MATH 241 FINAL EXAM ANSWERS December 15, 2003

INSTRUCTIONS: Work each problem 1-8 on a separate answer sheet. Be sure to put your name, your TA's name and the problem number on each sheet. Show all your work. Clearly indicate your answers by circling each answer. No graphing or programmable calculators are allowed, but you may use an ordinary calculator and an 8.5×11 sheet of notes. After completing the exam, write out and sign the honor pledge on one of your answer sheets.

1. [25] Let L_1 be the line with equation $\frac{x-1}{3} = \frac{y}{4} = z + 1$ and let L_2 be the line with equation $x = 1 - 2t$, $y = t$, $z = -1 + 3t$.
   a) Find the point of intersection of L_1 and L_2.
   b) Find the distance between L_1 and $(0, 1, 0)$.
   c) Find the cosine of the angle between L_1 and L_2.

2. [25] The position of a particle at time t is $\mathbf{r}(t) = t\mathbf{i} - t^2\mathbf{j} + \frac{2}{3}t^3\mathbf{k}$. Let C be the curve parameterized by this $\mathbf{r}(t)$, $1 \le t \le 3$.
   a) Find the particle's velocity and acceleration at any time t.
   b) Find the tangential component of acceleration a_T when $t = 1$.
   c) Find the curvature κ and unit tangent vector $\mathbf{T}$ of C when $t = 1$.
   d) Find the length of C.

3. [25] Answer both parts.
   a) Find the equation of the tangent plane to the surface $e^{xy}z - z^2 + 2 = 0$ at the point $(0, 0, 2)$.
   b) The surface $e^{xy}z - z = 0$ does *not* have a tangent plane at the point $(0, 0, 0)$. Explain why.

4. [25] Let D be the solid region in the first octant below the paraboloid $z = 4 - x^2 - y^2$. Suppose D has mass density $\delta(x, y, z) = 1 + z$.
   a) Write down an integral in rectangular coordinates which calculates the total mass of D.
   b) Write down an integral in cylindrical coordinates which calculates the total mass of D.
   c) Write down an integral in spherical coordinates which calculates the total mass of D.
   d) Evaluate one of the integrals a,b, or c above.

5. [25] Evaluate

$$\int_C y^2\,dx - xy\,dy,$$

where C is the the closed triangle with vertices at $(0, 0)$, $(2, 0)$, and $(0, 4)$, oriented counterclockwise.

6. [25] Let Σ be the portion of the cylinder $x^2 + y^2 = 1$ between the planes $z = 0$ and $z = 2$, and let $\mathbf{F} = xy^2\mathbf{i} + x^2\mathbf{j}$. Evaluate the flux of $\mathbf{F}$ through Σ, i.e., $\iint_\Sigma \mathbf{F} \cdot \mathbf{n}\,dS$, where $\mathbf{n}$ is the unit outward normal vector to Σ. Explain your method of calculation.

7. [25] Find the surface area of the portion of the plane $2x + 3y + z = 25$ which lies above the elliptical region $(x + y)^2 + (x + 3y)^2 \le 4$.

8. [25] Let $f(x, y) = x^3y - 3xy^2 + 2x$. Suppose you want to find the maximum and minimum values of f on the circle $x^2 + y^2 = 1$.

a) Write down explicit equations that x and y must satisfy at the point(s) where f achieves its maximum and minimum values on the circle.

b) Find the maximum and minimum values of f on the circle $x^2 + y^2 = 1$ and the points where they are achieved. You may use results from the following MATLAB sessions. (Note that because of round-off error, numbers which really should be real sometimes show up in MATLAB as complex numbers with a miniscule imaginary part.)

c) Jenny from Prof. Rosenberg's class and Jack from Prof. King's class tried to solve this with the MATLAB sessions below, which only differ in the one line indicated. Explain why their different methods gave the same values for `xc` and `yc`.

```
>> syms x y lam
>> f = x^3*y-3*x*y^2+2*x; g = x^2+y^2;
>> fx=diff(f,x); fy=diff(f,y); gx = diff(g,x); gy=diff(g,y);
>> ff = inline(vectorize(f));
>> [xa, ya] = solve(fx,fy);
>> xa=double(xa); ya=double(ya);
>> [xa, ya, ff(xa,ya)]
ans =
        0                -0.8165              0
        0                 0.8165              0
 1.0466 + 1.0466i        0 + 0.3651i   1.6746 + 1.6746i
-1.0466 + 1.0466i        0 - 0.3651i  -1.6746 + 1.6746i
 1.0466 - 1.0466i        0 - 0.3651i   1.6746 - 1.6746i
-1.0466 - 1.0466i        0 + 0.3651i  -1.6746 - 1.6746i
```

At this point Jenny gives the command `>> [xc, yc] = solve(fx*gy-fy*gx, g-1);` and Jack gives the command `>> [lamc, xc, yc] = solve(fx-lam*gx, fy-lam*gy, g-1);`. After that, their MATLAB sessions are identical:

```
>> xc = double(xc); yc = double(yc);
>> [xc, yc, ff(xc,yc)]
 0.2913 - 0.0000i   0.9566 + 0.0000i  -0.1935 - 0.0000i
-0.2913 + 0.0000i   0.9566 + 0.0000i   0.1935 + 0.0000i
 0.9930 + 0.0000i   0.1182 - 0.0000i   2.0601 + 0.0000i
-0.9930 - 0.0000i   0.1182 - 0.0000i  -2.0601 - 0.0000i
 0.3949 + 0.0000i  -0.9187 - 0.0000i  -0.2667 - 0.0000i
-0.3949 - 0.0000i  -0.9187 - 0.0000i   0.2667 + 0.0000i
 0.0000 - 2.1885i  -2.4061 - 0.0000i   0.0000 + 8.4132i
-0.0000 + 2.1885i  -2.4061 - 0.0000i  -0.0000 - 8.4132i
```
