so that  $gh$  and  $hg$  are the identity. Define  $\phi = g\eta$ . Then  $\phi f\psi^{-1}(x) = g\eta f\psi^{-1}(x) = gh(x, 0) = (x, 0)$ .

Note that if  $f$  is a submersion and  $Z \subset Y$  is a  $k$  dimensional submanifold, then  $f^{-1}(Z)$  is a  $k + m - n$  dimensional submanifold of  $X$ . You see this by choosing a chart  $\phi$  with  $Z \cap V = \phi^{-1}(\mathbb{R}^k \times 0)$ . Then  $f^{-1}(Z) \cap U = \psi^{-1}(\mathbb{R}^k \times 0 \times \mathbb{R}^{m-n})$ . An example of a submersion is the projection  $Y \times W \rightarrow Y$  form manifolds  $Y$  and  $W$ . In fact it is a theorem that any proper submersion is locally such a projection. We can prove this eventually if you like.

Let's see examples of immersions. Consider the torus  $T^2$  thought of as  $\mathbb{R}^2/\mathbb{Z}^2$ . Let  $\alpha$  be a real number and define  $f_\alpha: \mathbb{R} \rightarrow T^2$  by  $f(t) = (t, \alpha t)$ . Then each  $f_\alpha$  is an immersion. If  $\alpha$  is irrational then  $f_\alpha$  is one to one and the image of  $f_\alpha$  is dense in the torus. If  $\alpha$  is rational, then  $f_\alpha$  is not one to one and the image is a submanifold of  $T^2$ .

Another immersion is  $S^1 \rightarrow \mathbb{R}^2$  given by  $(x, y) \mapsto (x, xy)$ . This immersion is not one to one, its image is a figure 8.

A map  $f: \mathbb{R} \rightarrow \mathbb{R}^n$  is an immersion if and only if the velocity  $f'(t) \neq 0$  for all  $t$ .

A map  $f(u, v): \mathbb{R}^2 \rightarrow \mathbb{R}^n$  is an immersion if and only if  $\partial f/\partial u$  and  $\partial f/\partial v$  are always linearly independent.

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A particularly nice kind of immersion is an immersion which is also a homeomorphism to its image. We call such an immersion an embedding and note that its image is a submanifold. Warning: As stated before, Bredon gives a totally screwy definition of a submanifold as a one to one immersion. No way.

**approximating a continuous function by a smooth function:** If  $f: X \rightarrow Y$  is a continuous function between smooth manifolds we can approximate it by a smooth function. At this point it is easiest to leave the proof of this for general  $Y$  until after we have shown any smooth manifold can be embedded in some  $\mathbb{R}^n$ . But we can prove the important case  $Y = \mathbb{R}$ .

So suppose  $f: X \rightarrow \mathbb{R}$  is continuous. We will show that  $f$  can be approximated arbitrarily closely by a smooth function. (To be precise, given any continuous  $\epsilon: X \rightarrow (0, \infty)$  there is a smooth  $g: X \rightarrow \mathbb{R}$  so that  $|f(x) - g(x)| < \epsilon(x)$  for all  $x \in X$ . You can let  $\epsilon$  be constant if you wish or let it approach 0 as you go toward the "edge" of  $X$ .)

We will actually prove a stronger result, which we call the relative version. This says that if  $f$  is already smooth on a neighborhood of some closed set  $C$ , then we may choose our approximation  $g$  so that  $g(x) = f(x)$  for all  $x$  in some (smaller) neighborhood of  $C$ . This is very common in differential topology. Most theorems have a stronger relative version. In fact there is a generic differential topology construction which can be summarized "if the relative version of something can be constructed in  $\mathbb{R}^n$  then it can be constructed in any smooth manifold". Later we will see examples of this generic argument, but we will use more specialized techniques to prove smooth approximation.

In our proof we will actually prove a weaker relative result but this weaker result can then be easily used to prove the stronger result. In this weak relative result we suppose that in fact  $f(x) = 0$  for all  $x$  in a neighborhood of  $C$ . But first:

**A cheap trick, locally finite open covers:** Suppose  $Z$  is a paracompact space and  $\{U_\alpha\}$  is an open cover of  $Z$ . I claim there are open  $U'_\alpha \subset U_\alpha$  so that  $\{U'_\alpha\}$  still covers  $Z$  but the  $U'_\alpha$  are locally finite, i.e., for each  $z \in Z$  there is a neighborhood  $V$  of  $z$  which intersects only finitely many  $U'_\alpha$ . The cheap trick is the proof. Take a partition of unity  $\psi_\alpha: U_\alpha \rightarrow [0, 1]$  and let  $U'_\alpha = \psi_\alpha^{-1}((0, 1])$ .

**proof of smooth approximation:** We have a continuous  $f: X \rightarrow \mathbb{R}$  and assume  $f$  is 0 on an open neighborhood  $V$  of  $C$ . The first step is to cover  $X - V$  with charts  $\phi_\alpha: U_\alpha \rightarrow V_\alpha \subset \mathbb{R}^n$ . After breaking up the  $U_\alpha$  in smaller pieces if necessary you can assume that the closure of each  $U_\alpha$  is compact and is either contained in  $X - C$  or contained in  $V$ . By the cheap trick above, we may also assume the cover  $\{U_\alpha\}$  is locally finite. Now choose a partition of unity for the cover  $U_\alpha$ , i.e., continuous  $\psi_\alpha: X \rightarrow [0, 1]$  so that:

- a) The support of each  $\psi_\alpha$  is contained in  $U_\alpha$  (and is consequently compact). Recall the support of  $\psi_\alpha$  is the closure of  $X - \psi_\alpha^{-1}(0)$ .
- b) Each point of  $x$  has a neighborhood which intersects the support of only finitely many  $\psi_\alpha$ . (Actually this is automatically true by local finiteness of the  $U_\alpha$ .)
- c)  $\sum_\alpha \psi_\alpha(x) = 1$  for each  $x \in X$ . (Note that by b), this sum always has only finitely many nonzero terms.)

What we will do is approximate each function  $\psi_\alpha(x)f(x)$  by a smooth  $g_\alpha$  with support in  $U_\alpha$ . (We take  $g_\alpha = 0$  if  $U_\alpha \subset V$  since in this case  $\psi_\alpha(x)f(x) = 0$ .) Then  $\sum_\alpha g_\alpha$  will be smooth and approximate  $\sum_\alpha \psi_\alpha f = f$  and will be 0 on  $C$ . To approximate  $\psi_\alpha(x)f(x)$  it suffices to approximate  $\psi_\alpha(\phi_\alpha^{-1}(y))f(\phi_\alpha^{-1}(y))$  by a smooth function  $h: V_\alpha \rightarrow \mathbb{R}$  with compact support, then set  $g_\alpha(x) = h(\phi_\alpha(x))$  for  $x \in U_\alpha$  and  $g_\alpha(x) = 0$  for  $x \notin U_\alpha$ .

So we have reduced to the following case: Suppose  $X$  is an open subset of  $\mathbb{R}^n$  and  $f: X \rightarrow \mathbb{R}$  is continuous with compact support  $K \subset X$ . Then for any  $\epsilon > 0$  and neighborhood  $V$  of  $K$  in  $X$  we want a smooth  $g: X \rightarrow \mathbb{R}$  with support contained in  $V$  so that  $|f(x) - g(x)| < \epsilon$  for all  $x \in X$ . For convenience we may extend  $f$  to be zero outside  $X$  and thus assume  $X = \mathbb{R}^n$ . Choose  $\delta > 0$  so that  $|f(x) - f(y)| < \epsilon$  whenever  $|x - y| \leq \delta$ , and so that the distance from  $K$  to  $\mathbb{R}^n - V$  is bigger than  $\delta$ . Take any smooth  $h: \mathbb{R}^n \rightarrow [0, \infty)$  with support contained in  $\{x \mid |x| \leq \delta\}$  and  $\int_{\mathbb{R}^n} h(x) dx = 1$ . For example  $h(x) = 0$  for  $|x| > \delta$  and  $h(x) = ce^{1/(|x|^2 - \delta^2)}$  for an appropriate  $c$  when  $|x| < \delta$ . Then the convolution  $g(x) = \int_{\mathbb{R}^n} h(x - y)f(y) dy$  will be such a good approximation.

**smooth Urysohn lemma:** Suppose  $A$  and  $B$  are disjoint closed subsets of  $X$ . By the Urysohn lemma there is a continuous  $h_0: X \rightarrow [0, 1]$  which is 0 on a neighborhood of  $A$  and 1 on a neighborhood of  $B$ . By the weak relative result we may approximate  $\sqrt{h_0}$  by a smooth  $h_1$  which is 0 on a neighborhood of  $A$  and we may approximate  $\sqrt{1 - h_0}$  by a smooth  $h_2$  which is 0 on a neighborhood of  $B$ . Then  $h = h_1^2 / (h_1^2 + h_2^2)$  is smooth and is 0 on a neighborhood of  $A$  and 1 on a neighborhood of  $B$ .

**smooth Tietze extension theorem variation:** Suppose  $V$  is an open neighborhood of a closed set  $C$  in  $X$ , and  $f: V \rightarrow \mathbb{R}$  is smooth. Take a smooth  $h: X \rightarrow [0, 1]$  which is 0 on a neighborhood of  $X - V$  and 1 on a neighborhood of  $C$ . Then  $hf$  is smooth and defined on all  $X$  and equals  $f$  on a neighborhood of  $C$ . Note a difference between this and the continuous Tietze extension theorem which only requires that  $f$  be defined on  $C$ . If you really wanted to, you could prove a smooth Tietze extension theorem where you only require  $f$  be defined and smooth on  $C$ . (We define a map  $f$  on a closed set  $C$  to be smooth if at each  $x \in C$  there is an open neighborhood  $U$  of  $x$  and a smooth  $g$  on  $U$  which agrees with  $f$  on  $C \cap U$ . Using a smooth partition of unity, this is equivalent to saying that  $f$  may be extended to a smooth function on a neighborhood of  $C$ .)

**weak relative approximation implies strong:** Now the weak relative version implies the stronger as follows. By our smooth Tietze extension theorem there is a smooth  $g_1: X \rightarrow \mathbb{R}$  which equals  $f$  on a neighborhood of  $C$ . Then  $f - g_1$  is 0 on a neighborhood of  $C$  and by the weak version may be approximated by a smooth  $g_2$  which is 0 on a neighborhood of  $C$ . Then  $g_1 + g_2$  is smooth, approximates  $f$ , and equals  $f$  on a neighborhood of  $C$ .

**smooth partitions of unity:** Suppose  $\{U_\alpha\}$  is an open cover of a smooth manifold  $X$ . By the above cheap trick, we may after shrinking the  $U_\alpha$  assume that this open cover is locally finite. Pick a continuous partition of unity  $\psi_\alpha: X \rightarrow [0, 1]$  for the cover. We may approximate each  $\sqrt{\psi_\alpha}$  by a smooth  $\eta_\alpha: X \rightarrow \mathbb{R}$  whose support is in  $U_\alpha$ . Let  $\eta = \sum_\alpha \eta_\alpha^2$  and let  $\mu_\alpha = \eta_\alpha^2 / \eta$ . Then  $\{\mu_\alpha\}$  is a smooth partition of unity for the cover  $\{U_\alpha\}$ .

**rant against an undeservably common version of partitions of unity:** Some authors, for example Bredon, approach partitions of unity by gratuitously changing the indexing set  $\mathcal{A} = \{\alpha\}$ . They take a subordinate locally finite open cover  $\{V_\beta\}$  with index set  $\mathcal{B}$ . There is a function  $\nu: \mathcal{B} \rightarrow \mathcal{A}$  so that  $V_\beta \subset U_{\nu(\beta)}$ . They then take a partition of unity  $\{\phi_\beta: V_\beta \rightarrow [0, 1]\}$ . This is easily converted to our notion of partition of unity by letting  $\psi_\alpha = \sum_{\beta \in \nu^{-1}(\alpha)} \phi_\beta$ . You then don't have the awkwardness in a proof of notationally accounting for the new index set  $\mathcal{B}$ .

**references:** If you really want to know what is going on with approximations, read Chapter 2 of Hirsch. But Hirsch tends to be quite abstract. Smooth partitions of unity are constructed in [BJ, 62-65] and [Br, 89-91] by taking very special charts.

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**critical points, critical values, regular points, regular values:** Suppose  $f: X \rightarrow Y$  is smooth and  $\dim Y = k$ . We say  $x \in X$  is a regular point of  $f$  if  $df_x: T_x X \rightarrow T_{f(x)} Y$  has rank  $k$ . We say  $x \in X$  is a critical point of  $f$  if  $df_x$  has rank less than  $k$ . A point  $y \in Y$  is a critical value of  $f$  if  $y = f(x)$  for some critical point  $x$  of  $f$ . A point  $y \in Y$  is a regular value if it is not a critical value. Note that by the implicit function theorem, if  $y$  is a regular value of  $f$  then  $f^{-1}(y)$  is a submanifold of  $X$  with codimension  $k$ , i.e., of dimension  $\dim X - k$ . (Note, of course  $f^{-1}(y)$  could be empty. We make the convention that if it suits us then the empty set is a manifold with whatever dimension we please.)

**Sard's theorem:** Suppose  $f: X \rightarrow Y$  is smooth. Sard's theorem says that the set of regular values of  $f$  is dense in  $Y$ . (It actually says the set of critical values has measure 0, but this implies the density of the regular values [Mi1, 16-19], [Br, 80-82], [BJ, 56-61], [H, 68-72].) The proof of Sard's theorem is of no interest to us. It does not contain anything we will find useful later. If anyone really wants to see the proof I can present it somewhere outside class time. But the theorem itself is immensely useful. Note that Sard's theorem actually requires some differentiability, in fact  $f$  must be  $k$  times differentiable for some  $k > \max(0, \dim X - \dim Y)$  [H, 69]. This is the only time we use higher differentiability, otherwise  $C^1$  will suffice for everything we do, probably until we get to deRham's theorem.

**a few consequences of Sard's theorem:** If  $\dim X < \dim Y$  then all points of  $X$  are critical, so  $f(X)$  is nowhere dense in  $Y$ . Suppose  $X$  and  $Y$  are submanifolds of  $\mathbb{R}^n$  and  $n > \dim X + \dim Y$ . Then for almost all  $v \in \mathbb{R}^n$ ,  $X$  is disjoint from the translation of  $Y$  by  $v$ . To prove this, consider the smooth map  $h: X \times Y \rightarrow \mathbb{R}^n$  given by  $h(x, y) = x - y$ . Then for every  $v \notin h(X \times Y)$ , we have  $X$  disjoint from the translation of  $Y$  by  $v$ .

**immersing and embedding a smooth manifold in some  $\mathbb{R}^n$ :** We will show that if  $f: X \rightarrow Y$  is continuous and  $\dim Y \geq 2 \dim X$  then  $f$  may be approximated by an immersion  $g$ . If  $f$  is proper (i.e., the inverse image of any compact set is compact) and  $\dim Y \geq 2 \dim X + 1$  then  $f$  may be approximated by an embedding  $g$ .

As usual, there are relative versions of the above results. If  $f$  already immerses or embeds a neighborhood of a closed subset  $C$ , then we may specify that  $g$  coincides with  $f$  on a neighborhood of  $C$ .

I will first outline a slick, but to me unsatisfying, proof. You first show that the space of smooth functions from one manifold to another satisfies the Baire category theorem, so the intersection of countably many open dense sets is dense. You then cover  $X$  by countably many compact sets  $K_i$ , each contained in a chart, whose image is also contained in a chart of  $Y$ . The set of smooth maps from  $X$  to  $Y$  which immerses  $K_i$  is certainly open. To show it is dense is just a local problem of approximating a map  $\mathbb{R}^k \rightarrow \mathbb{R}^n$  by a map which immerses a given compact  $K$  if  $n \geq 2k$ . We'll see how to do this below. So in the end, the immersions are an intersection of countably many open dense sets, and are hence dense.

I will do another proof instead. For now we will suppose  $Y = \mathbb{R}^n$ . Later we can use this case to prove the general case. The outline of the proof for  $Y = \mathbb{R}^n$  is as follows. First of all, we may as well suppose that  $f$  is smooth, since continuous functions to  $\mathbb{R}$  can be approximated by smooth functions. Let  $K \subset X$  be compact. We show there is a map  $h: X \rightarrow \mathbb{R}^n \times \mathbb{R}^m$  which embeds a neighborhood of  $K$  and approximates the map  $(f, 0)$ . We then show there is an  $n$  dimensional subspace  $L$  of  $\mathbb{R}^n \times \mathbb{R}^m$  close to  $\mathbb{R}^n \times 0$  so that the composition of  $h$  with orthogonal projection to  $L$  still embeds a neighborhood of  $K$  (if  $n > 2 \dim X$ ) or just immerses a neighborhood of  $K$  (if  $n = 2 \dim X$ ). Now composing with (the isomorphism) which orthogonally projects  $L$  to  $\mathbb{R}^n \times 0 = \mathbb{R}^n$  gives an approximation of  $f$  which embeds or immerses a neighborhood of  $K$ . Of course if  $X$  is compact, we are through since we could take  $K = X$ . If not, we still have  $X = \bigcup_{i=1}^{\infty} K_i$  where each  $K_i$  is compact,  $K_i \subset \text{Int} K_{i+1}$ . In case  $f$  is proper, we take  $K_i = f^{-1}(\text{the ball of radius } i \text{ about the origin})$ . Let  $W_i = K_i - \text{Int} K_{i-1}$  which is compact. By the above argument we may approximate  $f$  by  $f_i$  which imbeds or immerses a neighborhood of  $W_i$ . Consider the open cover  $U_i = \text{Int} K_{2i+1} - K_{2i-2}$  and take a smooth partition of unity  $\psi_i$ . Note that if  $x \in \text{Int} W_{2i}$  then  $x$  is in just the one open set  $U_i$ , so  $\psi_i$  equals 1 on  $W_{2i}$ . Then  $f' = \sum_{i=1}^{\infty} \psi_i f_{2i}$  approximates  $f$  and embeds or immerses each  $W_{2i}$ . Repeat the same argument with  $f'$  replacing  $f$  and the indices of the  $K_i$  shifted by 1 and we approximate  $f'$  by  $g$  which imbeds or immerses each  $W_{2i+1}$ . Since an approximation of an immersion is still an immersion, it also still

immerses each  $W_{2i}$ . Consequently it immerses  $X$ . In case  $f$  is proper, then  $i - 1 - \epsilon < |g(x)| < i + \epsilon$  for  $x \in W_i$  and if  $n \geq 2 \dim X + 1$  then  $g$  embeds a neighborhood of each  $W_i$  from which we conclude  $g$  is an embedding on all  $X$ .

Now for some details. We may cover  $K$  by a finite number of charts  $\phi_i: U_i \rightarrow \mathbb{R}^k$ ,  $i = 1, \dots, \ell$ . Let  $U_0 = X - K$  and choose a smooth partition of unity  $\psi_i: X \rightarrow [0, 1]$ . I claim that the map  $h: X \rightarrow \mathbb{R}^n \times \mathbb{R}^{(1+k)\ell}$  given by

$$h(x) = (f(x), \epsilon\psi_1(x), \epsilon\psi_1(x)\phi_1(x), \dots, \epsilon\psi_\ell(x), \epsilon\psi_\ell(x)\phi_\ell(x))$$

embeds a neighborhood of  $K$ . It is one to one on the neighborhood  $X - \psi_0^{-1}(1)$  of  $K$  since if  $h(x) = h(y)$  and  $\psi_0(x) \neq 1$  then  $\psi_i(x) = \psi_i(y) \neq 0$  for some  $i \geq 1$  and hence  $\phi_i(x) = \phi_i(y)$  so  $x = y$ . It also immerses the same neighborhood since if  $\psi_i(x) \neq 0$  then  $h\phi_i^{-1}(y) = (\dots, u(y), u(y)y, \dots)$  and  $u(y) \neq 0$  (for  $u = \epsilon\psi_i\phi_i^{-1}$  and  $p = \phi_i(x)$ ). Such a function must have maximal rank since the Jacobian contains a row  $\nabla u$  and a  $k \times k$  minor  $u(y)I_k + y\nabla u$  which then row reduces to the  $k \times k$  identity  $I_k$ .

Now let us find the projection. It suffices to show the following result. Suppose  $h: X \rightarrow S \subset \mathbb{R}^m$  embeds a neighborhood of  $K$  to a linear subspace  $S$  of  $\mathbb{R}^m$ . If  $\dim S > 2 \dim X$  then it suffices to show that for a dense set  $A$  of unit vectors  $v \in S$  the map  $x \mapsto h(x) - (h(x) \cdot v)v$  immerses a neighborhood of  $K$ . If  $\dim S > 2 \dim X + 1$  then it suffices to show that for a dense set  $A$  of unit vectors  $v \in S^{m-1}$  the map  $x \mapsto h(x) - (h(x) \cdot v)v$  embeds a neighborhood of  $K$ . Note that  $x \mapsto h(x) - (h(x) \cdot v)v$  is the composition of  $h$  and orthogonal projection to the orthogonal complement  $S'$  of  $v$  in  $S$ , which has one lower dimension than  $S$ . So we may repeat this process as often as we like as long as there are enough dimensions. In particular, if  $T$  is a given subspace of  $\mathbb{R}^m$  with  $\dim T \geq 2 \dim X$  we may always choose our  $v$  to be nearly perpendicular to  $T$  and hence in the end we compose  $h$  with orthogonal projection to a subspace very close to  $T$  and still get an immersion or embedding near  $K$ .

So let us show the projection result given above. Let  $V$  be an open neighborhood of  $K$  which  $h$  embeds. If  $v$  is a unit vector let  $P_v(y) = y - (y \cdot v)v$  be orthogonal projection to the orthogonal complement of  $v$ . For any linear transformation  $P$ , we have  $dP = P$ , so  $d(P_v h) = dP_v dh = P_v dh$ . Note that the kernel of  $P_v$  is one dimensional and is spanned by  $v$ . Since  $dh$  has rank  $k$  on  $V$  we know that  $d(P_v h)$  will have rank  $k$  on  $V$  if and only if the kernel of  $P_v$  intersects the image of  $dh$  nontrivially, i.e., if and only if  $v$  is in the image of  $dh$ .  $TX$  is a manifold with dimension  $2 \dim X$ . We have  $TS = S \times S$  and consider the projection  $\rho: TS \rightarrow S$  which is not the bundle projection, i.e., translates each vector to start at the origin. Then  $\rho dh: TX \rightarrow S$  is a smooth map and since  $\dim S > \dim TX$  there is a dense set of  $v$  not in the image of  $dh$ . Since we may scale any such  $v$  to a unit vector not in the image of  $dh$ , we have a dense set of unit vectors  $v$  so that  $P_v h$  is an immersion on  $V$ . To see embedding, consider the open subset  $Z$  of the manifold  $X \times X$  given by  $Z = V \times V - \Delta(V) = \{(x, y) \in V \times V \mid x \neq y\}$ . Let  $Y$  be the unit sphere in  $S$ ,  $Y = \{x \in S \mid |x| = 1\}$ . We have a smooth map  $\mu: Z \rightarrow Y$  given by  $\mu(x, y) = (h(x) - h(y))/|h(x) - h(y)|$ . If  $\dim S > 2 \dim X + 1$  then  $\dim Y = \dim S - 1 > 2 \dim X = \dim Z$ . So there is a dense set of  $v$  not in the image of  $\mu$  and also not in the image of  $\rho dh$ . Note that if  $v$  is not in the image of  $\mu$  then  $-v$  is also not in the image (since  $\mu(x, y) = -\mu(y, x)$ ) and hence  $P_v h$  is one to one on  $V$ . Hence  $P_v h$  embeds some (possibly smaller) neighborhood of  $K$  (for example, it embeds any compact neighborhood of  $K$ ).

**references:** Hirsch [H, 23-26] proves the result I did in class, embedding a compact  $n$  manifold in  $\mathbb{R}^{2n+1}$ . Bredon [Br, 91-92] does this also and in addition proves the noncompact case. Hirsch's proof of the general approximation result is quite abstract [H, 55]. In [BJ, 66-71] a different sort of proof is given of the approximation result, essentially building up the approximation chart by chart.

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**vector fields:** A vector field  $v$  on a manifold  $X$  is a choice of a tangent vector  $v(x) \in T_x(X)$  at each point  $x$  of  $X$ . In other words it is a section of the tangent bundle  $\pi: TX \rightarrow X$ , that is, a map  $v: X \rightarrow TX$  so that  $\pi v$  is the identity on  $X$ . Unless stated otherwise we assume all our vector fields are smooth. One example of a vector field is the 0 vector field which is 0 at each point. This is also called the zero section. If  $v$  is a vector field on  $X$  and  $f: X \rightarrow \mathbb{R}$  is smooth then  $v(f)$  is also a smooth function on  $X$  where we define  $v(f)(x)$  to be  $v(x)(f)$  where the vector  $v(x)$  is thought of as a derivation. If  $U \subset \mathbb{R}^n$  is open, we may also think of a vector field on  $U$  in the traditional way as being a map  $v: U \rightarrow \mathbb{R}^n$  since this gives a vector field  $x \mapsto (x, v(x)) \in U \times \mathbb{R}^n = TU$ . We may also commonly denote this vector field as  $\sum_{i=1}^n v_i(x) \partial / \partial x_i$  which thinks of vectors as a derivations. So  $v(f) = \sum_{i=1}^n v_i(x) \partial f / \partial x_i$ .

**the flow generated by a vector field:** Consider first a vector field  $v: U \rightarrow \mathbb{R}^n$  on an open subset of  $\mathbb{R}^n$ . By the theory of ODEs (namely uniqueness of solution of an initial value problem), for each  $x \in U$  there is a unique parameterized curve  $\phi(x, t)$  for  $t \in (a_x, b_x)$  so that:

- $\phi(x, 0) = x$ .
- $d\phi(x, t)/dt|_{t=t_0} = v(\phi(x, t_0))$  for each  $t_0 \in (a_x, b_x)$ .
- $(a_x, b_x)$  is some interval around 0, which we assume to be as large as possible.
- $\phi(x, t)$  depends smoothly on  $x$  and  $t$ , in particular  $\phi: A \rightarrow U$  is smooth where  $A$  is some open subset of  $U \times \mathbb{R}$  and  $A \cap x \times \mathbb{R} = x \times (a_x, b_x)$ .

In particular, we let  $\phi(x, t)$  be the unique solution to the initial value problem  $y(0) = x, y'(t) = v(y(t))$ . Note that for fixed  $s \in (a_x, b_x)$  the curves  $\phi(x, s+t)$  and  $\phi(\phi(x, s), t)$  both satisfy the same initial value problem  $y(0) = \phi(x, s), y'(t) = v(y(t))$  so by uniqueness we must have:

- $\phi(x, s+t) = \phi(\phi(x, s), t)$ .

In particular this means that if  $b_x \neq \infty$  and  $t \rightarrow b_x$  then  $\phi(x, t)$  either approaches the boundary of  $U$  or goes off to infinity. For if  $\phi(x, t)$  remained bounded and did not approach the boundary of  $U$  as  $t \rightarrow b_x$ , there would be a sequence of  $t_i \rightarrow b_x$  so  $y_i = \phi(x, t_i) \rightarrow y \in U$ . But then  $\phi(x, t_i + t) = \phi(y_i, t)$  so we could have extended our solution  $\phi(x, t)$  to be valid for  $t < t_i + b_{y_i}$ . But  $t_i + b_{y_i} > b_x$  for large  $i$ , violating the maximality of the interval  $(a_x, b_x)$ . Similarly, if  $a_x \neq -\infty$  and  $t \rightarrow a_x$  then  $\phi(x, t)$  either approaches the boundary of  $U$  or goes off to infinity.

If  $h: U \rightarrow V$  is a diffeomorphism of  $U$  to another open  $V \subset \mathbb{R}^n$ , then we have a vector field  $v'$  on  $V$  defined by  $v'(y) = dh(v(h^{-1}(y)))$ . I claim that if  $\phi(x, t)$  is the flow generated by  $v$ , then  $\phi'(y, t) = h\phi(h^{-1}(y), t)$  is the flow generated by  $v'$ . This is because

$$d\phi'(y, t)/dt = dh\phi(h^{-1}(y), t)/dt = dh(d\phi(h^{-1}(y), t)/dt) = dh(v(\phi(h^{-1}(y), t))) = v'(\phi'(y, t))$$

Consequently, the flow generated by a vector field on a manifold is independent of the chart we use to define it, so we may define the flow generated by a vector field on a smooth manifold.

If  $v$  is a vector field on  $X$  then the flow generated by  $v$  is the smooth map  $\phi: A \rightarrow X$  where  $A$  is an open neighborhood of  $X \times 0$  in  $X \times \mathbb{R}$  so that

- $\phi(x, 0) = x$ .
- $d\phi(x, t)/dt|_{t=t_0} = v(\phi(x, t_0))$  for each  $(x, t_0) \in A$ .
- $A \cap x \times \mathbb{R} = x \times (a_x, b_x)$  where  $(a_x, b_x)$  is some interval around 0, which we assume to be as large as possible.
- $\phi(x, s+t) = \phi(\phi(x, s), t)$ .
- If  $b_x \neq \infty$  and  $t \rightarrow b_x$  then  $\phi(x, t)$  goes off to infinity, i.e., is eventually outside any compact subset of  $X$ .
- If  $a_x \neq -\infty$  and  $t \rightarrow a_x$  then  $\phi(x, t)$  goes off to infinity, i.e., is eventually outside any compact subset of  $X$ .

We say a vector field  $v$  has compact support if there is a compact  $K \subset X$  so that  $v(x) = 0$  for all  $x \notin K$ . For example, any vector field on a compact manifold  $X$  has compact support. Note that if  $v(x) = 0$  then  $\phi(x, t) = x$  for all  $t \in \mathbb{R}$  and  $a_x = -\infty$  and  $b_x = \infty$  since this constant map solve the IVP. So if  $v$  has compact support it is impossible for  $\phi(x, t)$  to approach infinity as  $t \rightarrow a_x$  or  $b_x$ . Consequently  $A = X \times \mathbb{R}$ .

Notice that if  $A = X \times \mathbb{R}$  (for example if  $v$  has compact support) then if we define  $h_t: X \rightarrow X$  by  $h_t(x) = \phi(x, t)$  then we have  $h_t h_s = h_{t+s}$  and  $h_0 =$  the identity, So each  $h_t$  is a diffeomorphism of  $X$  with inverse  $h_{-t}$ . In fact the map  $t \rightarrow h_t$  gives a group homomorphism from  $\mathbb{R}$  to the group of diffeomorphisms of  $X$ . This is also called a smooth action of  $\mathbb{R}$  on  $X$ . This will be familiar to those of you going into dynamical systems.

**nothing interesting happens away from critical values:** Let us give an example of using vector fields to construct useful diffeomorphisms. Suppose  $f: X \rightarrow \mathbb{R}$  is a proper smooth function and  $f$  has no critical values in the interval  $[a, b]$ . Let  $X_s = \{x \in X \mid f(x) \leq s\} = f^{-1}((-\infty, s])$ . We will show there is a diffeomorphism  $h: X \rightarrow X$  so that  $h(X_a) = X_b$ . In fact we will find a smooth family of diffeomorphisms  $h_t$  for  $t \in [0, 1]$  so that  $h_t(X_a) = X_{a+t(b-a)}$ .

Let  $L = f^{-1}([a, b])$  and pick a compact neighborhood  $K$  of  $L$ . First, for each  $p \in f^{-1}([a, b])$  there is a neighborhood  $U_p$  of  $p$  in  $X$  and a vector field  $v_p$  on  $U_p$  so that  $v_p(f) = 1$ . Just choose a chart

$\phi_p: U'_p \rightarrow V'_p \subset \mathbb{R}^n$ . Let  $g_p: V'_p \rightarrow \mathbb{R}$  be  $g_p = f\phi_p^{-1}$ . Let  $V_p$  be the set of regular points of  $g_p$ , i.e., the set of  $y \in V'_p$  so that  $\nabla g_p(y) \neq 0$ . Define a vector field  $w_p$  on  $V_p$  by  $w_p = \nabla g_p / |\nabla g_p|^2$ . Then  $w_p(g_p) = w_p \cdot \nabla g_p = 1$ . Let  $U_p = \text{Int}K \cap \phi_p^{-1}(V_p)$  and  $v_p = d\phi_p^{-1}(w_p)$  then  $v_p(f) = w_p(g_p) = 1$ . If  $p \notin L$  let  $U_p = X - L$  and  $v_p = 0$ .

An alternative method of constructing these local  $v_p$  is as follows. If  $p \in L$  then  $df_p \neq 0$  so there is a vector  $u \in T_p X$  so  $df(u) \neq 0$ . Extend  $u$  in any way to a smooth vector field  $u_p$  on a neighborhood  $U'_p$  of  $p$ . Shrink  $U'_p$  to a neighborhood  $U_p$  of  $p$  so that  $df(u_p) \neq 0$  on all  $U_p$ . Now let  $v_p = u_p/df(u_p)$  where we understand  $df(u_p) \in T\mathbb{R}$  to be a scalar. Then  $v_p(f) = df(v_p) = 1$ .

Pick a smooth partition of unity  $\{\psi_p\}$  for the cover  $\{U_p\}$ . Let  $v = \sum \psi_p v_p$ . Then the support of  $v$  is in  $K$  and hence is compact. Also, if  $x \in L$  then at  $x$  we have  $v(f) = \sum \psi_p(x) v_p(f) = \sum \psi_p(x) = 1$ . Similarly, for any  $x \in X$  we also have  $0 \leq v(f) \leq 1$ .

Let  $\phi: X \times \mathbb{R} \rightarrow X$  be the flow generated by  $v$ . Note that if  $\phi(x, t) \in L$  then  $df\phi(x, t)/dt = v(f) = 1$  which means that if  $x \in L$  then  $f\phi(x, t) = f(x) + t$  as long as  $a - f(x) \leq t \leq b - f(x)$ . Also you always have  $0 \leq df\phi(x, t)/dt \leq 1$  which means  $f(x) \leq f\phi(x, t) \leq f(x) + t$  if  $t \geq 0$  and  $f(x) + t \leq f\phi(x, t) \leq f(x)$  if  $t \leq 0$ .

Define  $h_t: X \rightarrow X$  by  $h_t(x) = \phi(x, t(b-a))$ . Note  $h_s h_t = h_{s+t}$  so  $h_t$  is a diffeomorphism since  $h_t^{-1} = h_{-t}$ . If  $x \in L$  then  $f h_t(x) = f(x) + t(b-a)$  as long as  $t$  satisfies  $a \leq f(x) + t(b-a) \leq b$ . For any  $x$  and  $t \geq 0$  we have  $f(x) \leq f h_t(x) \leq f(x) + t(b-a)$ . So if  $f(x) \leq a$  then  $f h_t(x) \leq a + t(b-a)$  for  $t \in [0, 1]$ . Thus  $h_t(X_a) \subset X_{a+t(b-a)}$ . If  $f(x) \leq a$  then  $f h_t^{-1} x = f h_{-t}(x) \leq f(x) \leq a$  for  $t \in [0, 1]$ . If  $a < f(x) \leq a + t(b-a)$  then if  $s = (f(x) - a)/(b-a)$  we have  $f h_{-s} x = a$  so

$$f h_t^{-1} x = f h_{s-t} h_{-s} x \leq f h_{-s} x = a$$

for  $t \in [0, 1]$ . Thus  $h_t^{-1}(X_{a+t(b-a)}) \subset X_a$ . Consequently,  $h_t(X_a) = X_{a+t(b-a)}$ .

**smooth function on subsets of manifolds:** Suppose  $X$  and  $Y$  are manifolds,  $A \subset X$  and  $f: A \rightarrow Y$ . We would like to make sense of what it means to say  $f$  is smooth. Two possibilities come to mind:

- $f$  is smooth if  $f$  can be extended to a smooth function on a neighborhood of  $A$ .
- $f$  is smooth if at each point  $p$  of  $A$  there is a neighborhood  $U$  of  $p$  in  $X$  and a smooth  $g: U \rightarrow Y$  so that  $g(x) = f(x)$  for all  $x \in U \cap A$ .

Though a) appears stronger and simpler, b) is preferable since it is easier to verify and after all, smoothness is a local property so the definition should be local, not the global definition in a). Fortunately, a) and b) are equivalent so we don't have to compromise. I'll prove equivalence in the case  $Y$  is an open subset of  $\mathbb{R}^n$ . Once we have some more techniques the case of general  $Y$  will follow easily.

For each  $p \in A$  we have a local extension  $g_p: U_p \rightarrow Y$ . Let  $U' = \bigcup U_p$  and let  $\{\psi_p\}$  be a partition of unity for the cover  $\{U_p\}$  of  $U'$ . Then  $g = \sum \psi_p g_p$  is a map from  $U'$  to  $\mathbb{R}^n$ . Let  $U = g^{-1}(Y)$ . Then  $g$  is a smooth extension of  $f$  to the neighborhood  $U$  of  $A$ .

Recall from 2/16 that if  $A$  is closed and  $Y = \mathbb{R}^n$  we may in fact extend  $f$  to a smooth function defined on all of  $X$ . However if  $Y \neq \mathbb{R}^n$  there may be some topological obstructions to finding a global extension. For example the identity map on the unit circle in  $\mathbb{R}^2$  does not extend to a smooth (or even a continuous) map from  $\mathbb{R}^2$  to the circle.

**isotopy extension theorem:** Suppose  $Y$  and  $X$  are smooth manifolds. Then an isotopy of embeddings of  $Y$  in  $X$  is a smooth family of embeddings  $g_t: Y \rightarrow X$  for  $t \in [0, 1]$ . In other words, we have a smooth map  $G: Y \times [0, 1] \rightarrow X$  so that if  $g_t(y) = G(y, t)$  then each  $g_t$  is an embedding. An isotopy of diffeomorphisms or a diffeotopy of  $X$  is a smooth family of diffeomorphisms  $g_t: X \rightarrow X$  for  $t \in [0, 1]$ . Usually we also want  $g_0$  to be the identity. Generally the standard practice is to just call these isotopies and omit the extra words "of embeddings" or "of diffeomorphisms". Be aware however that there is potentially a source of confusion. For example you could have an isotopy of embeddings  $g_t: \mathbb{R} \rightarrow \mathbb{R}$  with  $g_0 =$  the identity so that  $g_t$  is not a diffeomorphism for  $t > 0$ . How about  $g_t(x) = \tan((1-t/2) \tan^{-1}(x))$ . The confusion disappears if you insist embeddings be proper.

Suppose  $g_t: Y \rightarrow X$  is an isotopy of embeddings and  $Y$  is compact. Then  $g_t$  can be covered by an isotopy of  $X$ , i.e., there is an isotopy  $h_t: X \rightarrow X$  so that  $h_0 =$  the identity and  $h_t g_0(y) = g_t(y)$  for all  $y \in Y$ . Following is a proof.

Let  $\mu: X \times \mathbb{R} \rightarrow \mathbb{R}$  and  $\eta: X \times \mathbb{R} \rightarrow X$  be projections. Define  $G: Y \times [0, 1] \rightarrow X \times \mathbb{R}$  by  $G(y, t) = (g_t(y), t)$  and for convenience, extend  $G$  to a smooth map  $G: Y \times (a, b) \rightarrow X \times \mathbb{R}$  with  $a < 0, b > 1$ . Note that (after perhaps modifying  $a$  and  $b$ )  $G$  is an embedding. Again for convenience, after composing with a

diffeomorphism of  $\mathbb{R}$  to  $(a, b)$  which is fixed on  $[0, 1]$  we may as well assume  $(a, b) = \mathbb{R}$ . Let  $Z = G(Y \times \mathbb{R})$ . We have a vector field  $(0, 1)$  (or if you prefer,  $\partial/\partial t$ ) on  $Y \times \mathbb{R}$ . Pushing forward by  $G$  gives a vector field  $v$  on  $Z$ ,  $v(G(y, t)) = dG(0, 1)$ . The outline of the proof is:

- 1) Extend  $v$  to a vector field  $v(x, s)$  on  $X \times \mathbb{R}$  so that  $v(x, s) = (w(x, s), 1) \in TX \times T\mathbb{R} = T(X \times \mathbb{R})$  where  $w$  has compact support. In other words,  $d\mu(v) = 1$  and there is a compact  $K$  so  $v(x, t) = (0, 1)$  for  $x \notin K$ .
- 2) Let  $\phi$  be the flow generated by  $w$ . Show that:
  - a)  $\phi((x, s), t)$  is defined for all  $x, s, t$ .
  - b)  $\mu\phi((x, s), t) = s + t$  for all  $x, s, t$ .
  - c)  $\phi(G(y, s), t) = G(y, s + t)$  for all  $y, s, t$ .
- 3) Note that 2b) implies that for fixed  $s$  and  $t$  the map  $f_{st}(x, s) = \phi((x, s), t - s)$  gives a diffeomorphism of  $X \times s$  with  $X \times t$  (with inverse  $f_{ts}$ ). Define the diffeomorphism  $h_t: X \rightarrow X$  by  $h_t(x) = \eta\phi((x, 0), t)$ . Then  $h_t$  will have the required properties.

So let us show all these things. To extend  $v$ , it suffices to extend  $v$  locally and then piece together with a partition of unity. Since  $Z$  is a submanifold, at any  $p \in Z$  you can choose local coordinates  $\varphi: U \rightarrow V \subset \mathbb{R}^n$  so that  $U \cap Z = \varphi^{-1}(V \cap \mathbb{R}^k \times 0)$ . After shrinking  $U$  we may as well suppose that  $V = V' \times V''$  for some  $V' \subset \mathbb{R}^k$  and  $V'' \subset \mathbb{R}^{n-k}$ . Note that  $v$  is tangent to  $Z$  so  $d\varphi(v)$  is a vector field  $(v', 0)$  where  $v'$  is a vector field on  $V'$ . So we may extend  $v$  to  $U$  by letting  $v(x) = d\varphi^{-1}(v'\pi(x), 0)$  where  $\pi: V' \times V'' \rightarrow V'$  is projection. This extension may not have unit speed in the  $\mathbb{R}$  direction however. But if this extension is of the form  $(w, w') \in TX \times T\mathbb{R}$  we can just replace it with the extension  $(w, 1)$ .

We have  $d(\mu\phi((x, s), t))/dt = d\mu d(\phi((x, s), t))/dt = d\mu v(\phi((x, s), t)) = 1$ . Thus  $\mu\phi((x, s), t) = \mu(x, s) + t = s + t$  so 2b) holds wherever  $\phi((x, s), t)$  is defined. Hence  $\phi((x, s), t) = (\phi'(x, s, t), s + t)$  for  $\phi' = \eta\phi$ .

Outside some compact subset  $K$ , our vector field  $v$  is  $(0, 1)$ . Hence  $\phi((x, s), t) = (x, s + t)$  for  $x \notin K$  since this solves the differential equation. Thus  $\phi'(x, s, t) \in K$  if  $x \in K$  and in particular,  $\phi((x, s), t)$  cannot go off to infinity in finite time so  $\phi$  is defined everywhere and 2a) holds.

To see 2c), note that  $dG(y, s + t)/dt = dG(d(y, s + t)/dt) = dG(0, 1) = v$  so  $G(y, s + t)$  satisfies the differential equation and thus 2c) holds.

To see 3,

$$h_t g_0(y) = \eta\phi((g_0(y), 0), t) = \eta\phi G(y, 0), t) = \eta G(y, t) = g_t(y)$$

**references:** Flows: [Br, 86-88], [BJ, 74-85], [H, 149-151].  $X_a$  diffeomorphic to  $X_b$ , [Mi2, 12-13]. Isotopy extension theorem: [H, 180], [BJ, 88-95].

**Exercise 1:** It is probably true that an isotopy  $g_t: Y \rightarrow X$  of proper embeddings can be covered by an isotopy of diffeomorphisms of  $X$ . See if you can prove this, or find a proof or counterexample in the literature. In [H, 181] he tries something like this only with another condition, bounded velocity. Perhaps you could fill in details of one of the following proofs:

- Pick a proper function  $h: X \rightarrow \mathbb{R}$ . Do the proof as in the compact case, but make sure that as you flow, the function  $h$  does not grow too fast. This will ensure the flow will be defined for all time.
- Use a variation of the compact proof to find a sequence of isotopies with compact support which cover more and more of the isotopy. Perhaps you can be skillful enough to also prove the sequence converges to an isotopy. More specifically, let  $\{K_i\}$  be a compact exhaustion of  $X$ . Let  $G(y, t) = g_t(y)$ . We are assuming  $G$  is proper. Let  $L_i = \{y \in Y \mid g_t(y) \in K_i \text{ for some } t\}$ . Note  $L_i$  is compact. After taking a subsequence, you may assume that  $G(L_i \times [0, 1]) \subset K_{i+1}$ . Suppose you have an isotopy  $h_t^i: X \rightarrow X$  with support in  $K_{i+1}$  so that  $h_t^i g_0(y) = g_t(y)$  for all  $(y, t) \in L_i \times [0, 1]$ . The trick is to find an isotopy  $h_t^{i+1}$  of  $X$  with support in  $K_{i+2} - K_i$  so that  $h_t^{i+1} h_t^i g_0(y) = g_t(y)$  for all  $(y, t) \in L_{i+1} \times [0, 1]$ . Then we can let  $h_t^{i+1} = h_t^i h_t^i$ . Perhaps you can do this cleverly enough for the  $h_t^i$  to converge. You'd want for every  $x$  to have an  $i$  so that  $h_t^i(x) \in \text{Int}K_i$  since then the infinite composition of isotopies is locally finite.

## 2/28

We will start with what we will eventually see is a preliminary version of the tubular neighborhood theorem. Suppose  $X \subset \mathbb{R}^n$  is a submanifold. Then there is a neighborhood  $U$  of  $X$  in  $\mathbb{R}^n$  and a smooth strong deformation retraction  $\rho: U \rightarrow X$ . In fact, we can define  $\rho$  to be the closest point map,  $\rho(x)$  is the closest point in  $X$  to the point  $x$ . The deformation retraction of  $\rho$  goes along straight lines,  $\rho_t(x) = (1 - t)x + t\rho(x)$ , then  $\rho_0 = \text{the identity}$  and  $\rho_1 = \rho$ .

In order to prove this, we need to show that there is a neighborhood  $U$  so that each point in  $U$  has a unique closest point in  $X$ , and the resulting well defined closest point map is smooth. Let us look carefully at the closest point map. Take any point  $p \in \mathbb{R}^n$ . Then the closest point to  $p$  in  $X$ , if it exists, is a critical point of the function  $f : X \rightarrow \mathbb{R}$  given by  $f(x) = |x - p|^2$ . If  $v \in T_x X$  then  $df(v) = 2(x - p) \cdot v$ , so if  $x$  is a critical point of  $f$  we must have  $p - x$  perpendicular to  $T_x X$ . This leads us to the following. Let  $N = \{(x, y) \in X \times \mathbb{R}^n \mid y \in (T_x X)^\perp\}$  be the normal bundle of  $X$  in  $\mathbb{R}^n$ . We have a smooth map  $h : N \rightarrow \mathbb{R}^n$  given by  $h(x, y) = x + y$ . I claim that  $dh_{(x,0)}$  is an isomorphism at any  $x \in X$ . This is clear because  $T_{(x,0)}N = T_x X \oplus T_x X^\perp$  and  $dh$  is just addition which maps  $T_x X \oplus T_x X^\perp$  isomorphically to  $T_x \mathbb{R}^n$ . So  $h$  restricts to a local diffeomorphism of some neighborhood  $E'$  of  $X \times 0$  in  $N$ . By restricting  $E'$  further we get a diffeomorphism of a neighborhood  $E$  of  $X \times 0$  in  $N$  to a neighborhood  $U$  of  $X$  in  $\mathbb{R}^n$ . Our closest point map  $\rho$  is given by  $\rho(x) = \pi h^{-1}(x)$  where  $\pi : N \rightarrow X$  is the bundle projection  $\pi(x, y) = x$ . If  $X$  is compact, we could take  $E = \{(x, y) \in N \mid |y| \leq d\}$  for some small  $d > 0$  but in the noncompact case we might need to take a compact exhaustion  $K_i$  of  $X$  and a decreasing sequence of  $d_i > 0$  so that  $E = \{(x, y) \in N \mid |y| \leq d_i$  if  $x \in \text{Int}K_i - \text{Int}K_{i-1}\}$ .

**references:** [Br, 92-94], [H, 109-110]. In [BJ, 123] this is done in more generality by using geodesics in a Riemannian manifold instead of our use of straight lines in  $\mathbb{R}^n$ . As we will see later we can get these more general results in other, simpler ways.

### 3/2

Let us see some consequences of our result from 2/28. Suppose  $f : X \rightarrow Y$  is continuous. We claimed on 2/16 that  $f$  could be approximated arbitrarily closely by a smooth function but only proved it for  $Y = \mathbb{R}^n$ . We can now prove it for general  $Y$ . Embed  $Y$  in some  $\mathbb{R}^n$  and by our results from last time there is a neighborhood  $U$  of  $Y$  and a smooth retraction  $\rho : U \rightarrow Y$ . We may approximate  $f$  by  $g' : X \rightarrow U$  and then  $g = \rho g' : X \rightarrow Y$  approximates  $f$ .

Here's another result. Suppose  $f : X \rightarrow Y$  is continuous. Then if  $g : X \rightarrow Y$  is close enough to  $f$ , then  $f$  and  $g$  are homotopic, i.e., there is a continuous  $H : X \times [0, 1] \rightarrow Y$  so that  $H(x, 0) = f(x)$  and  $H(x, 1) = g(x)$  for all  $x \in X$ . The proof is to embed  $Y$  in some  $\mathbb{R}^n$  with a neighborhood  $U$  and retraction  $\rho : U \rightarrow Y$  and let  $H(x, t) = \rho((1-t)f(x) + tg(x))$ .

Similarly, suppose  $f : X \rightarrow Y$  is an embedding with  $X$  compact. Then if  $g : X \rightarrow Y$  is  $C^1$  close enough to  $f$ , then  $g$  is an embedding and  $f$  and  $g$  are  $\epsilon$  isotopic, i.e., they are isotopic via a small isotopy. The proof is again to embed  $Y$  in some  $\mathbb{R}^n$  with a neighborhood  $U$  and retraction  $\rho : U \rightarrow Y$  and let  $H(x, t) = \rho((1-t)f(x) + tg(x))$ . Then each  $h_t = f + t(g - f)$  is an immersion because  $dh_t = df + td(g - f) \approx df$  since  $f$  and  $g$  are  $C^1$  close. But then each  $h_t$  is one to one because it is locally one to one and close to  $f$  and  $X$  is compact. So each  $h_t$  is an embedding.

It is useful to look at the 2/28 result from the following point of view. We found a neighborhood  $V$  of the zero section of the normal bundle  $\pi : N \rightarrow X$  of  $X$  in  $\mathbb{R}^n$  and an embedding  $\eta : V \rightarrow \mathbb{R}^n$  so that:

- a)  $\eta(x) = x$  for all  $x \in X$  (and we identify  $X$  with the zero section).
- b)  $\eta(V)$  is a neighborhood of  $X$ .
- c)  $d\eta_x$  is "the identity" on the zero section. What this means is that at each  $x \in X$ , if we take a vector  $y \in T_x X^\perp$  it determines a tangent vector in  $T_{(x,0)}N$  which we also call  $y$ , namely the velocity of the curve  $t \mapsto (x, ty)$ . Then we note that  $d\eta(y) = y$ .

We call such an  $\eta$  a tubular neighborhood of  $X$  in  $\mathbb{R}^n$ . We will also call the image of  $\eta$  a tubular neighborhood, although generally if we do so we make the additional assumption that for each  $x \in X$ ,  $V \cap \pi^{-1}(x)$  is a disc about the origin in the vector space  $\pi^{-1}(x)$ . Note the retraction  $\rho$  is given by  $\rho = \pi \eta^{-1}$ .

### 3/5

To extend the notion of tubular neighborhood to submanifolds  $X$  of a manifold  $Y$  where we don't know what perpendicular means we need to use the more general definition of the normal bundle as a quotient bundle. The normal bundle  $\pi : N \rightarrow X$  of  $X$  in  $Y$  is the quotient bundle  $TY|_X / TX$ . So a normal vector at a point  $x \in X$  will be an equivalence class  $[y]$  of vectors  $y \in T_x Y$  where we say  $[y] = [y']$  if and only if  $y - y' \in T_x X$ .

If  $Y$  is a submanifold of  $\mathbb{R}^n$  the normal bundle of  $X$  in  $Y$  is  $N = \{(x, [y]) \in X \times \mathbb{R}^n \mid y \in T_x Y\}$ . We may identify  $N$  with  $N' = \{(x, y) \in X \times \mathbb{R}^n \mid y \in T_x Y \text{ and } y \perp T_x X\}$ . In particular, consider the map  $N' \rightarrow N$  which sends  $(x, y)$  to  $(x, [y])$ . This restricts to an isomorphism on each vector space. Its inverse is

the map  $(x, [y]) \rightarrow (x, y')$  where  $y' = y$  minus the orthogonal projection of  $y$  to  $T_x X$  or equivalently,  $y'$  is the orthogonal projection of  $y$  to  $T_x X^\perp$ .

A tubular neighborhood of  $X$  in  $Y$  will be an embedding  $\eta: V \rightarrow Y$  from neighborhood  $V$  of the zero section of  $N$  so that:

- a)  $\eta(x) = x$  for all  $x \in X$  (and we identify  $X$  with the zero section of  $N$ ).
- b)  $\eta(V)$  is a neighborhood of  $X$ .
- c)  $d\eta_x$  is “the identity” on the zero section. What this means is that at each  $x \in X$ , if we take a vector  $[y] \in T_x Y / T_x X$  it determines a tangent vector in  $T_{(x,0)} N$  which we also call  $[y]$ , namely the equivalence class of the velocity of the curve  $t \mapsto [ty]$ . Then we ask that  $[d\eta([y])] = [y]$ , i.e.,  $d\eta([y]) - y \in T_x X$ , or equivalently, the velocity vector  $d(\eta([ty]))/dt|_{t=0}$  minus  $y$  is in  $T_x X$ .

Let me do a simple example. Let  $X = \{(x, 1) \in \mathbb{R}^2\}$  and  $Y = \mathbb{R}^2$ . We have  $N = X \times (\mathbb{R}^2/\mathbb{R} \times 0)$ . Define  $\eta: N \rightarrow \mathbb{R}^2$  by  $\eta((x, 1), [(u, v)]) = (xe^{-v}, e^v)$ . Note this formula is and must be independent of  $u$  since it must be independent of the representative of equivalence classes in  $\mathbb{R}^2/\mathbb{R} \times 0$ . Each vector space in  $N$  is mapped to a level curve  $xy = c$ ,  $y > 0$ , so this definitely is not the tubular neighborhood we originally considered on 2/28 which would be  $(x, 1 + v)$ . Consider a normal vector  $[(u, v)]$  at  $(x, 1) \in X$ . Then

$$d\eta([(u, v)]) = d/dt(\eta((x, 1), [(tu, tv)]))|_{t=0} = d(xe^{-vt}, e^{vt})/dt|_{t=0} = (-vx, v)$$

But  $[(-vx, v)] = [(u, v)]$  so  $d\eta$  is the identity on the zero section.

I claim that tubular neighborhoods of  $X$  in  $Y$  exist. The easiest way to see this is to embed  $Y$  in some  $\mathbb{R}^n$  and take a tubular neighborhood  $U$  of  $Y$  with retraction  $\rho: U \rightarrow Y$ . The normal bundle of  $X$  in  $Y$  is isomorphic to  $N = \{(x, y) \in X \times \mathbb{R}^n \mid y \in T_x Y \text{ and } y \perp T_x X\}$ . Let  $V' \subset N$  be  $V' = \{(x, y) \in N \mid x + y \in U\}$  and define  $\eta: V' \rightarrow Y$  by  $\eta(x, y) = \rho(x + y)$ . Then you can verify that, after restricting to a smaller neighborhood  $V$  of  $X \times 0$  in  $N$ ,  $\eta$  is a tubular neighborhood.

**Exercise 1:** Verify that the above  $\eta$  is a local diffeomorphism near the zero section  $X \times 0$  of  $N$ . Then fill in the rest of the details showing that it is a tubular neighborhood.

**Riemannian metrics on a smooth vector bundle:** Suppose  $\pi: E \rightarrow X$  is a smooth vector bundle over a smooth manifold  $X$ . A Riemannian metric  $g$  on the bundle is a smooth choice of inner product  $g_x$  on each fiber  $\pi^{-1}(x)$ . If you understand what it says, smoothness of  $g_x$  is just smoothness of the corresponding section of the bundle  $(E \otimes E)^*$  since an inner product on a vector space  $V$  is just a vector in  $(V \otimes V)^*$  with certain properties (symmetric, positive definite). But an easier way to explain the smoothness is to take a bundle chart  $\phi: \pi^{-1}(U) \rightarrow U \times \mathbb{R}^n$ . Since any inner product  $g$  on  $\mathbb{R}^n$  has the form  $g(z, y) = z^T A y$  for a unique symmetric matrix  $A$  with all positive eigenvalues, for each  $x \in U$  we get such a matrix  $A_x$  so that  $g_x(\phi^{-1}(x, z), \phi^{-1}(x, y)) = z^T A_x y$ . Then we ask that the map  $x \mapsto A_x$  be smooth.

Note that if  $g_1$  and  $g_2$  are two inner products on a vector space  $V$  and  $t_1, t_2 > 0$ , then  $t_1 g_1 + t_2 g_2$  is also an inner product on  $V$ . Consequently we may construct a Riemannian metric on any smooth vector bundle by piecing together local metrics with a partition of unity. To be precise, cover  $X$  with  $\{U_\alpha\}$  with bundle charts  $\phi_\alpha: \pi^{-1}(U_\alpha) \rightarrow U_\alpha \times \mathbb{R}^n$ . Pick a smooth partition of unity  $\{\psi_\alpha\}$  for the cover. Define a local Riemannian metric on each  $\pi^{-1}(U_\alpha)$  by  $g_{\alpha x}(\phi_\alpha^{-1}(x, z), \phi_\alpha^{-1}(x, y)) = z^T y$ . Now define  $g_x = \sum_\alpha \psi_\alpha(x) g_{\alpha x}$ .

### 3/7

**transversality:** There are a number of forms transversality takes, but they all boil down to some variation of saying that, after a slight deformation, the intersection of something  $k$  dimensional with something  $m$  dimensional in an  $n$  dimensional manifold will have dimension  $k + m - n$  (or else be empty). In particular, if  $k + m < n$  the intersection will be empty.

Let us start with two submanifolds  $X$  and  $Z$  of a manifold  $Y$ . We say that  $X$  is transverse to  $Z$  at a point  $x \in X \cap Z$  if  $T_x X + T_x Z = T_x Y$ , i.e., any vector in  $T_x Y$  is a sum of vectors in  $T_x X$  and  $T_x Z$ . By linear algebra, this is equivalent to saying that  $\dim(T_x X \cap T_x Z) = \dim X + \dim Z - \dim Y$ . If we have an inner product on  $T_x Y$  it is also equivalent to saying that  $T_x X^\perp \cap T_x Z = 0$ . We say that  $X$  is transverse to  $Z$  if  $X$  is transverse to  $Z$  at all  $x \in X \cap Z$ . We denote this  $X \pitchfork Z$ .

**a characterization of submanifolds:** First we note that  $X \subset Z$  is a  $k$  dimensional smooth submanifold of an  $n$  dimensional smooth manifold  $Z$  if and only if for each  $x \in X$  there is a neighborhood  $U$  of  $x$  in  $Z$  and a smooth  $f: U \rightarrow \mathbb{R}^{n-k}$  so that 0 is a critical value of  $f$  and  $f^{-1}(0) = U \cap X$ . One direction is immediate,

if  $X$  is smooth there is by definition (1/29) a chart  $\phi: U \rightarrow \mathbb{R}^n$  and a  $k$  dimensional linear subspace  $L \subset \mathbb{R}^n$  so that  $\phi^{-1}(L) = U \cap X$ . Let  $f = \eta\phi$  where  $\eta: \mathbb{R}^n \rightarrow \mathbb{R}^{n-k}$  is linear with kernel  $L$ . The other direction is the implicit function theorem. Suppose we have such an  $f$ . After shrinking  $U$  we assume we have a chart  $\phi: U \rightarrow V \subset \mathbb{R}^n$ . Then  $d(f\phi^{-1})$  has rank  $n - k$  so by the implicit function theorem, after rearranging the order of the coordinates in  $\mathbb{R}^n$  and shrinking  $U$  and  $V$  there is a  $V' \subset \mathbb{R}^k$  and a smooth  $g: V' \rightarrow \mathbb{R}^{n-k}$  so that  $\phi(f^{-1}(0)) = (f\phi^{-1})^{-1}(0) = \{(x, g(x)) \in V' \times \mathbb{R}^{n-k}\}$ . Define  $\psi: V' \times \mathbb{R}^{n-k} \rightarrow V' \times \mathbb{R}^{n-k}$  by  $\psi(x, y) = (x, y - g(x))$ . Then  $\psi\phi(f^{-1}(0)) = \{(x, 0) \in V' \times \mathbb{R}^{n-k}\} = L \cap V' \times \mathbb{R}^{n-k}$ . So the chart  $\psi\phi$  shows that  $X$  is a submanifold at  $x$ .

Note that we used this point of view of submanifolds implicitly when we noted earlier (2/19) that the inverse image of a regular value is a submanifold. Note also that if  $c$  is a regular value of  $f$  and  $x \in X = f^{-1}(c)$  then  $T_x X = df_x^{-1}(0)$ , (since clearly  $T_x X \subset df_x^{-1}(0)$  and they have the same dimension).

**codimension of a submanifold:** Often if  $X \subset Y$  is a submanifold a more useful quantity than the dimension of  $X$  is the codimension of  $X$ , defined to be  $\text{codim} X = \dim Y - \dim X$ .

**transverse intersections are submanifolds:** We claim that if  $X$  is transverse to  $Y$  then  $X \cap Y$  is a submanifold of  $Z$  of dimension  $\dim X + \dim Y - \dim Z$ . Note we make the convention that the empty manifold has any dimension we please so we include the possibility that  $X \cap Y$  is empty. Let  $k = \dim X$ ,  $m = \dim Y$ , and  $n = \dim Z$ . Take any  $x \in X \cap Y$  and choose a neighborhood  $U$  of  $x$  in  $Z$  and smooth  $f: U \rightarrow \mathbb{R}^{n-k}$  and  $g: U \rightarrow \mathbb{R}^{n-m}$  so that  $0$  is a regular value of  $f$  and  $0$  is a regular value of  $g$  and  $U \cap X = f^{-1}(0)$  and  $U \cap Y = g^{-1}(0)$ . Now  $(d(f, g)_x)^{-1}(0) = df_x^{-1}(0) \cap dg_x^{-1}(0) = T_x X \cap T_x Y$  so  $d(f, g)_x$  has rank  $n - \dim(T_x X \cap T_x Y) = n - (k + m - n)$  so  $x$  is a regular point of  $(f, g)$ . So  $U \cap X \cap Y = (f, g)^{-1}(0)$  is a submanifold of dimension  $k + m - n$ . So  $X \cap Y = (f, g)^{-1}(0)$  is a submanifold of dimension  $k + m - n$ . This is more neatly expressed as  $\text{codim}(X \cap Y) = \text{codim} X + \text{codim} Y$ .

**generic intersections are transverse:** Suppose  $X$  and  $Y$  are submanifolds of  $Z$ . Then we claim that after an arbitrarily small isotopy of  $X$  to some  $X'$ ,  $X'$  and  $Y$  are transverse. A proof will come later.

**transversality of a map and a submanifold:** Suppose  $f: X \rightarrow Z$  is smooth and  $Y \subset Z$  is a smooth submanifold. Then we say  $f$  is transverse to  $Y$ , or  $f \pitchfork Y$  if for every  $x \in f^{-1}(Y)$  we have  $df_x(T_x X) + T_{f(x)} Y = T_{f(x)} Z$ . In particular, if  $f$  is an embedding this is the same as saying  $f(X) \pitchfork Y$ , but it applies to more situations. I claim that if  $f \pitchfork Y$  then  $f^{-1}(Y)$  is a submanifold of  $X$  whose codimension equals the codimension of  $Y$  in  $Z$ . So  $\dim f^{-1}(Y) = \dim Y + \dim X - \dim Z$ . To see why, choose any  $p \in f^{-1}(y)$ . Choose a neighborhood  $U$  of  $f(p)$  and a smooth  $g: U \rightarrow \mathbb{R}^k$  so that  $0$  is a regular value of  $g$  and  $g^{-1}(0) = U \cap Y$ . Note that  $f^{-1}(U \cap Y) = (g \circ f)^{-1}(0)$ . Also  $\ker(d(g \circ f)_p) = df_p^{-1}(\ker dg_{f(p)}) = df_p^{-1}(T_{f(p)} Y)$ . But  $df_p(T_p X) + T_{f(p)} Y = T_{f(p)} Z$  implies that  $\dim X - \dim(df_p^{-1}(T_{f(p)} Y)) = \dim Z - \dim Y = k$ . Consequently,  $d(g \circ f)_p$  has rank  $k$  and so  $d(g \circ f)$  has rank  $k$  on some neighborhood  $V$  of  $p$  in  $X$ . So  $V \cap f^{-1}(Y) = V \cap (g \circ f)^{-1}(0)$  is a submanifold with codimension  $k$ .

**transversality of two maps:** We say that two maps  $f: X \rightarrow Z$  and  $g: Y \rightarrow Z$  are transverse if for every  $p \in X$  and  $q \in Y$  with  $f(p) = g(q)$  we have  $df_p(T_p X) + dg_q(T_q Y) = T_{f(p)} Z$ . Note if  $g$  is an embedding this is the same as saying  $f$  is transverse to  $g(Y)$ . So this notion includes all previous notions of transversality. If  $f \pitchfork g$  then the pullback  $\{(x, y) \in X \times Y \mid f(x) = g(y)\}$  is a submanifold of  $X \times Y$ .

**perturbing a map to be transverse:** Suppose  $f: X \rightarrow Z$  and  $g: Y \rightarrow Z$  are smooth. Then I claim that we may approximate  $f$  arbitrarily closely by a map  $f'$  so that  $f' \pitchfork g$ . This approximation can be a  $C^1$  approximation or in fact  $C^\infty$  if you wish. First I'll do the case  $Z = \mathbb{R}^n$ . Define  $h: X \times Y \rightarrow \mathbb{R}^n$  by  $h(x, y) = f(x) - g(y)$ . Let  $\epsilon \in \mathbb{R}^n$  be a regular value of  $h$ . Then I claim that  $(f - \epsilon) \pitchfork g$ . Take any  $p \in X$  and  $q \in Y$  so that  $g(q) = f(p) - \epsilon$ . Now  $T_{(p,q)} X \times Y = T_p X \oplus T_q Y$  and  $dh_{(p,q)}(u, v) = df_p(u) - dg_q(v)$ . Since  $h(p, q) = \epsilon$  we know that  $dh_{(p,q)}$  is onto and thus  $T_{f(p)} Z = df_p(T_p X) + dg_q(T_q Y)$ . Since  $df = d(f - \epsilon)$  we thus see  $(f - \epsilon) \pitchfork g$ .

Now let us do the case of general  $Z$ . embed  $Z$  in some  $\mathbb{R}^n$  and let  $U$  be a tubular neighborhood of  $Z$  with closest point retraction  $\rho: U \rightarrow Z$ . Let  $\pi: E \rightarrow Y$  be the bundle induced by  $g$  from the normal bundle of  $Z$ . That is,  $E = \{(y, u) \in Y \times \mathbb{R}^n \mid u \perp T_{g(y)} Z\}$ . We have a neighborhood  $V$  of the zero section of  $E$  and a map  $g': V \rightarrow \mathbb{R}^n$  given by  $g'(y, u) = g(y) + u$  so that  $\rho g'(y, u) = g(y)$ . Also take  $V$  as big as possible, i.e., so  $\rho^{-1}g(y) = g'(\{(y, u) \in V\})$  for all  $y \in Y$ . I claim that a map  $h: X \rightarrow U$  is transverse to  $g'$  if and only if  $\rho h: X \rightarrow Z$  is transverse to  $g$ . To see this, suppose  $h \pitchfork g'$  and suppose  $\rho h(p) = g(q)$ . Then  $h(p) = g'(q, u)$  for a unique  $u$ . So  $dh(T_p X) + dg'(T_{(q,u)} E) = T_{h(p)} \mathbb{R}^n$ . Compose with  $d\rho$  and we get

$d\rho h(T_p X) + d\rho g'(T_{(q,u)} E) = d\rho(T_{h(p)} \mathbb{R}^n) = T_{\rho h(p)} Z$ . But  $\rho g'(y, u) = g(y)$  so  $d\rho g'(T_{(q,u)} E) = dg(T_q Y)$ . So  $\rho h \pitchfork g$ . The other direction is similar. So using this, we can approximate  $f: X \rightarrow U$  by a map  $f'': X \rightarrow U$  so that  $f'' \pitchfork g'$ . (Strictly speaking we have only proven this when  $X$  is compact because then for small enough  $\epsilon$  we know  $(f - \epsilon)(X) \subset U$ . We do the usual mumbling about the noncompact case.) But then  $\rho f'' \pitchfork g$  so letting  $f' = \rho f''$  we are done.

Specializing to the cases where  $g$  or  $f$  or both is an embedding we get the following results:

- If  $Y \subset Z$  then we may approximate  $f$  by  $f'$  so  $f' \pitchfork Y$ .
- If  $X \subset Z$  we may  $\epsilon$  isotop  $X$  to an  $X'$  so that  $g \pitchfork X'$ .
- If  $X \subset Z$  and  $Y \subset Z$  we may  $\epsilon$  isotop  $X$  to an  $X'$  so that  $X' \pitchfork Y$ .

**references:** [BJ, 143-151] does a more complicated transversality, for example finding sections of a bundle transverse to a submanifold. [H, 74-78] proves things very abstractly as usual, using Baire category. [Br, 114-118] is closer to the approach here but again goes via bundle transversality.

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**transverse intersections are linear:** Note that in proving that if  $X \pitchfork Y$  then  $X \cap Y$  is a submanifold, we also showed that there is a chart in which both  $X$  and  $Y$  are linear. In particular if  $X$  has codimension  $k$  and  $Y$  has codimension  $m$  then at any  $p \in X \cap Y$  we took local equations  $f: U \rightarrow \mathbb{R}^k$  for  $X$  and  $g: U \rightarrow \mathbb{R}^m$  for  $Y$  and then  $(f, g)$  were local equations for  $X \cap Y$ . But then if  $h: U \cap X \cap Y \rightarrow \mathbb{R}^{n-k-m}$  is a chart for  $X \cap Y$  and we extend  $h$  to  $U$  then  $d(f, g, h)$  has maximal rank  $n$  at  $p$  and hence after shrinking  $U$  gives a chart. But then in this chart  $X$  is given by  $x_1 = x_2 = \dots = x_k = 0$  and  $Y$  is given by  $x_{k+1} = x_{k+2} = \dots = x_{k+m} = 0$  so both  $X$  and  $Y$  are linear.

**extension of vector to a vector field:** Suppose  $X \pitchfork Y$ ,  $p \in X \cap Y$  and  $v \in T_p(X \cap Y)$  then there is a vector field  $u$  on  $Z$  so that  $u$  is tangent to  $X$ ,  $u$  is tangent to  $Y$ , and  $u(p) = v$ . It suffices to find  $u$  locally since we may piece together such local  $u$  with a partition of unity. But this is easy to do locally since we may take local coordinates where  $X$  and  $Y$  are linear subspaces. Then we may let  $u$  be a constant vector field.

**transversality implies perturbation doesn't change things:** Suppose  $X$  and  $Y$  are compact and  $f_t: X \rightarrow Z$  and  $g_t: Y \rightarrow Z$ ,  $t \in [0, 1]$  are isotopies of embeddings and  $f_t \pitchfork g_t$  for all  $t$ . Then we claim there is an isotopy of diffeomorphisms  $h_t: Z \rightarrow Z$  with compact support so that  $f_t(X) = h_t(f_0(X))$  and  $g_t(Y) = h_t(g_0(Y))$  for all  $t$ . In particular, if  $X$  and  $Y$  are compact transverse submanifolds of  $Z$  and  $X'$  and  $Y'$  are small perturbations of  $X$  and  $Y$  then there is a diffeomorphism  $h: Z \rightarrow Z$  so that  $h(X) = X'$  and  $h(Y) = Y'$ . This follows since  $X'$  and  $Y'$  are  $\epsilon$  isotopic to  $X$  and  $Y$  respectively.

Another result is this. Suppose  $Y$  is compact,  $g_t: Y \rightarrow Z$ ,  $t \in [0, 1]$  is an isotopy of embeddings, and  $f_t: X \rightarrow Z$  is a proper homotopy so that  $f_t \pitchfork g_t$  for all  $t$ . Then there is an isotopy of diffeomorphisms  $h_t: X \rightarrow X$  so that  $h_t(f_0^{-1}(g_0(Y))) = f_t^{-1}(g_t(Y))$  for all  $t$ . The proper homotopy condition just means that the map  $(x, t) \mapsto f_t(x)$  from  $X \times [0, 1]$  to  $Z$  is proper. In particular this holds automatically if  $X$  is compact.

Let us prove these results. They are similar enough that we will prove them both at once. First, just as we did in the isotopy extension theorem we extend  $f_t$  and  $g_t$  so  $t \in \mathbb{R}$  and for technical reasons  $f_t = f_2$ ,  $g_t = g_2$  if  $t > 2$  and  $f_t = f_{-1}$ ,  $g_t = g_{-1}$  if  $t < -1$ . Let  $\pi_1: Z \times \mathbb{R} \rightarrow Z$ ,  $\pi_2: Z \times \mathbb{R} \rightarrow \mathbb{R}$ ,  $\pi_3: X \times \mathbb{R} \rightarrow X$ , and  $\pi_4: X \times \mathbb{R} \rightarrow \mathbb{R}$  be projections. Define  $F: X \times \mathbb{R} \rightarrow Z \times \mathbb{R}$  and  $G: Y \times \mathbb{R} \rightarrow Z \times \mathbb{R}$  by  $F(x, t) = (f_t(x), t)$  and  $G(y, t) = (g_t(y), t)$ . Note that  $G$  is an immersion because in local coordinates,  $dG = \begin{pmatrix} dg_t & dg_t/dt \\ 0 & 1 \end{pmatrix}$ , but  $G$  is then an embedding. Likewise in the first theorem  $F$  is an embedding.

We also have  $F \pitchfork G$ . To see this, pick any  $(x_0, t_0)$  and  $(y_0, t_0)$  so that  $F(x_0, t_0) = G(y_0, t_0) = (z_0, t_0)$ . Then

$$dF(T_{(x_0, t_0)} X \times \mathbb{R}) = dF(T_{x_0} X \times 0) + dF(0 \times T_{t_0} \mathbb{R}) = (df_{t_0} T_{x_0} X) \times 0 + \text{Span}(df_t/dt(x_0, t_0), 1)$$

$$dG(T_{(y_0, t_0)} Y \times \mathbb{R}) = dG(T_{y_0} Y \times 0) + dG(0 \times T_{t_0} \mathbb{R}) = (dg_{t_0} T_{y_0} Y) \times 0 + \text{Span}(dg_t/dt(y_0, t_0), 1)$$

So  $dF(T_{(x_0, t_0)} X \times \mathbb{R}) + dG(T_{(y_0, t_0)} Y \times \mathbb{R})$  contains  $(df_{t_0} T_{x_0} X) \times 0 + (dg_{t_0} T_{y_0} Y) \times 0$  which by transversality is  $T_{z_0} Z \times 0$ . Since it also contains  $(df_t/dt(x_0, t_0), 1)$  it contains all  $T_{z_0} Z \times T_{t_0} \mathbb{R} = T_{(z_0, t_0)} Z \times \mathbb{R}$  and hence  $F \pitchfork G$ .

Let  $Y'' = G(Y)$  and  $K = F^{-1}(Y'')$ . Note that since  $F \pitchfork G$ , we know  $K$  is a submanifold of  $X$ . Pick any  $(x_0, t_0) \in K$ . I claim there is a vector  $v \in T_{(x_0, t_0)}K$  so that  $d\pi_4 v = 1$ . Define  $y_0 \in Y$  and  $z_0 \in Z$  by  $F(x_0, t_0) = G(y_0, t_0) = (z_0, t_0)$ . By transversality,  $T_{(x_0, t_0)}K = dF_{(x_0, t_0)}^{-1}T_{(z_0, t_0)}Y''$ . Since  $df_{t_0}(T_{x_0}X) + dg_{t_0}(T_{y_0}Y) = T_{z_0}Z$  there are  $v_1 \in T_{x_0}(X)$  and  $v_2 \in T_{y_0}(Y)$  so that  $df_{t_0}v_1 + dg_{t_0}v_2 = df_t/dt(x_0, t_0) - dg_t/dt(y_0, t_0)$ . Then

$$dF_{(x_0, t_0)}(-v_1, 1) = (df_t/dt(x_0, t_0) - df_{t_0}v_1, 1) = (dg_t/dt(y_0, t_0) + dg_{t_0}v_2, 1) = dG(v_2, 1) \in T_{(z_0, t_0)}Y''$$

so we may let  $v = (-v_1, 1)$ .

Now to prove the second result, extend  $v$  in any way to a vector field  $u'$  on  $X \times \mathbb{R}$  so that  $u'$  is tangent to  $K$  and  $u'(x_0, t_0) = v$ . Then let  $u = u'/d\pi_4 u'$  defined on a neighborhood of  $(x_0, t_0)$ . So  $u$  is a local vector field tangent to  $K$  with  $d\pi_4 u = 1$ . Piece together such local  $u$  with a partition of unity and we get a vector field  $u$  on all  $X \times \mathbb{R}$  tangent to  $K$  and with  $d\pi_4 u = 1$ . Moreover, since  $F$  is proper we know  $K \cap X \times [-1, 2]$  is compact so we can specify that  $u(x, t) = (0, 1)$  for all  $x$  outside some compact neighborhood of  $\pi_3(K)$  or for  $|t| > 3$ , which will ensure that the flow for  $u$  will be defined for all time. Let  $\psi$  be the flow for  $u$ . We may then define  $h_t: X \rightarrow X$  by  $h_t(x) = \pi_3 \psi((x, 0), t)$ . As an exercise you should check that  $h$  has the required properties.

Now let us prove the first result. We want a vector field  $w$  on  $Z \times \mathbb{R}$  so that  $d\pi_2(w) = 1$ ,  $w$  is tangent to  $X'' = F(X \times \mathbb{R})$ ,  $w$  is tangent to  $Y'' = G(Y \times \mathbb{R})$ , and  $w(z, t) = (0, 1)$  for  $z$  outside a compact set. It suffices to construct such a  $w$  locally. Note we may start out with  $w' = dF(u)$  on  $X''$  where  $u$  is as in the proof of the second result. Since  $X'' \pitchfork Y''$  we may extend this to a vector field  $w'$  on a neighborhood of any point of  $X'' \cap Y''$  so that  $w'$  is still tangent to both  $X''$  and  $Y''$ . Setting  $w = w'/d\pi_2 w'$  gives us our desired local vector field. To get  $h_t: Z \rightarrow Z$  do the usual flow construction. Let  $\phi$  be the flow for  $w$  and we let  $h_t(z) = \pi_1 \phi((z, 0), t)$ . Since  $w$  is tangent to  $X''$  we know that  $\phi((z, s), t) \in X''$  if  $(z, s) \in X''$  and similarly for  $Y''$ . Since  $d\pi_2(w) = 1$  we know  $\pi_2 \phi((z, s), t) = s + t$ . Since  $w(z, t) = (0, 1)$  for  $(z, t)$  outside a compact set we know  $\phi((z, s), t) = (z, s + t)$  for  $z$  outside some compact set. Also  $h_t$  is a diffeomorphism because  $h_t^{-1}(z) = \pi_1 \phi((z, t), -t)$ . Thus  $h_t$  has the required properties.

Here is another proof of the second result which works if  $X$  is compact. Consider the isotopies of embeddings  $F_t: X \rightarrow Z \times X$  and  $G_t: Y \times X \rightarrow Z \times X$  given by  $F_t(x) = (f_t(x), x)$  and  $G_t(y, x) = (g_t(y), x)$ . Note that  $F_t \pitchfork G_t$ . So by the first result there is an isotopy  $H_t: Z \times X \rightarrow Z \times X$  so that  $H_t(F_0(X)) = F_t(X)$  and  $H_t(G_0(Y)) = G_t(Y)$ . Note that  $F_t(X) \cap G_t(Y) = F_t(f_t^{-1}g_t(Y))$ . Let  $h_t: X \rightarrow X$  be the isotopy  $h_t(x) = F_t^{-1}H_t F_0(x)$ . We have

$$\begin{aligned} h_t(f_0^{-1}(g_0(Y))) &= F_t^{-1}H_t F_0 f_0^{-1}(g_0(Y)) = F_t^{-1}H_t(F_0(X) \cap G_0(Y)) \\ &= F_t^{-1}(H_t F_0(X) \cap H_t G_0(Y)) = F_t^{-1}(F_t(X) \cap G_t(Y)) = f_t^{-1}g_t(Y) \end{aligned}$$

and thus the result is proven.

**Manifolds with boundary:** Manifolds with boundary are like manifolds except that charts map to  $R_+^n = \{x \in \mathbb{R}^n \mid x_1 \geq 0\}$ . If  $X$  is a manifold with boundary then the boundary of  $X$ , denoted  $\partial X$ , is the set of points of  $X$  which a chart maps to  $\{x_1 = 0\}$ . Note then that if  $X$  is an  $n$  dimensional manifold with boundary, then  $X - \partial X$  is an ordinary manifold without boundary since charts map to the open subset  $\{x \in \mathbb{R}^n \mid x_1 > 0\}$  of  $\mathbb{R}^n$ . Also  $\partial X$  is itself an  $n - 1$  dimensional manifold with boundary since the charts map it to open subsets of  $\{x \in \mathbb{R}^n \mid x_1 = 0\} = \mathbb{R}^{n-1}$ .

Most of what we have done with manifolds also can be done with manifolds with boundary, with little or no modification of the proofs.

- There is the tangent space (but resist the temptation to make it a half space at the boundary).
- You have submanifolds. Sometimes you want a submanifold  $X$  of  $Y$  to have boundary compatible with  $Y$ , to be precise you want local charts so that  $X$  corresponds to a subspace of  $\mathbb{R}^n$  which is not contained in  $\{x_1 = 0\}$ . In other words,  $\partial X = X \cap \partial Y$  and  $X \pitchfork \partial Y$ . Such a submanifold is called neat. Even better is a proper submanifold where you also require  $X$  to be a closed subset. This has new meaning even in the case of manifolds without boundary.
- You can approximate any map  $X \rightarrow Y$  by an embedding if  $\dim Y \geq 2 \dim X + 1$ . Indeed, if  $f(\partial X) \subset \partial Y$  is a proper map you can approximate  $f$  by a proper embedding if  $\dim Y \geq 2 \dim X + 1$ .

- You have transversality but no longer does perturbation not change the behavior. You also often would want transversality of both  $X$  and  $\partial X$ . Here is a theorem you can get. Suppose  $X$  and  $Y$  are compact and  $f_t: X \rightarrow Z$  and  $g_t: Y \rightarrow Z$  are isotopies so that for each  $t$ ,  $f_t \pitchfork g_t$ ,  $f_t|_{\partial X} \pitchfork g_t$ ,  $f_t \pitchfork g_t|_{\partial Y}$ , and  $f_t|_{\partial X} \pitchfork g_t|_{\partial Y}$ . Moreover  $f_t(X) \cap \partial Z$  and  $g_t(Y) \cap \partial Z$  are empty for each  $t$ . Then there is an isotopy  $h_t$  of  $Z$  so that  $f_t(X) = h_t(f_0(X))$  and  $g_t(Y) = h_t(g_0(Y))$  for all  $t$ .
- You have isotopies, but you need to be more careful to get an isotopy extension theorem. If  $g_t: X \rightarrow Y$  is an isotopy of embeddings of a compact manifold with boundary, it can be covered by an isotopy of  $Y$  if either  $g_t(X) \cap \partial Y$  is empty for all  $t$  or  $g_t(X)$  is a proper submanifold of  $Y$  for all  $t$ . The problem is, you do not want to have points moving in and out of the boundary as time goes on, as this would make it impossible to cover the isotopy. You also do not want the “angle”  $X$  makes with  $\partial Y$  to be sometimes 0 and sometimes not, as this could not be covered.
- If  $X$  is a proper submanifold of  $Y$ , you have tubular neighborhoods of  $X$  in  $Y$  and you have uniqueness of tubular neighborhoods. If  $X \subset Y - \partial Y$ , you have tubular neighborhoods of  $X$  in  $Y$  but they will not actually be neighborhoods since they will not contain open neighborhoods of points of  $\partial X$ . You do still have uniqueness though in the compact case.
- You can still assign a flow to a vector field on  $X$ , but it will not be defined on a neighborhood of  $X \times 0$  in  $X \times \mathbb{R}$  unless the vector field is tangent to  $\partial X$ . For example consider the vector field  $(1, 0)$  on the unit disc  $X = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ . The flow is  $\psi((x, y), t) = (x + t, y)$  defined for  $-\sqrt{1 - y^2} - x \leq t \leq \sqrt{1 - y^2} - x$ . So if  $x = \sqrt{1 - y^2}$  the flow is not defined for  $t > 0$  (since the vector field points away from  $X$  at those points). On the other hand, the flow for the vector field  $(-y, x)$  is defined for all  $t$ .

**collar neighborhoods of the boundary:** There is an embedding  $c: \partial X \times [0, \epsilon) \rightarrow X$  onto a neighborhood of  $\partial X$  so that  $c(x, 0) = x$ . Such a  $c$  is called a collar. Collars are a sort of tubular neighborhood and they are unique. Uniqueness means that if  $c_0: \partial X \times [0, \epsilon_0) \rightarrow X$  and  $c_1: \partial X \times [0, \epsilon_1) \rightarrow X$  are any two collars, there is a smooth family of collars  $c_t: \partial X \times [0, \epsilon) \rightarrow X$  where  $\epsilon = \min\{\epsilon_0, \epsilon_1\}$ . Note by the isotopy extension theorem that this can be covered by an isotopy of  $X$  if  $\partial X$  is compact. The proof of existence and uniqueness of collars is the same as that for tubular neighborhoods of submanifolds. If  $X$  is a proper submanifold of  $Y$  there is a collar of  $Y$  which restricts to a collar of  $X$ .

**references:** Manifolds with boundary and theorems relating to them are discussed in [H, 30-31, 109-118], [Mi1, 12-13], [BJ, 129-141]. Usually you prove results for manifolds without boundary and just convince yourself of the corresponding result for manifolds with boundary which would have essentially the same proof, but would be filled with annoying details.

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**manifolds with corners:** If  $X$  and  $Y$  are manifolds with boundary, then  $X \times Y$  is technically no longer a manifold with boundary, since at points of  $\partial X \times \partial Y$  your charts map to  $\{x_1 \geq 0, x_2 \geq 0\}$ . One could deal with this by developing a notion of manifolds with corners where charts map to  $\{x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0\}$ . But this approach, while very occasionally useful, is not at all popular. One objection is that you cannot topologically distinguish corner points from ordinary boundary points. Instead, you observe that there is a standard way of rounding off corners, for example using the complex map  $z \mapsto z^2$  the first quadrant is mapped to the upper half plane. Thus the map  $(x_1, \dots, x_n) \mapsto (x_1^2 - x_2^2, 2x_1x_2, x_3, x_4, \dots, x_n)$  is a homeomorphism from  $\{x_1 \geq 0, x_2 \geq 0\}$  to  $\{x_1 \geq 0\}$  which is a diffeomorphism away from the corner  $\{x_1 = x_2 = 0\}$ . Composing charts to  $\{x_1 \geq 0, x_2 \geq 0\}$  with this map changes corner charts to ordinary charts and thus gives a standard way of smoothing out the corners of  $X \times Y$  and other situations where corners are introduced. It turns out that any way you smooth out a corner which does not change the differentiable structure on the complement of the corner will give you a manifold diffeomorphic to that obtained by any other such method. So up to diffeomorphism there is only one way to smooth corners.

**gluing two manifolds together:** Suppose  $X$  and  $Y$  are two  $n$  dimensional manifolds. Suppose  $A \subset \partial X$  and  $B \subset \partial Y$  are compact  $n - 1$  dimensional submanifolds and  $h: A \rightarrow B$  is a diffeomorphism. Then  $X \cup_h Y$  is the manifold obtained as follows. As a topological space,  $X \cup_h Y$  is the quotient space of the disjoint union of  $X$  and  $Y$  where we identify each  $x \in A$  with  $h(x) \in B$ . We define a smooth structure on  $X \cup_h Y$  by the following charts. Pick any collars  $c: \partial X \times [0, \epsilon) \rightarrow X$  and  $d: \partial Y \times [0, \epsilon) \rightarrow Y$ . Charts around a point  $x \in X - A$  are just charts in  $X$  with domain restricted to  $X - A$ . Charts around a point  $y \in Y - B$  are

just charts in  $Y$  with domain restricted to  $Y - B$ . Charts around a point  $x \in A - \partial A$  are obtained by taking a chart  $\phi': U' \rightarrow \mathbb{R}^{n-1}$  in  $A$  so  $U' \cap \partial A$  is empty. We define a chart  $\phi$  by  $\phi(c(x, t)) = (\phi'(x), t)$  and  $\phi(d(h(x), t)) = (\phi'(x), -t)$ . Finally around points  $x \in \partial A$  we define a chart by rounding the corner, (only this time it is a  $360^\circ$  corner as opposed to the  $90^\circ$  corner we encountered in the cartesian product so we would use  $z \mapsto \sqrt{z}$ ). You could also look at this process as making  $90^\circ$  corners at  $\partial A \subset \partial X$  and at  $\partial B \subset \partial Y$ , and then after gluing there is a  $180^\circ$  corner, i.e., no corner there. By uniqueness of collars and rounding corners, the resulting smooth manifold is independent of the choices of collars, i.e., different choices would give diffeomorphic smooth structures via a diffeomorphism which restricts to a diffeomorphism of the canonical smooth structure on  $X - \partial A$  and restricts to a diffeomorphism of the canonical smooth structure on  $Y - \partial B$ .

**closed manifold:** A common terminology is to say  $X$  is a closed  $n$  manifold if  $X$  is a compact  $n$  dimensional manifold with empty boundary.

**unoriented bordism:** We put an equivalence relation on the set of closed  $n$  manifolds and obtain an abelian group which is called  $\mathfrak{N}_n$ , the  $n$ -th unoriented bordism group. The group operation is disjoint union which we denote by  $+$ . If  $X$  and  $Y$  are closed  $n$  manifolds, we say  $X \sim Y$  if there is a compact  $n + 1$  dimensional manifold  $W$  so that  $\partial W$  is diffeomorphic to the disjoint union of  $X$  and  $Y$ . This is an equivalence relation. For  $X \sim X$  we may take  $W = X \times [0, 1]$ . If  $X \sim Y$  then the same  $W$  shows  $Y \sim X$ . Finally if  $\partial W = X + Y$  and  $\partial V = Y + Z$  we may form a new  $n + 1$  manifold  $U$  by gluing  $W$  to  $V$  along  $Y$  and then  $\partial U = X + Z$ .

We say the bordism class of  $X$  is the equivalence class of  $X$  under  $\sim$ . This forms an abelian group with group operation disjoint union and 0 element the empty manifold. If you don't like the idea of the empty manifold being  $n$  dimensional and so prominent, you could instead represent 0 by the  $n$  sphere  $S^n$  which is equivalent to the empty manifold since  $S^n$  is the boundary of the  $n + 1$  disc. Note that  $X + X = 0$  so inverses are easy, every element has order 2.

You can calculate that:

- $\mathfrak{N}_0 = \mathbb{Z}/2\mathbb{Z}$  generated by the one point manifold.
- $\mathfrak{N}_1 = 0$  since any closed 1 manifold is a union of circles which bounds a union of discs.
- $\mathfrak{N}_2 = \mathbb{Z}/2\mathbb{Z}$  generated by the real projective plane. You need to know classification of compact surfaces to see this.

To most easily give all  $\mathfrak{N}_n$  it is convenient to take  $\mathfrak{N}_* = \bigoplus_{n=0}^{\infty} \mathfrak{N}_n$ . Then  $\mathfrak{N}_*$  is in fact a ring whose product is just cartesian product. This is what we call a graded ring since the product takes  $\mathfrak{N}_n \times \mathfrak{N}_m$  to  $\mathfrak{N}_{n+m}$ . A powerful theorem of Thom [MS, 197] says that the graded ring  $\mathfrak{N}_*$  is freely generated (as a  $\mathbb{Z}/2\mathbb{Z}$  algebra) by manifolds  $Y^2, Y^4, Y^5, Y^6, Y^8, \dots$  where  $Y^n$  is a certain closed  $n$  manifold and  $n \neq 2^k - 1$ . If  $k$  is even you may take  $Y^k$  to be real projective  $k$  space, but you need something else in odd dimensions, since  $\mathbb{R}P^k$  bounds a compact manifold if  $k$  is odd.

**exercise:** Show that the connected sum  $X \sharp Y$  is equivalent in  $\mathfrak{N}_n$  to  $X + Y$ .

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**transversality of sections:** We showed that if  $\pi: E \rightarrow X$  is a smooth bundle,  $Y \subset E$ , and  $\sigma: X \rightarrow E$  is a section, then we may approximate  $\sigma$  by a section  $\sigma'$  so that  $\sigma' \pitchfork Y$ . The proof is to  $C^1$  approximate  $\sigma$  by a map  $f$  transverse to  $Y$ , then observe that if the approximation is close enough that  $\pi \circ f$  is a diffeomorphism of  $X$  close to the identity, then observe that  $\sigma' = f \circ (\pi \circ f)^{-1}$  is a section close to  $\sigma$  and transverse to  $Y$ .

**Mod 2 Euler characteristic:** Let  $X$  be a closed manifold. Let  $\sigma: X \rightarrow T(X)$  be a smooth section (i.e., a smooth vector field) transverse to the zero section. We define the mod 2 Euler characteristic of  $X$  to be the number of zeroes of  $\sigma$  mod 2. To see this is independent of the choice of  $\sigma$ , suppose we have  $\sigma_i$  transverse to the zero section,  $i = 0, 1$ . Consider the bundle on  $X \times [0, 1]$  with total space  $T(X) \times [0, 1]$ , and consider the section  $\sigma(x, t) = ((1-t)\sigma_0(x) + t\sigma_1(x), t)$ . Approximate  $\sigma$  by a section  $\sigma'$  transverse to the 0 section. Define sections  $\sigma'_t$  of  $T(X)$  by  $\sigma'(x, t) = (\sigma'_t(x), t)$ . Then  $\sigma'_i$  approximates  $\sigma_i$  for  $i = 0, 1$  and hence by transversality,  $\sigma_0^{-1}(0)$  and  $\sigma'_0^{-1}(0)$  have the same number of points. Likewise  $\sigma_1^{-1}(0)$  and  $\sigma'_1^{-1}(0)$  have the same number of points. But  $\sigma'^{-1}(0)$  by transversality is a proper compact one dimensional submanifold of  $X \times [0, 1]$  and hence its boundary has an even number of points. But its boundary consists of all the points in  $\sigma'_0^{-1}(0) \times 0$  and  $\sigma'_1^{-1}(0) \times 1$  and hence  $\sigma_0^{-1}(0)$  and  $\sigma_1^{-1}(0)$  have the same parity.

**Euler characteristic:** In fact we may assign an index  $i(p) = \pm 1$  to each transverse 0 of  $\sigma$  and then define the Euler characteristic of  $X$  to be the sum of  $i(p)$  over all zeroes of  $\sigma$  and thus obtain an integer

Euler characteristic. To get the index, in local bundle coordinates the section looks like  $\sigma(x) = (x, f(x))$  for  $x \in U \subset \mathbb{R}^n$  and  $f: U \rightarrow \mathbb{R}^n$ . If  $f(p) = 0$  we define  $i(p)$  to be the sign of the determinant of  $df_p$ . This is independent of coordinates chosen because if  $h$  is a gluing map between two charts we get the corresponding  $f$  is  $dh \circ f \circ h^{-1}$  and its Jacobian matrix is  $dh \circ df \circ dh^{-1}$  which has the same sign of the determinant. A similar, but more delicate argument as in the mod 2 case shows the independence of  $\sigma$ .

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If  $Z$  is any topological space, define the  $k$ -th bordism group of  $Z$  to be  $\mathfrak{N}_k(Z) = \{f: X \rightarrow Z \mid X \text{ is a closed } k \text{ dimensional manifold}\} / \sim$  where we put an equivalence relation on the maps as follows. We say  $f \sim g$  where  $f: X \rightarrow Z$  and  $g: Y \rightarrow Z$  if there is a compact  $k+1$  dimensional manifold  $W$  and a continuous  $F: W \rightarrow Z$  so that  $\partial W = X \cup Y$  and  $F|_X = f$  and  $F|_Y = g$ . The group operation on  $\mathfrak{N}_k(Z)$  is disjoint union. The zero element is the empty map from the empty manifold, or if you are uncomfortable with this, it is a constant map from the  $k$  sphere (assuming  $Z$  is nonempty). We define  $\mathfrak{N}_*(Z) = \bigoplus_{k=0}^{\infty} \mathfrak{N}_k(Z)$ . Let  $[f]$  denote the equivalence class of a map  $f: X \rightarrow Z$  in  $\mathfrak{N}_*(Z)$ .

Note that homotopic maps are equivalent in  $\mathfrak{N}_*(Z)$ , for if  $f_t: X \rightarrow Z$  is a homotopy,  $t \in [0, 1]$  then the map  $F: X \times [0, 1] \rightarrow Z$  given by  $F(x, t) = f_t(x)$  gives an equivalence between  $f_0$  and  $f_1$ .

If  $h: Z \rightarrow V$  is continuous we get a map  $h_*: \mathfrak{N}_*(Z) \rightarrow \mathfrak{N}_*(V)$  defined by  $h_*([f]) = [h \circ f]$ . Of course  $h_*$  maps  $\mathfrak{N}_k(Z)$  to  $\mathfrak{N}_k(V)$ .

**exercise:** Show that if  $h$  and  $h'$  are homotopic then  $h_* = h'_*$ .

If  $Z$  and  $V$  are closed manifolds and  $h: Z \rightarrow V$  is smooth we also get a map  $h^*: \mathfrak{N}_*(V) \rightarrow \mathfrak{N}_*(Z)$  in the other direction defined as follows. Take any  $[f] \in \mathfrak{N}_k(V)$  for  $f: X \rightarrow V$ . If we perturb  $f$  slightly to  $f'$  then  $f$  and  $f'$  are homotopic and hence  $[f] = [f']$ . So we may as well, after perturbation, assume that  $f \pitchfork h$ . Let  $Y$  be the pullback of  $f$  and  $h$ ,

$$Y = \{(x, z) \in X \times Z \mid f(x) = h(z)\}$$

Note that if  $\Delta = \{(v, v) \in V \times V\}$  is the diagonal, then  $Y = (f, h)^{-1}(\Delta)$ . But  $f \pitchfork h$  is equivalent to  $(f, h) \pitchfork \Delta$ . So  $Y$  is a manifold of dimension  $\dim(X \times Z) + \dim \Delta - \dim(V \times V) = \dim X + \dim Z - \dim V$ . We have a map  $g: Y \rightarrow Z$  given by  $g(x, z) = z$ . We define  $h^*([f]) = [g]$ .

**exercise:** Show that  $f \pitchfork h$  is equivalent to  $(f, h) \pitchfork \Delta$ .

To see that  $h^*$  is well defined, we must show that  $[g]$  is independent of the chosen representative  $f$ . So suppose  $[f_0] = [f_1]$ , and  $f_i \pitchfork h$ . Let  $Y_i = \{(x, z) \in X_i \times Z \mid f_i(x) = h(z)\}$  be the pullback of  $f_i$  and  $h$ . Define  $g_i: Y_i \rightarrow Z$  by  $g_i(x, z) = z$ . We must show that  $[g_0] = [g_1]$ .

There is compact  $k+1$  manifold  $W$  and a map  $F: W \rightarrow V$  with  $\partial W = X_0 + X_1$ ,  $F|_{X_0} = f_0$  and  $F|_{X_1} = f_1$ . Approximate  $F$  by a smooth map  $F'$  so  $F' \pitchfork h$ . Let  $U$  be the pullback of  $F'$  and  $h$ ,

$$U = \{(w, z) \in W \times Z \mid F'(w) = h(z)\}$$

Let  $f'_0 = F'|_{X_0}$  and  $f'_1 = F'|_{X_1}$ . Note that  $\partial U = Y'_0 + Y'_1$  where  $Y'_i = U \cap (X_i \times Z) =$  the pullback of  $f'_i$  and  $h$ . Let  $g'_i: Y'_i \rightarrow Z$  be induced by projection  $g'_i(x, z) = z$ . Since  $f'_i$  approximates  $f_i$  then by transversality we know that  $Y'_i$  is isotopic to  $Y_i$  and hence  $[g'_i] = [g_i]$ . (To be precise, choose isotopies  $H_i: Y_i \times [0, 1] \rightarrow X_i \times Z$  so that  $H_i(Y_i \times 0) = Y_i$  and  $H_i(Y_i \times 1) = Y'_i$  then the composition of  $H_i$  with projection to  $Z$  gives a bordism equivalence between  $g_i$  and  $g'_i$ .) But projection of  $U$  to  $Z$  gives a bordism equivalence between  $g'_0$  and  $g'_1$ . So  $g_0 \sim g'_0 \sim g'_1 \sim g_1$  and thus  $h^*$  is well defined.

**exercise:** Show that if  $h_0$  and  $h_1$  are homotopic then  $h_0^* = h_1^*$ . Conclude that we may define  $h^*$  for a continuous map by approximating  $h$  by a smooth map.

A similar construction shows that if  $Z$  is a closed  $n$  dimensional manifold then there is a symmetric bilinear form  $\cap: \mathfrak{N}_*(Z) \otimes \mathfrak{N}_*(Z) \rightarrow \mathfrak{N}_*(Z)$  which maps  $\mathfrak{N}_i(Z) \otimes \mathfrak{N}_j(Z)$  to  $\mathfrak{N}_{i+j-n}(Z)$ . Take  $[f] \in \mathfrak{N}_i(Z)$  and  $[g] \in \mathfrak{N}_j(Z)$ . You may as well suppose  $f \pitchfork g$ . Then represent  $[f] \cap [g]$  by the map of the pullback of  $f$  and  $g$  to  $Z$ . This is also  $g_* g^*([f]) = f_* f^*([g])$ .

### Connections of unoriented bordism with homology and cohomology with $\mathbb{Z}/2\mathbb{Z}$ coefficients:

For those exposed to homology and cohomology we have connections as follows. Warning- **all homology and cohomology below is with  $\mathbb{Z}/2\mathbb{Z}$  coefficients**. Also  $f_*$  and  $f^*$  could represent maps on bordism, homology, or cohomology depending on context.

There is a homomorphism  $\theta: \mathfrak{N}_*(Z) \rightarrow H_*(Z)$  where if  $f: X \rightarrow Z$  then  $\theta([f]) = f_*([X])$  where  $[X]$  is the fundamental class of  $X$ . Thom showed in ??? as a consequence of computing  $\mathfrak{N}_*(Z)$  that  $\theta$  is onto. In fact he showed that  $\mathfrak{N}_*(Z) = \mathfrak{N}_* \otimes H_*(Z)$  so that  $\theta(1 \otimes \alpha) = \alpha$  for  $1 \in \mathfrak{N}_0$  and  $\theta(\beta \otimes \alpha) = 0$  for  $\beta \in \mathfrak{N}_k, k > 0$ . This  $\theta$  commutes with the induced maps  $h_*$  on  $H_*$  and  $\mathfrak{N}_*$  (in fancy language, it is a natural transformation of functors).

If  $Z$  is a closed  $n$  manifold, then Poincare duality gives an isomorphism of  $D: H_k(Z) \rightarrow H^{n-k}(Z) = \text{Hom}(H_{n-k}(Z), \mathbb{Z}/2\mathbb{Z})$ . In terms of bordism, Poincare duality is given by sending  $[f]$  to the homomorphisms  $[g] \mapsto [f] \cap [g]$ . (So  $D\theta[f](\theta[g]) = \eta([f] \cap [g])$  where  $\eta: \mathfrak{N}_0(Z) \rightarrow \mathbb{Z}/2\mathbb{Z}$  counts the number of points mod 2 in the 0 manifold mapping to  $Z$ .) The map  $h^*$  on bordism is the map  $h^*$  on cohomology, composed with Poincare duality,  $D\theta h^* = h^* D\theta$ . Cup product is the composition of  $\cap$  with Poincare duality.  $D\theta[f] \cup D\theta[g] = D\theta([f] \cap [g])$ .

**exercise:** Prove all these results (but not the Thom representability result which is difficult).

**exercise:** Prove there is a Mayer-Vietoris exact sequence for unoriented bordism, if  $A$  and  $B$  are open subsets of a space, then there is a long exact sequence

$$\dots \rightarrow \mathfrak{N}_k(A \cap B) \rightarrow \mathfrak{N}_k(A) \oplus \mathfrak{N}_k(B) \rightarrow \mathfrak{N}_k(A \cup B) \rightarrow \mathfrak{N}_{k-1}(A \cap B) \rightarrow \dots$$

You may need to ask for a hint as to what the map  $\mathfrak{N}_k(A \cup B) \rightarrow \mathfrak{N}_{k-1}(A \cap B)$  should be (it is analogous to the map in homology).

**exercise:** Prove there is a long exact sequence for a pair  $(Z, A)$  in unoriented bordism

$$\dots \rightarrow \mathfrak{N}_k(A) \rightarrow \mathfrak{N}_k(Z) \rightarrow \mathfrak{N}_k(Z, A) \rightarrow \mathfrak{N}_{k-1}(A) \rightarrow \dots$$

You need to define the relative bordism groups  $\mathfrak{N}_k(Z, A)$ . They are equivalence classes of maps  $f: X \rightarrow Z$  from a compact  $k$  manifold  $X$ , such that  $f(\partial X) \subset A$ . Two maps  $f: X \rightarrow Z$  and  $g: Y \rightarrow Z$  are equivalent if there is a compact  $k+1$  manifold  $W$  and a map  $F: W \rightarrow Z$  so that  $X$  and  $Y$  are disjoint subsets of  $\partial W$ ,  $F|_X = f$ ,  $F|_Y = g$ , and  $F(\partial W - X - Y) \subset A$ . It may help your visualization to put corners in  $W$  at  $\partial X$  and  $\partial Y$ .

### 3/26

**Morse theory:** Let  $f: X \rightarrow \mathbb{R}$  be a proper smooth map. Let  $X_a = f^{-1}((-\infty, a])$ . We saw on 2/23 that if there are no critical values of  $f$  in  $[a, b]$  then  $X_a$  is diffeomorphic to  $X_b$ . So the only changes to  $X$  occur on intervals containing critical values. The strategy of Morse theory is to take a nice function  $f$  where you have a good understanding of its critical points and then build up  $X$  in a finite number of steps (if  $X$  is compact) one step for each critical point.

**nondegenerate critical points and their index:** We say a critical point  $p$  of  $f: X \rightarrow \mathbb{R}$  is nondegenerate if the Hessian of  $f$  at  $p$  is nonsingular. In other words, take any chart  $\phi: U \rightarrow \mathbb{R}^n$  around  $p$ . For convenience, suppose  $\phi(p) = 0$ . By Taylor's theorem we have  $f \circ \phi^{-1}(x) = f(p) + x^T H x / 2 +$  terms of degree 3 or more where the Hessian  $H$  is the matrix of partial derivatives  $H = (\partial^2 f / \partial x_i \partial x_j)$ . If we chose a different chart with gluing map  $h$  then the Hessian in the new chart will be  $dh^T H dh$  so in particular, the nonsingularity of  $H$  is independent of the coordinates chosen. But also  $dh^T H dh$  has the same number of negative eigenvalues as  $H$ , so this is independent of the coordinates also. (To see this, convince yourself that the index is the largest dimension of a subspace on which the quadratic form  $x \mapsto x^T H x$  is negative definite.) We define the index of a critical point to be the number of negative eigenvalues of the Hessian.

**Morse functions:** We say a smooth  $f: X \rightarrow \mathbb{R}$  is a Morse function if every critical point of  $f$  is nondegenerate.

The crucial result is that if  $p$  is a nondegenerate critical point of  $f$  with index  $k$ , then there is a chart  $\phi$  around  $p$  so that

$$f \circ \phi^{-1}(x) = f(p) - x_1^2 - x_2^2 - \dots - x_k^2 + x_{k+1}^2 + \dots + x_n^2$$

In particular, nondegenerate critical points are isolated, but more importantly they have a very standard form which we can study. Let us see why.

First, if  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  is any smooth function then there are smooth  $f_i: \mathbb{R}^n \rightarrow \mathbb{R}$  so that  $f(x) = f(0) + \sum_{i=1}^n x_i f_i(x)$ . (Just let  $f_i(x) = \int_0^1 \partial f / \partial x_i(tx) dt$ .) Applying this result to each  $f_i$  we get smooth  $f_{ij}$  so that

$f(x) = f(0) + \sum_{i=1}^n c_i x_i + \sum_{i,j=1}^n x_i x_j f_{ij}(x)$  for some constants  $c_i$ . Taking derivatives and evaluating at 0 we see  $c_i = \partial f / \partial x_i(0)$  and  $f_{ij}(0) + f_{ji}(0) = \partial^2 f / \partial x_i \partial x_j(0)$ . By replacing  $f_{ij}$  by  $f_{ij} + f_{ji}$  if  $j < i$  we may as well suppose that  $f_{ij} = 0$  if  $j > i$ .

Now suppose  $f$  has a nondegenerate critical point at 0. Then all  $c_i = 0$ , and after an orthogonal change of coordinates we may as well assume the Hessian of  $f$  at 0 is diagonal which means that  $f_{ij}(0) = 0$  if  $i \neq j$  and  $f_{ii}(0) \neq 0$ . After reordering the coordinates we may as well assume that  $f_{ii}(0) < 0$  for  $i \leq k$  and  $f_{ii}(0) > 0$  for  $i > k$ . We now simplify the expression for  $f$  by completing the square.

$$f(x) = f(0) + f_{11}(x)z_1^2 + \sum_{n \geq i \geq j \geq 2} x_i x_j \hat{f}_{ij}(x)$$

$$z_1 = x_1 + \sum_{i=2}^n x_i f_{i1}(x) / (2f_{11}(x))$$

$$\hat{f}_{ij} = f_{ij} - f_{i1}f_{j1} / (2f_{11}) \quad i \neq j$$

$$\hat{f}_{ii} = f_{ii} - f_{i1}^2 / (4f_{11})$$

Note  $\hat{f}_{ij}(0) = 0$  if  $i \neq j$  and  $\hat{f}_{ii}(0) = f_{ii}(0) \neq 0$ . Continuing in this fashion we eventually get

$$f(x) = f(0) + \sum_{i=1}^n \hat{f}'_{ii}(x)z_i^2$$

for some  $\hat{f}'_{ii}$  with  $\hat{f}'_{ii}(0) = f_{ii}(0)$ . Now take new coordinates  $y_i = \sqrt{|\hat{f}'_{ii}(x)|}z_i$  and we have

$$f(y) = f(0) - \sum_{i=1}^k y_i^2 + \sum_{i=k+1}^n y_i^2$$

These new coordinates  $y$  are a smooth change of coordinates by the inverse function theorem since the derivative matrix at 0 is upper triangular with diagonal entries  $\sqrt{|\hat{f}'_{ii}(x)|}$ .

Because of this local description we have a very concrete idea of how the sub level sets  $X_a = f^{-1}((-\infty, a])$  change as we pass a nondegenerate critical point. In particular if  $f$  is a proper Morse function with just one critical point in  $f^{-1}([a, b])$  with critical value in  $(a, b)$  and with index  $k$ , then  $X_b = X_a \cup D^k \times D^{n-k}$  where we glue  $S^{k-1} \times D^{n-k}$  to  $X_a$  by some embedding into  $\partial X_a = f^{-1}(a)$ .

**reference:** See [H, 142-147, 156-160] or [Mi2, 4-20]

### 3/28

**approximating by a Morse function:** Any  $f: X \rightarrow \mathbb{R}$  can be approximated arbitrarily closely by a Morse function. We will prove a form of this which is weaker if  $X$  is noncompact, in that we will assume  $f$  is proper and given  $\epsilon > 0$  we will find a Morse function  $g$  so that  $|f(x) - g(x)| < \epsilon$  at all  $x \in X$ . First, we may as well suppose that  $f$  is smooth. Now take a bounded embedding  $h$  of  $X$  to some  $\mathbb{R}^{n-1}$ . For example, approximate the constant 0 map by an embedding. Then  $(f, h): X \rightarrow \mathbb{R}^n$  is a proper embedding to some submanifold of  $\mathbb{R}^n$  which we will also call  $X$ . Then our map  $f$  is given by the inner product  $f(x) = x \cdot e_1$  with the vector  $e_1 = (1, 0, \dots, 0)$ . We will show that for almost all vectors  $v \in \mathbb{R}^n$  the map  $g_v(x) = x \cdot v$  from  $X$  to  $\mathbb{R}$  is a Morse function. Then we may find  $v$  close to  $e_1$  so  $g_v$  is Morse. Then  $g_v/e_1 \cdot v$  is Morse and approximates  $f$  (and moreover is still proper).

So it remains to show  $g_v$  is almost always Morse. Let  $E = \{(x, y) \in X \times \mathbb{R}^n \mid y \perp T_x X\}$  be the normal bundle of  $X$  and consider the map  $q: E \rightarrow \mathbb{R}^n$  given by  $q(x, y) = y$ . Let  $v$  be a regular value of  $q$ . Then I claim  $g_v$  is a Morse function. Let  $p$  be a critical point of  $g_v$ . If  $w \in T_p X$  then  $0 = dg_v(w) = v \cdot w$  so  $v \in T_p X^\perp$ , hence  $(p, v) \in E$ .

After a translation and orthogonal transformation we may as well assume for simplicity that  $p = 0$ ,  $T_p X = \mathbb{R}^k \times 0 \subset \mathbb{R}^k \times \mathbb{R}^{n-k} = \mathbb{R}^n$  and in a neighborhood  $U$  of  $p$ ,  $X$  is given as a graph  $X \cap U = \{(x, r(x))\}$  for  $x$  near 0 for some smooth  $r: \mathbb{R}^k \rightarrow \mathbb{R}^{n-k}$ . I claim that the normal vector space at a point  $(x, r(x))$  is  $\{(-(dr_x)^T w, w) \mid w \in \mathbb{R}^{n-k}\}$ . This is because any tangent vector at  $(x, r(x))$  has the form  $(u, dr_x u)$  and  $(u, dr_x u) \cdot (-(dr_x)^T w, w) = u^T (-(dr_x)^T w) + (dr_x u)^T w = 0$ . So  $\{(-(dr_x)^T w, w) \mid w \in \mathbb{R}^{n-k}\}$  is contained in the normal space. But it has the right dimension  $n - k$  and hence must be the whole normal space. So we have local coordinates  $(x, w) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$  on  $E$  near  $(p, v)$ , namely  $(x, w) \mapsto ((x, r(x)), (-(dr_x)^T w, w))$ . In

these coordinates, the map  $q$  is  $q(x, w) = (-(dr_x)^T w, w)$  and the point  $(p, v)$  has coordinates  $(0, v')$ . Since  $v$  is a regular value of  $q$ , we know  $dq_{(0, v')}$  is invertible but we calculate

$$dq = \begin{pmatrix} -H_{v'} & * \\ 0 & I \end{pmatrix}$$

where the  $ij$ -th coordinate of  $H_{v'}$  is  $\sum_{\ell=1}^{n-k} v'_\ell \partial^2 r_\ell / \partial x_i \partial x_j$ , in other words the Hessian of  $v' \cdot r$ . But  $v' \cdot r = g_v$  since

$$g_v(x) = (x, r(x)) \cdot v = (x, r(x)) \cdot (0, v') = v' \cdot r(x)$$

So the Hessian of  $g_v$  at  $p$  must be nonsingular, hence  $p$  is nondegenerate so  $g_v$  is a Morse function.

**references:** Milnor proves this in [Mi2, 31-38] by another method, using the distance function rather than the dot product. Hirsch in [H, 147-148] proves this in a fancy way using transversality in the 1-jet bundle. The  $k$ -jet bundle  $J^k(X, Y)$  is just the vector bundle over  $X \times Y$  whose vector space at any point  $(x, y)$  is the set of degree  $k$  Taylor polynomials of germs of functions from  $X$  to  $Y$  which map  $x$  to  $y$ . A smooth function from  $X$  to  $Y$  gives a section of this bundle restricted to its graph, obtained by taking the Taylor series at each point. Then you can show that Morse functions are those where this section is transverse to the 0 section. A fancy jet transversality theorem then gives density of Morse functions. Alternately, you can easily describe Morse function by their 2-jet, then the nonMorse function hit a subset of the 2 jets (where the linear terms are 0, but the Hessian of the quadratic terms is singular) but this has codimension  $n+1$  so by transversality it is missed by a generic function, i.e., generic functions are Morse.

#### 4/2

If  $f: X \rightarrow \mathbb{R}$  is a Morse function then a gradient-like vector field for  $f$  is a vector field  $v$  on  $X$  so that:

- $df_p(v(p)) > 0$  if  $p$  is not a critical point of  $f$ .
- If  $x$  is a critical point of  $f$  then there is a coordinate chart  $\psi: U \rightarrow \mathbb{R}^n$  with  $U$  a neighborhood of  $p$  in  $X$  so that
  - a)  $\psi(p) = 0$
  - b)  $f\psi^{-1}(y) = f(p) - \sum_{i=1}^k y_i^2 + \sum_{i=k+1}^n y_i^2$ .
  - c)  $d\psi(v(\psi^{-1}(y))) = (-y_1, \dots, -y_k, y_{k+1}, \dots, y_n)$ , in other words,  $v$  is half the gradient of  $f$  in nice local coordinates.

It is easy to construct gradient-like vector fields, just construct locally and piece together with a partition of unity. From now on let  $v$  be a gradient-like vector field for  $f$ , and let  $\phi(x, t)$  be the flow generated by  $v$ . Following are some properties:

- If  $p$  is not a critical point of  $f$  then  $f\phi(x, t) > f(x)$  if  $t > 0$  and  $f\phi(x, t) < f(x)$  if  $t < 0$ . (This follows from the  $df_p(v(p)) > 0$  condition).
- If  $p$  is a critical point then for the nice coordinate chart  $\psi$  given above we have  $\psi(\phi(\psi^{-1}(y), t)) = (y_1 e^{-t}, \dots, y_k e^{-t}, y_{k+1} e^t, \dots, y_n e^t)$ .

If  $p$  is a critical point of  $f$  we let

$$D_S(p) = \{x \in X \mid \lim_{t \rightarrow \infty} \phi(x, t) = p\}$$

$$D_U(p) = \{x \in X \mid \lim_{t \rightarrow -\infty} \phi(x, t) = p\}$$

We call  $D_S(p)$  the stable disc at  $p$  and call  $D_U(p)$  the unstable disc at  $p$ . Let  $c = f(p)$ . Note  $D_S(p) \subset f^{-1}((-\infty, c]) = X_c$  and  $D_U(p) \subset f^{-1}([c, \infty))$  They are submanifolds of  $X$  and intersect transversely at the single point  $p$ . If  $X$  is compact without boundary and  $p$  has index  $k$  then  $D_S(p)$  is diffeomorphic to  $\mathbb{R}^k$  and  $D_U(p)$  is diffeomorphic to  $\mathbb{R}^{n-k}$ . To see all this, note that in a nice coordinate chart  $\psi$  around  $p$  we have  $\psi(D_S(p)) = \{y_{k+1} = \dots = y_n = 0\}$  and  $\psi(D_U(p)) = \{y_1 = \dots = y_k = 0\}$ . In particular, for a small  $\epsilon > 0$  we then have  $D_S(p) \cap f^{-1}([c - \epsilon, c])$  is a closed  $k$  dimensional disc  $D^k$  with boundary sphere  $D_S(p) \cap f^{-1}(c - \epsilon)$ . Likewise  $D_U(p) \cap f^{-1}([c, c + \epsilon])$  is a closed  $n - k$  dimensional disc  $D^{n-k}$  with boundary sphere  $D_U(p) \cap f^{-1}(c + \epsilon)$ . We see  $D_S(p)$  is a submanifold because for each  $x \in D_S(p)$  there is a  $t_0 > 0$  so that  $f\phi(x, t_0) > c - \epsilon$ , and then the map  $z \mapsto \phi(z, t_0)$  is a diffeomorphism from a neighborhood of  $x$  in  $X$  to a neighborhood of  $\phi(x, t_0)$  in  $X$  which takes  $D_S(p)$  to itself. Since  $D_S(p)$  is a submanifold near  $\phi(x, t_0)$  it is

thus a submanifold near  $x$ . A similar argument shows that  $D_U(p)$  is a submanifold. In the case  $X$  is closed (i.e., compact without boundary), then  $\phi(x, t)$  is defined for all  $x$  and  $t$  and thus

$$D_S(p) = D_S(p) \cap f^{-1}([c - \epsilon, c]) \cup \{\phi(x, t) \mid t \leq 0, x \in D_S(p) \cap f^{-1}(c - \epsilon)\}$$

which is a closed disc  $D^k$  glued to  $S^{k-1} \times (-\infty, 0]$  along their boundaries, which is diffeomorphic to  $\mathbb{R}^k$ . Likewise,  $D_U(p)$  is diffeomorphic to  $\mathbb{R}^{n-k}$ .

Still to be written up: cancellation of critical points.

#### 4/4

General note. If  $X$  has boundary then we ask that a Morse function  $f$  on  $X$  be locally constant on the boundary. Usually, in fact, some components of the boundary will be the minima of  $f$  and the other components will be the maxima. For example the function  $f(x, y, z) = z$  on the cylinder  $\{x^2 + y^2 = 1, 0 \leq z \leq 4\}$ .

We started by showing that we may assume, after perturbing our gradient vector field slightly, that  $D_S(p) \pitchfork D_U(q)$ , and in fact if  $a$  is any regular value of  $f$  with  $f(p) > a > f(q)$  then  $D_S(p) \cap f^{-1}(a)$  is transverse to  $D_U(q) \cap f^{-1}(a)$  in  $f^{-1}(a)$ . Though not necessary, we will suppose  $X$  is compact. Order the critical points  $q_1, q_2, \dots, q_\ell$  so that  $f(q_i) \leq f(q_{i+1})$ . By induction we may assume this is true for all pairs of critical points  $q_i, q_j$  with  $j < m$ . Pick  $b$  so that  $f(q_m) > b$  and for all  $i < m$  either  $f(q_i) = f(q_m)$  or  $f(q_i) < b$ . Note that if  $f(q_i) = f(q_m)$  for  $i < m$  then  $D_S(q_m) \cap D_U(q_i)$  is empty since  $D_U(q_i) \subset f^{-1}([f(q_m), \infty))$  and  $D_S(q_m) \subset f^{-1}((-\infty, f(q_m)])$ . So we need not worry about those  $i$ . Let  $S = D_S(q_m) \cap f^{-1}(b)$  and  $K_i = D_U(q_i) \cap f^{-1}(b)$ . Then the  $K_i$  are disjoint smooth submanifolds of  $f^{-1}(b)$  (and in the nonboundary case their union is all of  $f^{-1}(b)$ ). There is a small isotopy of  $S$  which makes  $S$  transverse to  $K_1$ . Then there is a small isotopy of this isotoped  $S$  which makes  $S$  transverse to  $K_2$ , but is still small enough that it remains transverse to  $K_1$ . Continuing in this fashion we get a small isotopy  $h_t$  of  $f^{-1}(b)$  with  $h_0 = \text{identity}$  so that  $h_1(S) \pitchfork K_i$  for each  $i$ . (Actually when you really fill in the details you would want to first get transversality to  $K_{m-1}$  which is closed, then  $K_{m-2}$  and so on down. Then at each stage you have transversality to a compact union of manifolds which adhere nicely to each other. But that's just detail and perhaps not even necessary.) For technical reasons, reparameterise the isotopy  $h_t$  so that  $h_t = h_0 = \text{identity}$  for  $t < \epsilon$  and  $h_t = h_1$  for  $t > 1 - \epsilon$ . For a small  $\delta > 0$  we have a diffeomorphism  $g: f^{-1}(b) \times [0, 1] \rightarrow f^{-1}([b, b + \delta])$  given by  $g(x, t) =$  the unique  $\phi(x, s)$  so that  $f\phi(x, s) = b + t\delta$ . We also have a diffeomorphism  $H: f^{-1}(b) \times [0, 1] \rightarrow f^{-1}(b) \times [0, 1]$  given by  $H(x, t) = (h_t^{-1}(x), t)$ . We will alter  $v$  to a new vector field  $v'$  by replacing it by  $dg dH dg^{-1}v$  on  $f^{-1}([b, b + \delta])$ . (check this) The effect of this is to deform  $D_S(q_m)$  so that its intersection with  $f^{-1}(b)$  is  $h_1(S)$ . The intersection  $D_U(q_i)$  with  $f^{-1}(b)$  is unchanged (since  $v = v'$  on  $X_b$ ) so now  $D_S(q_m) \pitchfork D_U(q_i)$  and in fact  $D_S(q_m) \cap f^{-1}(a)$  is transverse to  $D_U(q_i) \cap f^{-1}(a)$  in  $f^{-1}(a)$  for every  $f(q_m) > a > f(q_i)$ , since flowing by  $\phi$  diffeomorphs this intersection to the intersection in  $f^{-1}(b)$ . A reference is in [Mi3, ?]

Still to be written up: Homology.

#### 4/6

### h-cobordism theorem:

to be written up.

**Poincare Conjecture:** The h-cobordism theorem implies the high dimensional Poincare conjecture. To be precise, suppose  $X$  is a closed simply connected manifold of dimension  $n$  with the homology of the  $n$  dimensional sphere. If  $n \geq 6$  (or  $n \geq 5$  with more work), then  $X$  is homeomorphic to the  $n$  dimensional sphere. While it is diffeomorphic to the sphere if  $n = 5, 6$ , it may not be diffeomorphic to a sphere if  $n \geq 7$ , since in these dimensions there are exotic spheres, manifolds homeomorphic to the sphere but not diffeomorphic to it. To prove this, let  $W = X - D_1 - D_2$  where  $D_i$  are two disjoint  $n$  discs in  $X$ . By general position we know that  $W$  is simply connected since mappings of the one or two dimensional disc to  $X$  will by general position miss the centers of the  $D_i$ . An algebraic topology calculation shows that  $H_*(W, \partial D_1) = 0$ , hence the h-cobordism theorem applies and  $W = \partial D_1 \times [0, 1] = S^{n-1} \times [0, 1]$ . So  $X = D_1 \cup S^{n-1} \times [0, 1] \cup D_2$  is homeomorphic to  $S^n$ .

You might not get a diffeomorphism to  $S^n$  however. You do have a diffeomorphism of  $D_1 \cup W$  to a disc  $D^n$  and of course  $D_2$  is diffeomorphic to  $D^n$ . These two discs are glued together by some diffeomorphism of  $S^{n-1}$ . If this diffeomorphism can be extended to  $D^n$  (for example if it is isotopic to the identity or reflection

through a hyperplane, as I recall this is if and only if due to Hatcher/Wagoner's work on pseudoisotopies) then  $X$  will be diffeomorphic to  $S^n$ . But if not then  $X$  is an exotic sphere.

The Poincare conjecture is easily true in dimensions  $\leq 2$ . In dimension 3 it was recently proven by Perelman and many experts believe the proof. In all these dimensions we get a diffeomorphism to a sphere. In dimension 4 it was proven by Mike Freedman in the 1980s, but it is still open whether  $X$  is diffeomorphic to  $S^4$ . Wikipedia has some nice articles which appear to be correct on "Generalized Poincare Conjecture" and "Exotic Sphere".

4/9,4/11

### finish proof of h-cobordism theorem:

**orientation of a vector space:** Let  $V$  be a finite dimensional real vector space. An orientation of  $V$  is an equivalence class of ordered bases of  $V$ . The equivalence relation is as follows. Suppose  $\{v_1, \dots, v_n\}$  and  $\{w_1, \dots, w_n\}$  are ordered bases. There are unique  $c_{ij}$  so that  $v_i = \sum_{j=1}^n c_{ij} w_j$ . We say these ordered bases are equivalent if  $\det[c_{ij}] > 0$ . A special exception is if  $V$  has dimension 0 where an orientation is just  $+$  or  $-$ . Thus every vector space has just two orientations.  $\mathbb{R}^n$  has a standard orientation coming from the standard basis  $\{e_1, e_2, \dots, e_n\}$ .

If  $h: V \rightarrow W$  is a linear isomorphism and  $\{v_1, \dots, v_n\}$  gives an orientation on  $V$  then we have an induced orientation on  $W$  given by  $\{h(v_1), h(v_2), \dots, h(v_n)\}$ . If we have already oriented  $W$  we then say  $h$  is orientation preserving if this induced orientation is the given orientation on  $W$ . Otherwise we say  $h$  is orientation reversing. For example if  $A$  is a nonsingular  $n \times n$  matrix then  $A: \mathbb{R}^n \rightarrow \mathbb{R}^n$  is orientation preserving if  $\det(A) > 0$  and is orientation reversing if  $\det(A) < 0$ .

Note that  $\{v_1, \dots, v_n\}$  and  $\{w_1, \dots, w_n\}$  are equivalent if and only if one can be deformed to the other in the space of ordered bases, i.e., there are continuous  $\alpha_i: [0, 1] \rightarrow V$  so that  $\alpha_i(0) = v_i$  and  $\alpha_i(1) = w_i$  and  $\{\alpha_1(t), \alpha_2(t), \dots, \alpha_n(t)\}$  is a basis of  $V$  for each  $t$ . If such  $\alpha_i$  exist then there are unique continuous  $c_{ji}(t)$  so that  $\alpha_i(t) = \sum_{j=1}^n c_{ji}(t) w_j$ . Then  $\det[c_{ij}(t)]$  is continuous and never 0 but  $[c_{ij}(1)] =$  the identity with positive determinant, so  $\det[c_{ij}(t)] > 0$  for all  $t$ . In particular  $\det[c_{ij}(0)] > 0$  implies  $\{v_1, \dots, v_n\}$  and  $\{w_1, \dots, w_n\}$  are equivalent. Conversely, suppose  $\{v_1, \dots, v_n\}$  and  $\{w_1, \dots, w_n\}$  are equivalent, and let  $C = [c_{ij}]$  where  $v_i = \sum_{j=1}^n c_{ij} w_j$ . I claim the space of  $n \times n$  matrices with positive determinant is path connected. Hence there is a continuous  $\beta: [0, 1] \rightarrow$  positive determinant  $n \times n$  matrices so that  $\beta(0) = I$  and  $\beta(1) = C$ . Letting  $\beta(t) = [c_{ij}(t)]$  and  $\alpha_i(t) = \sum_{j=1}^n c_{ij}(t) w_j$  we deform one basis to the other.

**connectedness of positive determinant matrices:** Any nonsingular matrix  $C$  can be uniquely written  $C = QR$  where  $Q$  is orthogonal and  $R$  is upper triangular with positive diagonal entries.  $Q$  is obtained by performing the Gram-Schmidt process on the columns of  $C$ . Then the deformation  $(QR, t) \mapsto Q(I + t(R - I))$  is a deformation retraction of the positive determinant matrices to the positive determinant orthogonal matrices which has the standard name  $SO(n)$ . So we must show  $SO(n)$  is connected. —To be continued.

### orientation of a vector bundle:

### orientation of a manifold:

**example- orienting  $S^2$ :** One way is to draw little oriented half circles at various points of  $S^2$  and convince yourself they vary continuously. Another way is to give an obviously continuous rule to pick the orientation. We say an ordered basis  $\{v_1, v_2\}$  of  $T_p(S^2)$  has the correct orientation if the three vectors  $\{v_1, v_2, p\}$  give the standard orientation of  $T_p(\mathbb{R}^3)$ . (This generalizes to the following. If  $X$  is oriented and  $Y$  is a submanifold of  $X$  whose normal bundle is oriented, then there is a natural induced orientation of  $Y$ . Just say  $\{v_1, \dots, v_k\} \in T_p(Y)$  has the correct orientation if we take  $\{w_1, \dots, w_\ell\}$  correctly orienting the normal bundle of  $Y$  at  $p$  and then  $\{v_1, \dots, v_k, w_1, \dots, w_\ell\}$  correctly orients  $T_p(X)$ .)

A third way to orient  $S^2$  is to find charts on  $S^2$  whose transition maps have positive Jacobian. We can do this as follows. Think of  $S^2 = \{(z, t) \in \mathbb{C} \mid |z|^2 + t^2 = 1\}$  and let  $\phi_1: S^2 - (0, -1) \rightarrow \mathbb{C}$  be  $\phi_1(z, t) = z/(1+t)$  (stereographic projection). let  $\phi_2: S^2 - (0, 1) \rightarrow \mathbb{C}$  be  $\phi_2(z, t) = \bar{z}/(1-t)$  (stereographic projection followed by complex conjugation). Then  $\phi_2 \phi_1^{-1}(z) = 1/\bar{z}$ . Letting  $z = x + iy$  we have  $\phi_2 \phi_1^{-1}(x, y) = (\frac{x}{(x^2+y^2)^2}, \frac{-y}{(x^2+y^2)^2})$

with Jacobian determinant  $\begin{pmatrix} \frac{y^2-x^2}{(x^2+y^2)^2} & \frac{-2xy}{(x^2+y^2)^2} \\ \frac{2xy}{(x^2+y^2)^2} & \frac{y^2-x^2}{(x^2+y^2)^2} \end{pmatrix}$  with determinant  $\frac{1}{(x^2+y^2)^2} > 0$ .

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**Bundles over  $S^1$ :** For any  $k$  there are only two bundles over  $S^1$ , up to bundle isomorphism, the trivial bundle and the Mobius band summed with a trivial bundle. The first is of course orientable and the second is not. To see this, suppose  $\pi: E \rightarrow S^1$  is a rank  $k$  bundle. Let  $f: [0, 1] \rightarrow S^1$  be  $f(t) = e^{2\pi ti}$ . Consider the induced bundle  $f^*(E) = \{(t, y) \in [0, 1] \times E \mid f(t) = \pi(y)\}$ . Since  $[0, 1]$  is contractible we know this bundle must be trivial so there is a bundle isomorphism  $g: [0, 1] \times \mathbb{R}^k \rightarrow f^*(E)$ . to be continued.

**orientability and the fundamental group:** Suppose  $x_0 \in X$ . Let  $\pi_1(X, x_0)$  be the fundamental group of  $X$ , this is the set of homotopy classes of maps  $(S^1, 1) \rightarrow (X, x_0)$ , i.e., loops starting and ending at  $x_0$ . to be continued.

**orientability of the connected sum:**

**orientability of compact surfaces:**

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$\omega_1 = 0$  iff orientable:

**Stieffel-Whitney classes:**

**alternating multilinear functions:**

4/18, 4/20

**differential forms on manifolds:** See Bredon 260-266.

**DeRham cohomology:**

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**Mayer-Vietoris sequence:**

See Bredon 287-288. If  $A$  and  $B$  are open subsets of a manifold  $X$  we get what is called a long exact sequence as follows:

$$\dots \xrightarrow{\beta_{k-1}} H_{\Omega}^{k-1}(A \cap B) \xrightarrow{\delta_{k-1}} H_{\Omega}^k(A \cup B) \xrightarrow{\alpha_k} H_{\Omega}^k(A) \oplus H_{\Omega}^k(B) \xrightarrow{\beta_k} H_{\Omega}^k(A \cap B) \xrightarrow{\delta_k} \dots$$

Exactness means that  $\ker \alpha_k = \text{Im } \delta_{k-1}$ ,  $\ker \beta_k = \text{Im } \alpha_k$ , and  $\ker \delta_k = \text{Im } \beta_k$  for all  $k$ . The map  $\alpha_k$  is defined by  $\alpha_k([\omega]) = [\omega|_A] \oplus [-\omega|_B]$ . The map  $\beta_k$  is defined by  $\beta_k([\mu] \oplus [\eta]) = [\mu|_{A \cap B} + \eta|_{A \cap B}]$ . To define the map  $\delta$ , first choose a smooth  $\phi: A \cup B \rightarrow [0, 1]$  so that  $\phi$  is 0 on a neighborhood of  $A - B$  and  $\phi$  is 1 on a neighborhood of  $B - A$  (using the smooth Urysohn lemma (2/16) on the closed subsets  $A - B$  and  $B - A$ ). Now define  $\delta_k(\sigma) = d\phi \wedge \sigma$ . Note that  $d\phi = 0$  wherever  $\phi$  is locally constant, in particular  $d\phi$  is 0 on a neighborhood of  $(A - B) \cup (B - A)$ . Consequently,  $d\phi \wedge \sigma$  is defined and smooth on all of  $A \cup B$  by setting it to be 0 on  $A - B$  and  $B - A$ .

We need to show these maps are independent of choices and the sequence is exact. To show  $\alpha_k$  is well-defined, if  $[\omega] = [\omega']$  then there is a  $k - 1$  form  $\tau$  so  $\omega + d\tau = \omega'$ . But then  $\omega'|_A = \omega|_A + d(\tau|_A)$  so  $[\omega'|_A] = [\omega|_A]$ . Similarly  $[\omega'|_B] = [\omega|_B]$  so  $\alpha_k$  is well defined. Similarly,  $\beta_k$  is well-defined. Now let us show  $\delta_k$  is well-defined. We have  $d\phi \wedge (\sigma + d\tau) = d\phi \wedge \sigma - d(d\phi \wedge \tau)$  so it is independent of the choice of the representative  $\sigma$ . If we choose a different  $\phi'$  then  $d\phi' \wedge \sigma = d\phi \wedge \sigma + d((\phi' - \phi)\sigma)$ . Note that  $(\phi' - \phi)\sigma$  is defined and smooth on all  $A \cup B$  since  $\phi' - \phi$  is 0 on a neighborhood of  $(A - B) \cup (B - A)$ .

**Mayer-Vietoris sequence is exact:** First let us show that  $\ker \beta = \text{Im } \alpha$  (for convenience, let us drop the subscript  $k$ ). Suppose  $\beta([\mu] \oplus [\eta]) = 0$  for some closed forms  $\mu$  on  $A$  and  $\eta$  on  $B$ . Then there is a form  $\sigma$  on  $A \cap B$  so that  $\mu|_{A \cap B} + \eta|_{A \cap B} = d\sigma$ . Consider the form  $\omega = (1 - \phi)\mu - \phi\eta - d\phi \wedge \sigma$  which is defined on  $A \cup B$ . (Define it to be  $\mu$  on  $A - B$  and  $-\eta$  on  $B - A$ .)  $\omega$  is closed because  $d\omega = -d\phi \wedge \mu - d\phi \wedge \eta - (-d\phi \wedge d\sigma) = 0$ . Then

$$[\mu] \oplus [\eta] - \alpha(\omega) = [\phi(\mu + \eta) + d\phi \wedge \sigma] \oplus [(1 - \phi)(\mu + \eta) - d\phi \wedge \sigma] = [d(\phi\sigma)] \oplus [d((1 - \phi)\sigma)] = 0$$

so  $\ker \beta \subset \text{Im } \alpha$ . On the other hand,  $\beta\alpha([\omega]) = [\omega|_{A \cap B}] + [-\omega|_{A \cap B}] = 0$ , so  $\ker \beta = \text{Im } \alpha$ .

Now let us show that  $\ker \alpha = \text{Im } \delta$ . Suppose  $\alpha([\omega]) = 0$  for some closed form  $\omega$  on  $A \cup B$ . Then  $\omega|_A = d\mu$  and  $\omega|_B = d\eta$ . So  $\omega = (1 - \phi)d\mu + \phi d\eta$ . Then

$$\omega - d((1 - \phi)\mu + \phi\eta) = \omega - (1 - \phi)d\mu - \phi d\eta + d\phi \wedge (\mu - \eta) = d\phi \wedge (\mu - \eta)$$

so  $[\omega] = \delta([\mu|_{A \cap B} - \eta|_{A \cap B}])$ . So  $\ker \alpha \subset \text{Im } \delta$ . But  $\alpha\delta(\sigma) = [d(\phi\sigma)] + [d((1-\phi)\sigma)] = 0 \oplus 0$  so  $\ker \alpha = \text{Im } \delta$ .

Finally let us show that  $\ker \delta = \text{Im } \beta$ . Suppose  $\delta([\sigma]) = 0$  for a closed form  $\sigma$  on  $A \cap B$ . Then  $d\phi \wedge \sigma = d\omega$  for some form  $\omega$  on  $A \cup B$ . Then  $\mu = -\omega + \phi\sigma$  is a closed form on  $A$  and  $\eta = \omega + (1-\phi)\sigma$  is a closed form on  $B$ . But then  $[\sigma] = \beta([\mu] \oplus [\eta])$ . So  $\ker \delta \subset \text{Im } \beta$ . But  $\delta\beta([\mu] \oplus [\eta]) = [d((\phi-1)\mu + \phi\eta)]$ . So  $\ker \delta = \text{Im } \beta$ .

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**five lemma:**

See Bredon, 181-182. Suppose we have a commutative diagram of abelian groups and homomorphisms as indicated below, where the rows are exact (i.e.,  $\ker \alpha_i = \text{Im } \alpha_{i-1}$  and  $\ker \beta_i = \text{Im } \beta_{i-1}$ ), and so  $\gamma_1, \gamma_2, \gamma_4$ , and  $\gamma_5$  are isomorphisms. Then  $\gamma_3$  is an isomorphism also.

$$\begin{array}{ccccccccc} A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & A_3 & \xrightarrow{\alpha_3} & A_4 & \xrightarrow{\alpha_4} & A_5 \\ \downarrow \gamma_1 & & \downarrow \gamma_2 & & \downarrow \gamma_3 & & \downarrow \gamma_4 & & \downarrow \gamma_5 \\ B_1 & \xrightarrow{\beta_1} & B_2 & \xrightarrow{\beta_2} & B_3 & \xrightarrow{\beta_3} & B_4 & \xrightarrow{\beta_4} & B_5 \end{array}$$

**finite handlebodies:** We say a manifold  $X$  without boundary is a finite handlebody if there is a proper positive Morse function on  $X$  with only finitely many critical points. Equivalently, we may build up  $X$  by starting from the empty manifold and adding finitely many handles, and then deleting the boundary. For example if  $X$  is compact and  $f: X \rightarrow \mathbb{R}$  is a Morse function then  $X \cap f^{-1}((-\infty, a))$  is a finite handlebody for any  $a$ . It will be convenient for us to prove things by induction on the number of handles (i.e., critical points).

**DeRham cohomology doesn't change if we  $\times \mathbb{R}$ :** We will show that if  $X$  is a finite handlebody and  $\pi: X \times \mathbb{R} \rightarrow X$  is projection, then  $\pi^*: H_\Omega^k(X) \rightarrow H_\Omega^k(X \times \mathbb{R})$  is an isomorphism. (In fact this is true for any manifold  $X$ .) The first step is to show that it is true for  $X = \mathbb{R}^n$ .

So let  $\omega$  be a closed  $k$  form on  $\mathbb{R}^n \times \mathbb{R}$ . We may write  $\omega$  uniquely as

$$\omega(x, t) = \sum_{|I|=k} f_I(x, t) dx_I + \sum_{|I|=k-1} g_I(x, t) dt \wedge dx_I$$

where  $I = (i_1, i_2, \dots, i_\ell)$  is a multiindex with  $i_1 < i_2 < \dots < i_\ell$  and  $|I| = \ell$ . Let  $\bar{g}_I(x, t) = \int_0^t g_I(x, s) ds$  and define  $\sigma = \sum_{|I|=k-1} \bar{g}_I(x, t) dx_I$ . Then

$$d\sigma(x, t) = \sum_{|I|=k-1} d\bar{g}_I \wedge dx_I = \sum_{|I|=k-1} g_I(x, t) dt \wedge dx_I + \sum_{|I|=k} h_I(x, t) dx_I$$

for some functions  $h_I$ . Then

$$\begin{aligned} 0 &= d\omega - d^2\sigma = d(\omega - d\sigma) = d(\sum_{|I|=k} (f_I(x, t) - h_I(x, t)) dx_I) \\ &= \sum_{|I|=k} (df_I - dh_I) \wedge dx_I = \sum_{|I|=k} (\partial f_I / \partial t - \partial h_I / \partial t) dt \wedge dx_I + \sum_{|I|=k+1} u_I dx_I \end{aligned}$$

for some functions  $u_I$ . So we have  $\partial f_I / \partial t - \partial h_I / \partial t = 0$  at all points which means  $f_I(x, t) - h_I(x, t) = v_I(x)$  for some functions  $v_I$ . Also  $u_I = 0$ . Consider the form  $\eta$  on  $\mathbb{R}^n$  given by  $\eta(x) = \sum_{|I|=k} v_I(x) dx_I$ . Then  $\pi^*\eta(x, t) = \eta(x) = \omega - d\sigma$ .  $\eta$  is closed because  $d\eta = \sum_{|I|=k+1} u_I dx_I = 0$ . So  $\pi^*$  is onto because  $[\omega] = [\omega - d\sigma] = \pi^*[\eta]$ . But consider  $\rho: \mathbb{R}^n \rightarrow \mathbb{R}^n \times \mathbb{R}$  given by  $\rho(x) = (x, 0)$ . Then  $\pi\rho$  is the identity so  $\rho^*\pi^*$  is the identity so  $\pi^*$  is one to one. Hence  $\pi^*$  is an isomorphism.

As a consequence,  $H_\Omega^k(\mathbb{R}^n) \approx H_\Omega^k(\mathbb{R}^0) = \begin{cases} 0 & \text{if } k \neq 0 \\ \mathbb{R} & \text{if } k = 0 \end{cases}$ . So every closed  $k > 0$  form on  $\mathbb{R}^n$  is exact. By the way, the above proof when applied to  $k = 1$  is a common multivariable calculus procedure for finding the potential function for a conservative vector field.

Next we show that  $\pi^*: H_\Omega^k(X) \rightarrow H_\Omega^k(X \times \mathbb{R})$  is an isomorphism if  $X = S^\ell \times \mathbb{R}^n$ . To see this, let  $A = (X - \text{north pole}) \times \mathbb{R}^n$  and  $B = (X - \text{south pole}) \times \mathbb{R}^n$ . Then  $A \cup B = X$  and  $A \cap B = S^{\ell-1} \times \mathbb{R}^{n+1}$ ,  $A = \mathbb{R}^{n+\ell}$ ,  $B = \mathbb{R}^{n+\ell}$  so by the Mayer-Vietoris sequence, the five lemma, and induction on  $\ell$  we see that  $\pi^*: H_\Omega^k(X) \rightarrow H_\Omega^k(X \times \mathbb{R})$  is an isomorphism. In particular, we have a commutative diagram with exact columns

$$\begin{array}{ccc}
H_{\Omega}^{k-1}(A) \oplus H_{\Omega}^{k-1}(B) & \xrightarrow{\pi^* \oplus \pi^*} & H_{\Omega}^{k-1}(A \times \mathbb{R}) \oplus H_{\Omega}^{k-1}(B \times \mathbb{R}) \\
\downarrow \beta_{k-1} & & \downarrow \beta'_{k-1} \\
H_{\Omega}^{k-1}(A \cap B) & \xrightarrow{\pi^*} & H_{\Omega}^{k-1}((A \cap B) \times \mathbb{R}) \\
\downarrow \delta_{k-1} & & \downarrow \delta'_{k-1} \\
H_{\Omega}^k(A \cup B) & \xrightarrow{\pi^*} & H_{\Omega}^k((A \cup B) \times \mathbb{R}) \\
\downarrow \alpha_k & & \downarrow \alpha'_k \\
H_{\Omega}^k(A) \oplus H_{\Omega}^k(B) & \xrightarrow{\pi^* \oplus \pi^*} & H_{\Omega}^k(A \times \mathbb{R}) \oplus H_{\Omega}^k(B \times \mathbb{R}) \\
\downarrow \beta_k & & \downarrow \beta'_k \\
H_{\Omega}^k(A \cap B) & \xrightarrow{\pi^*} & H_{\Omega}^k((A \cap B) \times \mathbb{R})
\end{array}$$

The first and fourth row maps are isomorphisms since  $A$  and  $B$  are diffeomorphic to  $\mathbb{R}^{n+\ell}$  and by induction on  $\ell$  the second and fifth row maps are isomorphisms. So by the five lemma, the third row map is an isomorphism.

In fact, since  $H_{\Omega}^k(A) = H_{\Omega}^k(B) = 0$  for  $k > 0$ , the exact sequence in the first column shows us that  $\delta_{k-1}: H_{\Omega}^{k-1}(A \cap B) \rightarrow H_{\Omega}^k(A \cup B)$  is an isomorphism if  $k > 1$ . We can use this exact sequence to compute  $H_{\Omega}^k(S^{\ell} \times \mathbb{R}^n) = \begin{cases} 0 & \text{if } k \neq 0, \ell \\ \mathbb{R} & \text{if } \ell \neq 0 \text{ and } k = 0, \ell. \\ \mathbb{R}^2 & \text{if } k = \ell = 0 \end{cases}$

To see this, we may as well assume  $n = 0$ . Note for any manifold  $X$  that  $H_{\Omega}^0(X) = \ker d = \mathbb{R}^k$  where  $k$  is the number of connected components of  $X$ , since  $\ker d =$  the space of locally constant functions on  $X$ . So the case  $\ell = 0$  is clear since  $S^0$  is two points. If  $\ell = 1$  we have the exact sequence

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H_{\Omega}^0(S^1 \times \mathbb{R}^n) & \xrightarrow{\alpha_0} & H_{\Omega}^0(A) \oplus H_{\Omega}^0(B) & \xrightarrow{\beta_0} & H_{\Omega}^0(A \cap B) & \xrightarrow{\delta_0} & H_{\Omega}^1(S^1 \times \mathbb{R}^n) & \xrightarrow{\alpha_1} & 0 \\
& & \parallel & & \parallel & & \parallel & & & & \\
& & \mathbb{R} & & \mathbb{R}^2 & & \mathbb{R}^2 & & & & 
\end{array}$$

from which we see that  $H_{\Omega}^1(S^1 \times \mathbb{R}^n) = \mathbb{Z}$ . If  $\ell > 1$  then  $\beta_0$  is onto, so we know  $\delta_0$  is the zero map, so  $\alpha_1$  is injective, but maps to the zero group, hence  $H_{\Omega}^1(S^{\ell} \times \mathbb{R}^n) = 0$  if  $\ell > 1$ . But we have the exact sequence for  $k \geq 1$

$$0 \xrightarrow{\beta_k} H_{\Omega}^k(A \cap B) \xrightarrow{\delta_k} H_{\Omega}^{k+1}(S^{\ell} \times \mathbb{R}^n) \xrightarrow{\alpha_{k+1}} 0$$

from which we see that

$$H_{\Omega}^{k+1}(S^{\ell} \times \mathbb{R}^n) \approx H_{\Omega}^k(A \cap B) \approx H_{\Omega}^k(S^{\ell-1} \times \mathbb{R}^{n+1})$$

which proves our result by induction.

Finally let us show the theorem for any finite handlebody  $X$ . We may write  $X = A \cup B$  where  $A$  is a finite handlebody with one less handle than  $X$  and  $B$  is the last handle added to  $X$ . Then  $A \cap B = S^{\ell-1} \times \mathbb{R}^{n-\ell+1}$  if  $B$  is an  $\ell$  handle. The theorem is true for  $B$  (since  $B = \mathbb{R}^n$ ) and is true for  $A \cap B$  by the above, and is true for  $A$  by induction on the number of handles. So by the Mayer-Vietoris sequence and the five lemma it is true for  $X$ .

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We did some of the calculation of  $H_{\Omega}^*(S^{\ell})$  given in the 4/25 notes. I also said some words relating cup products to the wedge of forms. I talked about the cup product on complex projective space  $\mathbb{C}\mathbb{P}^n$ . In particular I told you that the cup product  $\alpha \cup \beta$  is Poincare dual to the transverse intersection of the Poincare duals of  $\alpha$  and  $\beta$ . I told you that  $H_{\Omega}^2(\mathbb{C}\mathbb{P}^n) = \mathbb{R}$  is generated by the Poincare dual  $\alpha$  to  $\mathbb{C}\mathbb{P}^{n-1} \subset \mathbb{C}\mathbb{P}^n$  and hence  $\alpha^k$  is Poincare dual to  $\mathbb{C}\mathbb{P}^{n-k}$ . I gave no reasons for anything though.

I also gave a conjecture for the 2-form corresponding to the generator  $\alpha$  of  $H_{\Omega}^2(\mathbb{C}\mathbb{P}^n) = \mathbb{R}$ . But anyway, Bredon gives this form on pages 292-294.

I continued calculating the deRham cohomology of the sphere. I started to tell you a  $\ell$  form generating  $H_{\Omega}^{\ell}(S^{\ell})$  and should have continued. If  $S^{\ell} = \{(x_0, x_1, \dots, x_{\ell}) \mid x_0^2 + \dots + x_{\ell}^2 = 1\}$  we may let  $\omega$  be the restriction of  $x_0 dx_1 \wedge dx_2 \wedge \dots \wedge dx_{\ell} - x_1 dx_0 \wedge dx_2 \wedge \dots \wedge dx_{\ell}$  to  $S^{\ell}$ . Note  $d\omega = 0$  for dimension reasons (it is an  $\ell + 1$  form on an  $\ell$  dimensional manifold). But by Stokes' theorem we can see that  $\omega \neq d\sigma$  for any form  $\sigma$ , since then  $\int_{S^{\ell}} \omega = \int_{\partial S^{\ell}} \sigma = 0$  since  $\partial S^{\ell}$  is empty. But  $S^{\ell} = \partial D^{\ell+1}$  where  $D^{\ell+1}$  is the unit disc in  $\mathbb{R}^{\ell+1}$ . So by Stokes' theorem,  $\int_{S^{\ell}} \omega = \int_{D^{\ell+1}} d\omega$ . but  $d\omega = 2dx_0 \wedge \dots \wedge dx_{\ell}$  so  $\int_{D^{\ell+1}} d\omega$  is twice the volume of the unit disc in  $\mathbb{R}^{\ell+1}$ , and in particular is nonzero.

I also proved Stokes' theorem, generally following the proof in Bredon, page 267-268.

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