MATH 241 FINAL EXAM ANSWERS December 15, 2003

INSTRUCTIONS: Work each problem 1-8 on a separate answer sheet. Be sure to put your name, your TA's name and the problem number on each sheet. Show all your work. Clearly indicate your answers by circling each answer. No graphing or programmable calculators are allowed, but you may use an ordinary calculator and an 8.5×11 sheet of notes. After completing the exam, write out and sign the honor pledge on one of your answer sheets.

1. [25] Let L_1 be the line with equation $\frac{x-1}{3} = \frac{y}{4} = z+1$ and let L_2 be the line with equation $x = 1-2t$, $y = t$, $z = -1+3t$.

a) Find the point of intersection of L_1 and L_2.

It is immmediate from the equations that both lines go through $(1, 0, -1)$. *So* $(1, 0, -1)$ *is the point of intersection.*

b) Find the distance between L_1 and $(0, 1, 0)$.

The vector from $(1, 0, -1)$ *to* $(0, 1, 0)$ *is* $(-1, 1, 1)$ *so the distance is*

$$\frac{||\ (-1,1,1) \times (3,4,1)\ ||}{||\ (3,4,1)\ ||} = \frac{||\ (-3,4,-7)\ ||}{||\ (3,4,1)\ ||} = \frac{\sqrt{3^2+4^2+7^2}}{\sqrt{3^2+4^2+1^2}} = \sqrt{\frac{74}{26}} = \sqrt{\frac{37}{13}}$$

c) Find the cosine of the angle between L_1 and L_2.

It is

$$\frac{(3,4,1)\cdot(-2,1,3)}{||\ (3,4,1)\ ||\,||\ (-2,1,3)\ ||} = \frac{1}{\sqrt{26}\sqrt{14}} = \frac{1}{\sqrt{364}}$$

2. [25] The position of a particle at time t is $\mathbf{r}(t) = t\mathbf{i} - t^2\mathbf{j} + \frac{2}{3}t^3\mathbf{k}$. Let C be the curve parameterized by this $\mathbf{r}(t)$, $1 \le t \le 3$.

a) Find the particle's velocity and acceleration at any time t.

The velocity is $\mathbf{r}'(t) = \mathbf{i} - 2t\mathbf{j} + 2t^2\mathbf{k}$ *and the acceleration is* $\mathbf{r}''(t) = -2\mathbf{j} + 4t\mathbf{k}$.

b) Find the tangential component of acceleration a_T when $t = 1$.

There are two formulae you could use here. One is $a_T = \mathbf{v}\cdot\mathbf{a}/\ ||\ \mathbf{v}\ ||$. *At time* $t = 1$ *we have* $\mathbf{v}(1) = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ *and* $\mathbf{a}(1) = -2\mathbf{j} + 4\mathbf{k}$ *so*

$$a_T = \frac{(1)(0) + (-2)(-2) + (2)(4)}{\sqrt{1^2+2^2+2^2}} = \frac{12}{3} = 4$$

You could also use $a_T = d(\text{speed})/dt$. *The speed is*

$$\sqrt{1^2 + (-2t)^2 + (2t^2)^2} = \sqrt{1 + 4t^2 + 4t^4} = 1 + 2t^2$$

So $a_T = 4t$. *When* $t = 1$ *we get* $a_T = 4$.

c) Find the curvature κ and unit tangent vector $\mathbf{T}$ of C when $t = 1$.

From b) we get $\| \mathbf{v} \| = 3$ *when* $t = 1$. *We then have*

$$\mathbf{T} = \mathbf{v} / \| \mathbf{v} \| = (1, -2, 2)/3 = (1/3, -2/3, 2/3)$$

We know $\kappa = a_N / \| \mathbf{v} \|^2$. *We have a couple ways to find* a_N. *We have*

$$a_N = \sqrt{\| \mathbf{a} \|^2 - a_T^2} = \sqrt{0^2 + 2^2 + 4^2 - 4^2} = 2$$

when $t = 1$. *We could also use*

$$a_N = \frac{\| \mathbf{v} \times \mathbf{a} \|}{\| \mathbf{v} \|} = \frac{\| (1, -2, 2) \times (0, -2, 4) \|}{3} = \frac{\| (-4, -4, -2) \|}{3} = 2$$

So in the end $\kappa = 2/3^2 = 2/9$.

d) Find the length of C.

The length is the integral of the speed which from part b) is $1 + 2t^2$. *So the length is*

$$\int_1^3 1 + 2t^2 \, dt = t + \frac{2}{3} t^3 \Big]_1^3 = 3 + 18 - (1 + 2/3) = \frac{58}{3} \quad \text{or} \quad 19\frac{1}{3}$$

3. [25] Answer both parts.

a) Find the equation of the tangent plane to the surface $e^{xy} z - z^2 + 2 = 0$ at the point $(0, 0, 2)$.

The gradient of $e^{xy} z - z^2 + 2$ *is* $(yze^{xy},\ xze^{xy},\ e^{xy} - 2z)$ *which is* $(0, 0, -3)$ *at* $(0, 0, 2)$. *So the tangent plane has equation* $z = 2$.

b) The surface $e^{xy} z - z = 0$ does *not* have a tangent plane at the point $(0, 0, 0)$. Explain why.

The gradient of $f(x, y, z) = e^{xy} z - z$ *is* $(yze^{xy},\ xze^{xy},\ e^{xy} - 1)$ *which is* $(0, 0, 0)$ *at the origin. Since the gradient of* f *is zero at the origin we are not guaranteed to have a tangent plane to* $f^{-1}(0)$ *at the origin. But that is not enough to show there is no tangent plane*[1]. *Solving* $0 = e^{xy} z - z = z(e^{xy} - 1)$ *we see that* $z = 0$ *or* $e^{xy} = 1$ *which means* $xy = 0$. *So this surface is the three planes* $z = 0$, $x = 0$, $y = 0$. *So there could be no tangent plane at the origin since any tangent plane would have to contain all three planes.*

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[1] *For example, the gradient of* z^2 *is 0 at the origin, but nevertheless* $z^2 = 0$ *has a tangent plane at the origin.*

4. [25] Let D be the solid region in the first octant below the paraboloid $z = 4 - x^2 - y^2$. Suppose D has mass density $\delta(x, y, z) = 1 + z$.

a) Write down an integral in rectangular coordinates which calculates the total mass of D.

The projection of D to the xy plane is the region in the first quadrant inside the circle $x^2 + y^2 = 4$. So the total mass is

$$\int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{4-x^2-y^2} 1 + z \, dzdydx$$

Of course there are five other possible orders of integration. Here is another one:

$$\int_0^4 \int_0^{\sqrt{4-z}} \int_0^{\sqrt{4-z-y^2}} 1 + z \, dxdydz$$

b) Write down an integral in cylindrical coordinates which calculates the total mass of D.

$$\int_0^{\pi/2} \int_0^2 \int_0^{4-r^2} (1 + z)r \, dzdrd\theta$$

c) Write down an integral in spherical coordinates which calculates the total mass of D.

The θ and ϕ limits are immediate, $0 \le \theta \le \pi/2$ and for any θ, we have $0 \le \phi \le \pi/2$. To figure out the maximum ρ we must solve for ρ in the equation of the paraboloid, $\rho \cos\phi = 4 - \rho^2 \sin^2\phi$. So $\rho^2 \sin^2\phi + \rho\cos\phi - 4 = 0$ and by the quadratic formula,

$$\rho = \frac{-\cos\phi \pm \sqrt{\cos^2\phi + 16\sin^2\phi}}{2\sin^2\phi} = \frac{-\cos\phi \pm \sqrt{1 + 15\sin^2\phi}}{2\sin^2\phi}$$

We do not take the negative square root since that would make $\rho < 0$. So in the end, the mass is:

$$\int_0^{\pi/2} \int_0^{\pi/2} \int_0^{\frac{-\cos\phi + \sqrt{1+15\sin^2\phi}}{2\sin^2\phi}} (1 + \rho\cos\phi)\rho^2 \sin\phi \, d\rho d\phi d\theta$$

d) Evaluate one of the integrals a,b, or c above.

The integral b) is obviously the easiest.

$$\int_0^{\pi/2} \int_0^2 \int_0^{4-r^2} (1 + z)r \, dzdrd\theta = \int_0^{\pi/2} \int_0^2 rz + \frac{rz^2}{2} \Big]_0^{4-r^2} drd\theta$$

$$= \int_0^{\pi/2} \int_0^2 r(4-r^2) + \frac{r(4-r^2)^2}{2}\, dr d\theta$$

Letting $u = 4 - r^2$, $du = -2r dr$ we get:

$$\int_0^{\pi/2} \int_0^2 r(4-r^2) + \frac{r(4-r^2)^2}{2}\, dr d\theta = -\frac{1}{2} \int_0^{\pi/2} \int_4^0 u + \frac{u^2}{2}\, du d\theta$$

$$= -\frac{1}{2} \int_0^{\pi/2} \frac{u^2}{2} + \frac{u^3}{6} \Big]_4^0 d\theta = \frac{1}{2} \int_0^{\pi/2} \frac{16}{2} + \frac{64}{6}\, d\theta = \frac{1}{2}\Big(8 + \frac{32}{3}\Big)\Big(\frac{\pi}{2}\Big) = \frac{14\pi}{3}$$

5. [25] Evaluate

$$\int_C y^2\, dx - xy\, dy,$$

where C is the the closed triangle with vertices at $(0,0)$, $(2,0)$, and $(0,4)$, oriented counterclockwise.

By Green's theorem, this is

$$\iint_R -y - 2y\, dA$$

where R is the region enclosed by C. The region R is the region in the first quadrant below the line $y = 4 - 2x$. So

$$\int_C y^2\, dx - xy\, dy = \int_0^2 \int_0^{4-2x} -3y\, dy dx = \int_0^2 -3y^2/2 \Big]_0^{4-2x} dx$$

$$= -\frac{3}{2} \int_0^2 16 - 16x + 4x^2\, dx = -\frac{3}{2}\Big(16x - 8x^2 + \frac{4x^3}{3}\Big) \Big]_0^2 = -\frac{3}{2}\Big(32 - 32 + \frac{32}{3}\Big) = -16$$

You could also calculate this directly. Notice that if C_1 is the line segment from $(0,0)$ to $(2,0)$ then $\int_{C_1} y^2\, dx - xy\, dy = 0$ since $y = 0$ on C_1. Also if C_2 is the line segment from $(0,4)$ to $(0,0)$ then $\int_{C_2} y^2\, dx - xy\, dy = \int_{C_2} y^2\, dx = 0$ since x and dx are 0 on C_1. So we only need calculate the integral on the line segment C_3 from $(2,0)$ to $(0,4)$. This is parameterized by $\mathbf{r}(t) = (2,0) + t(-2,4)$, $0 \le t \le 1$.

$$\int_C y^2\, dx - xy\, dy = \int_{C_3} y^2\, dx - xy\, dy = \int_0^1 ((4t)^2, -(2-2t)(4t)) \cdot (-2,4)\, dt$$

$$= \int_0^1 -2(16t^2) - 4(2-2t)(4t)\, dt = \int_0^1 -32t\, dt = -16t^2 \Big]_0^1 = -16$$

6. [25] Let Σ be the portion of the cylinder $x^2+y^2=1$ between the planes $z=0$ and $z=2$, and let $\mathbf{F}=xy^2\mathbf{i}+x^2\mathbf{j}$. Evaluate the flux of $\mathbf{F}$ through Σ, i.e., $\iint_\Sigma \mathbf{F}\cdot\mathbf{n}\,dS$, where $\mathbf{n}$ is the unit outward normal vector to Σ. Explain your method of calculation.

The straightforward way to solve this is to break Σ up into two pieces Σ_1 with $y\geq 0$ and Σ_2 with $y\leq 0$. The calculation isn't pretty and we'll look at alternatives later, but here it is. Note Σ_1 is the graph $y=\sqrt{1-x^2}$ for $0\leq z\leq 2$ and $-1\leq x\leq 1$. Then switching the roles of z and y we have

$$\mathbf{n}\,dS = (-\frac{\partial\sqrt{1-x^2}}{\partial x},\ 1,\ -\frac{\partial\sqrt{1-x^2}}{\partial z})\ dzdx = (\frac{x}{\sqrt{1-x^2}},\ 1,\ 0)\ dzdx$$

and so

$$\iint_{\Sigma_1}\mathbf{F}\cdot\mathbf{n}\,dS = \int_{-1}^{1}\int_0^2 (x(1-x^2),\ x^2, 0)\cdot(\frac{x}{\sqrt{1-x^2}},\ 1,\ 0)\ dzdx$$

$$= \int_{-1}^{1}\int_0^2 x^2\sqrt{1-x^2}+x^2\ dzdx = \int_{-1}^{1} 2x^2\sqrt{1-x^2}+2x^2\ dx$$

The $\int_{-1}^1 2x^2\ dx = 4/3$ is easy, but we'll need trig substitutions to integrate the other term. Let $x=\sin t$, for $-\pi/2\leq t\leq \pi/2$, then $dx=\cos t\ dt$ and

$$\int_{-1}^{1} 2x^2\sqrt{1-x^2}\ dx = \int_{-\pi/2}^{\pi/2} 2\sin^2 t\cos^2 t\ dt = \int_{-\pi/2}^{\pi/2}\frac{\sin^2(2t)}{2}\ dt$$

$$= \int_{-\pi/2}^{\pi/2}\frac{1-\cos(4t)}{4}\ dt = \frac{t}{4}-\frac{\sin(4t)}{16}\Big]_{-\pi/2}^{\pi/2} = \frac{\pi}{4}$$

So the flux through Σ_1 is $\frac{\pi}{4}+\frac{4}{3}$. Similarly we calculate the flux through Σ_2 where $y=-\sqrt{1-x^2}$. Since the normal points in the direction making y decrease we have

$$\mathbf{n}\,dS = (\frac{\partial(-\sqrt{1-x^2})}{\partial x},\ -1,\ \frac{\partial(-\sqrt{1-x^2})}{\partial z})\ dzdx = (\frac{x}{\sqrt{1-x^2}},\ -1,\ 0)\ dzdx$$

So

$$\iint_{\Sigma_2}\mathbf{F}\cdot\mathbf{n}\,dS = \int_{-1}^{1}\int_0^2 (x(1-x^2),\ x^2, 0)\cdot(\frac{x}{\sqrt{1-x^2}},\ -1,\ 0)\ dzdx$$

$$= \int_{-1}^{1}\int_0^2 x^2\sqrt{1-x^2}-x^2\ dzdx = \int_{-1}^{1} 2x^2\sqrt{1-x^2}-2x^2\ dx = \frac{\pi}{4}-\frac{4}{3}$$

Adding these together we get

$$\iint_\Sigma \mathbf{F}\cdot\mathbf{n}\,dS = \frac{\pi}{4}+\frac{4}{3}+\frac{\pi}{4}-\frac{4}{3} = \frac{\pi}{2}$$

There is another way to solve this using the divergence theorem. The surface Σ is not closed, so we can not use the divergence theorem directly, but we can close up Σ by adding *the two discs* Σ_3 *and* Σ_4 *where* Σ_3 *is the portion of the plane* $z = 0$ *inside* $x^2 + y^2 = 1$ *and* Σ_4 *is the portion of the plane* $z = 2$ *inside* $x^2 + y^2 = 1$*. Note that the fluxes through* Σ_3 *and* Σ_4 *are both zero since their normals are* $\pm\mathbf{k}$ *which is perpendicular to* $\mathbf{F} = xy^2\mathbf{i} + x^2\mathbf{j}$ *so* $\mathbf{F}\cdot\mathbf{n} = 0$ *on* Σ_3 *and* Σ_4*. By the divergence theorem*

$$\iint_{\Sigma\cup\Sigma_3\cup\Sigma_4} \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_D y^2\,dV$$

where D *is the solid region between* $z = 0$ *and* $z = 2$ *and inside* $x^2 + y^2 = 1$*. So*

$$\iint_\Sigma \mathbf{F}\cdot\mathbf{n}\,dS = \iint_{\Sigma\cup\Sigma_3\cup\Sigma_4} \mathbf{F}\cdot\mathbf{n}\,dS = \int_0^{2\pi}\int_0^1\int_0^2 r^2\sin^2\theta\; r\,dzdrd\theta$$

$$= \int_0^{2\pi}\int_0^1 r^3 z\sin^2\theta\,\Big]_0^2 drd\theta = \int_0^{2\pi}\int_0^1 2r^3\sin^2\theta\,drd\theta = \int_0^{2\pi} \frac{r^4\sin^2\theta}{2}\Big]_0^1 d\theta$$

$$= \int_0^{2\pi}\frac{\sin^2\theta}{2}\,d\theta = \int_0^{2\pi}\frac{1-\cos(2\theta)}{4}\,d\theta = \frac{\theta}{4} - \frac{\sin(2\theta)}{8}\Big]_0^{2\pi} = \frac{\pi}{2}$$

A third alternative is to view Σ *as a parameterized surface, with parameterization* $h(\theta, z) = (\cos\theta, \sin\theta, z)$, $0 \le z \le 2$, $0 \le \theta \le 2\pi$*. Then by the newest edition of Ellis&Gulick if you have that, or by notes handed out in Dr. King's section and perhaps in Dr. Rosenberg's section as well,*

$$\mathbf{n}\,dS = \pm\partial h/\partial\theta \times \partial h/\partial z\,dzd\theta$$

$$= \pm(-\sin\theta, \cos\theta, 0)\times(0, 0, 1)\,dzd\theta = \pm(\cos\theta, \sin\theta, 0)\,dzd\theta$$

We take the plus sign since we want $\mathbf{n}$ *to point outward. So*

$$\iint_\Sigma \mathbf{F}\cdot\mathbf{n}\,dS = \int_0^{2\pi}\int_0^2 (\cos\theta\sin^2\theta, \cos^2\theta, 0)\cdot(\cos\theta, \sin\theta, 0)\,dzd\theta$$

$$= \int_0^{2\pi}\int_0^2 \cos^2\theta\sin^2\theta + \sin\theta\cos^2\theta\,dzd\theta = \int_0^{2\pi}\int_0^2 \frac{\sin^2(2\theta)}{4} + \sin\theta\cos^2\theta\,dzd\theta$$

$$= \int_0^{2\pi}\int_0^2 \frac{1-\cos(4\theta)}{8} + \sin\theta\cos^2\theta\,dzd\theta = \int_0^{2\pi}\frac{1-\cos(4\theta)}{4} + 2\sin\theta\cos^2\theta\,d\theta$$

$$= \frac{\theta}{4} - \frac{\sin(4\theta)}{16} - \frac{2\cos^3\theta}{3}\Big]_0^{2\pi} = \frac{\pi}{2}$$

7. [25] Find the surface area of the portion of the plane $2x+3y+z=25$ which lies above the elliptical region $(x+y)^2+(x+3y)^2 \le 4$.

This plane is the graph $z = 25 - 2x - 3y$ so the surface area is

$$\iint_R \sqrt{(-2)^2+(-3)^2+1}\,dA = \iint_R \sqrt{14}\,dA$$

where R is the region $(x+y)^2+(x+3y)^2 \le 4$. We can integrate over R using the coordinate change $u = x+y$, $v = x+3y$. Then $\frac{\partial(u,v)}{\partial(x,y)} = (1)(3)-(1)(1) = 2$. So $\frac{\partial(x,y)}{\partial(u,v)} = 1/2$. So

$$\iint_R \sqrt{14}\,dA = \int_{-2}^{2}\int_{-\sqrt{4-v^2}}^{\sqrt{4-v^2}} \frac{\sqrt{14}}{2}\,dudv = \int_0^{2\pi}\int_0^2 \frac{\sqrt{14}r}{2}\,drd\theta$$

$$= \int_0^{2\pi} \frac{\sqrt{14}r^2}{4}\Big]_0^2 d\theta = \int_0^{2\pi} \sqrt{14}\,d\theta = 2\sqrt{14}\pi$$

You could also calculate $\int_{-2}^{2}\int_{-\sqrt{4-v^2}}^{\sqrt{4-v^2}} \frac{\sqrt{14}}{2}\,dudv$ immediately by noticing that it is $\frac{\sqrt{14}}{2}$ times the area of the disc of radius 2, or $\frac{\sqrt{14}}{2}\pi 2^2 = 2\sqrt{14}\pi$.

8. [25] Let $f(x,y) = x^3y - 3xy^2 + 2x$. Suppose you want to find the maximum and minimum values of f on the circle $x^2+y^2=1$.

a) Write down explicit equations that x and y must satisfy at the point(s) where f achieves its maximum and minimum values on the circle.

Let $g(x,y) = x^2+y^2$. By the method of Lagrange multipliers, since f and g are differentiable everywhere and ∇g is never zero on the circle $g=1$, explicit equations are:

$$3x^2y - 3y^2 + 2 = \lambda(2x)$$

$$x^3 - 6xy = \lambda(2y)$$

$$x^2+y^2 = 1$$

for some λ. Optionally, since the first two equations say ∇f is a multiple of ∇g, we can replace them by the equation $\nabla f \times \nabla g = 0$. This reduces to the single equation $f_xg_y - f_yg_x = 0$. So you could equally well write down explicit equations:

$$(3x^2y - 3y^2 + 2)2y - (x^3 - 6xy)2x = 0$$

$$x^2+y^2 = 1$$

b) Find the maximum and minimum values of f on the circle $x^2 + y^2 = 1$ and the points where they are achieved. You may use results from the following MATLAB sessions. (Note that because of round-off error, numbers which really should be real sometimes show up in MATLAB as complex numbers with a miniscule imaginary part.)

By part a) and the MATLAB output below, we see that possible max/min are given by the points `(xc,yc)`. *Thus possibilities are about* $(\pm.2913, .9566)$, $(\pm.993, .1182)$ *and* $(\pm.3949, -.9187)$. *The last column of the table gives the values of* f *at these points, so we see that the maximum is at about* $(.993, .1182)$ *where* $f(.993, .1182) \approx 2.0601$ *and the minimum is at about* $(-.993, .1182)$ *where* $f(-.993, .1182) \approx -2.0601$. *Note that the computation of* `(xa,ya)` *is unnecessary for this problem. These are the critical points of* f *which we would only need if we wanted the max/min of* f *on the disc* $x^2 + y^2 \leq 1$.

c) Jenny from Prof. Rosenberg's class and Jack from Prof. King's class tried to solve this with the MATLAB sessions below, which only differ in the one line indicated. Explain why their different methods gave the same values for `xc` and `yc`.

```
>> syms x y lam
>> f = x^3*y-3*x*y^2+2*x; g = x^2+y^2;
>> fx=diff(f,x); fy=diff(f,y); gx = diff(g,x); gy=diff(g,y);
>> ff = inline(vectorize(f));
>> [xa, ya] = solve(fx,fy);
>> xa=double(xa); ya=double(ya);
>> [xa, ya, ff(xa,ya)]
ans =
        0                -0.8165                0
        0                 0.8165                0
  1.0466 + 1.0466i        0 + 0.3651i    1.6746 + 1.6746i
 -1.0466 + 1.0466i        0 - 0.3651i   -1.6746 + 1.6746i
  1.0466 - 1.0466i        0 - 0.3651i    1.6746 - 1.6746i
 -1.0466 - 1.0466i        0 + 0.3651i   -1.6746 - 1.6746i
```

At this point Jenny gives the command `>> [xc, yc] = solve(fx*gy-fy*gx, g-1);` and Jack gives the command `>> [lamc, xc, yc] = solve(fx-lam*gx, fy-lam*gy, g-1);`. After that, their MATLAB sessions are identical:

```
>> xc = double(xc); yc = double(yc);
>> [xc, yc, ff(xc,yc)]
  0.2913 - 0.0000i   0.9566 + 0.0000i  -0.1935 - 0.0000i
 -0.2913 + 0.0000i   0.9566 + 0.0000i   0.1935 + 0.0000i
  0.9930 + 0.0000i   0.1182 - 0.0000i   2.0601 + 0.0000i
 -0.9930 - 0.0000i   0.1182 - 0.0000i  -2.0601 - 0.0000i
  0.3949 + 0.0000i  -0.9187 - 0.0000i  -0.2667 - 0.0000i
 -0.3949 - 0.0000i  -0.9187 - 0.0000i   0.2667 + 0.0000i
  0.0000 - 2.1885i  -2.4061 - 0.0000i   0.0000 + 8.4132i
 -0.0000 + 2.1885i  -2.4061 - 0.0000i  -0.0000 - 8.4132i
```

Jack's code solved the first set of equations we gave in part a). Jenny solved the second set. In our answer to part a) we showed that these two sets of equations were equivalent. (Jack finds points on the circle $g = 1$ where ∇f is a multiple of ∇g. Jenny finds points on the circle $g = 1$ where ∇f and ∇g are parallel. Since ∇g is never 0 on the circle, this is the same thing[2]*.)*

[2] *If the problem were different and ∇g were 0 on the level set then Jenny would have found those points where $\nabla g = 0$ but Jack would not. Jack would have had to find those points in a separate computation.*