Math 340 Final Exam December 19, 2002

1. (10) Find a basis for the vector space of lower triangular 2×2 matrices. What is the dimension of this vector space?

A basis is $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ *since any lower triangular matrix is a linear combination of these three matrices and they are linearly independent. The dimension is 3 since there are 3 basis elements.*

2. (20) Suppose a matrix A has row echelon form $\begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

a) What is rank(A)?

The rank is two since there are two pivots.

b) Find, if possible, a basis for the null space of A.

The vectors $(x, y, z, w)^T$ *in the null space satisfy* $x + 2y + w = 0$ *and* $z + 3w = 0$ *so* $(x, y, z, w) = y(-2, 1, 0, 0) + w(-1, 0, -3, 1)$. *So the two linearly independent vectors* $(-2, 1, 0, 0)^T, (-1, 0, -3, 1)^T$ *form a basis of the null space of A.*

c) Find, if possible, a basis for the column space of A.

The column space is unknown since row operations change the column space and we have no idea which row operations were done to A. Some of you pointed out correctly that the first and third columns of A form a basis of the column space of A since those are the pivot columns. In fact, any two columns of A except the first two will be a basis of the column space of A.

3. (30) Consider the curve C parameterized by $\mathbf{x}(t) = 4\sin t\mathbf{i} + 3\sin t\mathbf{j} + 5\cos t\mathbf{k}$, $0 \le t \le \pi$.

a) Find the tangential and normal components of acceleration a_T and a_N as functions of time.

$v(t) = 4\cos t\mathbf{i} + 3\cos t\mathbf{j} - 5\sin t\mathbf{k}$ *and* $a(t) = -4\sin t\mathbf{i} - 3\sin t\mathbf{j} - 5\cos t\mathbf{k}$. *The speed* $||v||$ *is a constant, 5. So* $a_T = d(speed)/dt = 0$. *Then* $a_N = \sqrt{||a||^2 - a_T^2} = ||a|| = 5$. *You could also have calculated* $a_T = v \cdot a/||v||$ *and* $a_N = ||v \times a||/||v||$.

b) Find the curvature κ when $t = \pi/3$.

$\kappa = a_N/||v||^2 = 5/25 = 1/5$

c) Find the length of C.

Length $= \int_0^\pi ||v(t)||\, dt = \int_0^\pi 5\, dt = 5\pi$.

d) Find $\int_C zdx + ydy - dz$

$$\int_C zdx + ydy - dz = \int_C (z, y, -1) \cdot \mathrm{d}s = \int_0^\pi (5\cos t, 3\sin t, -1) \cdot (4\cos t, 3\cos t, -5\sin t)\, dt$$

$$= \int_0^\pi 20\cos^2 t + 9\sin t\cos t + 5\sin t\, dt$$

$$= \int_0^\pi 10 + 10\cos(2t) + 9\sin t\cos t + 5\sin t\, dt$$

$$= 10t + 5\sin(2t) + 9\sin^2 t/2 - 5\cos t\ ]_0^\pi = 10\pi + 10$$

4. (20) Find all points on the surface $x^2+3yz = 5$ where the tangent plane is perpendicular to the line through the points $(1, 2, 3)$ and $(2, 1, 0)$. Find an equation of the tangent plane at one of those points.

The tangent plane to the level set is perpendicular to $\nabla(x^2 + 3yz)$ *so we want* $(2x, 3z, 3y)$ *to be parallel to* $(2, 1, 0) - (1, 2, 3) = (1, -1, -3)$. *Two vectors in* $\mathbb{R}^3$ *are parallel when their cross product is 0, so we could take the cross product and set it to 0, or else just solve* $(2x, 3z, 3y) = t(1, -1, -3)$. *I will use the latter approach so we see that* $t = 2x$ *so* $3z = -2x$ *and* $3y = -6x$. *So* $z = -2x/3$ *and* $y = -2x$. *Plugging in to* $x^2 + 3yz = 5$ *we get* $x^2 + 3(-2x)(-2x/3) = 5$ *or* $5x^2 = 5$. *Thus* $x = \pm 1$. *So the only points are* $(1, -2, -2/3)$ *and* $(-1, 2, 2/3)$. *At the first point the equation of the tangent plane is* $(x - 1) - (y + 2) - 3(z + 2/3) = 0$.

5. (20) Let S be the portion of the surface $z = 3 - \sqrt{x^2 + y^2}$ in the first octant. Let C be the boundary of S, oriented counterclockwise when viewed from above. Let $\mathbf{F}(x.y, z) = \sin x\mathbf{i} - xz\mathbf{j} + xy\mathbf{k}$.

a) Describe C, (a clear sketch is sufficient).

b) Find $\int_C \mathbf{F} \cdot \mathrm{d}s$.

We can use Stokes' Theorem to evaluate this, with the upward normal. $\mathrm{curl}\mathbf{F}(x, y, z) = 2x\mathbf{i} - y\mathbf{j} - z\mathbf{k}$ *and* $\mathbf{n}\, dS = (-\partial z/\partial x, -\partial z/\partial y, 1)\, dxdy$. *Then if* D *is the projection of* S *to the* xy *plane,*

$$\int_C \mathbf{F} \cdot \mathrm{d}s = \int\int_S \mathrm{curl}\mathbf{F} \cdot \mathrm{d}S$$

$$= \int\int_D (2x, -y, -(3 - \sqrt{x^2 + y^2}) \cdot (x/\sqrt{x^2 + y^2}, y/\sqrt{x^2 + y^2}, 1) dxdy$$

$$= \int\int_D 2x^2/r - y^2/r - 3 + r dxdy$$

$$= \int_0^{\pi/2} \int_0^3 2r^2 \cos^2\theta - r^2 \sin^2\theta - 3r + r^2\, dr d\theta$$

$$= \int_0^{\pi/2} \int_0^3 3r^2 \cos^2\theta - 3r\, dr d\theta$$

$$= \int_0^{\pi/2} r^3 \cos^2\theta - 3r^2/2]_0^3\, d\theta$$

$$= \int_0^{\pi/2} 27\cos^2\theta - 27/2\, d\theta$$

$$= \int_0^{\pi/2} (27/2)\cos(2\theta)\, d\theta$$

$$= (27/4)\sin(2\theta)]_0^{\pi/2} = 0$$

Some people almost suceeded in evaluating the line integral directly. The easiest way is to note that $\sin x\mathbf{i}$ *is conservative so* $\int_C \sin x dx = 0$. *Thus* $\int_C \mathbf{F}\cdot \mathrm{d}s = \int_C -xzdy + xydz$ *which can be readily evaluated by parameterizing each of the three pieces of* C, *or as follows without calculation. The integral* $\int_{C_1} -xzdy + xydz$ *is 0 for the quarter circle* C_1 *in the* xy *plane since both* z *and* dz *are 0. It is zero for the line in the* yz *plane since* $x = 0$. *It is zero for the line in the* xz *plane since both* y *and* dy *are 0.*

6. (20) Let D be the solid region inside the cylinder $r = 2$, above the surface $z = x^2 + y^2$, and below the surface $z = 10 + x^2 - 3y^2 + y$. Let S be the boundary of D oriented pointing outward from D. Let $\mathbf{F}(x, y, z) = z\mathbf{i} - x\mathbf{j} + y^2\mathbf{k}$.

a) Find $\int\int_S \mathbf{F}\cdot \mathrm{d}S$.

$\mathrm{div}\mathbf{F} = 0$ *so* $\int\int_S \mathbf{F}\cdot \mathrm{d}S = \int\int\int_D \mathrm{div}\mathbf{F}\, dV = 0$.

b) Let S' be obtained from S by deleting the surface $z = x^2+y^2$, $r \le 2$. Find $\int\int_{S'} \mathbf{F}\cdot \mathrm{d}S$.

Let S'' *be the surface* $z = r^2$, $r \le 2$, *oriented downward. Then* $0 = \int\int_S \mathbf{F}\cdot \mathrm{d}S = \int\int_{S'} \mathbf{F}\cdot \mathrm{d}S + \int\int_{S''} \mathbf{F}\cdot \mathrm{d}S$ *So* $\int\int_{S'} \mathbf{F}\cdot \mathrm{d}S = -\int\int_{S''} \mathbf{F}\cdot \mathrm{d}S$. *But*

$$\int\int_{S''} \mathbf{F}\cdot \mathrm{d}S = \int_0^{2\pi}\int_0^2 (r^2, -x, y^2)\cdot(2x, 2y, -1)\, r dr d\theta$$

$$= \int_0^{2\pi}\int_0^2 2r^4\cos\theta - 2r^3\cos\theta\sin\theta - r^3\sin^2\theta\, dr d\theta$$

$$= \int_0^{2\pi} .4r^5\cos\theta - .5r^4\cos\theta\sin\theta - .25r^4\sin^2\theta]_0^2\, d\theta$$

$$= \int_0^{2\pi} 12.8\cos\theta - 8\cos\theta\sin\theta - 4\sin^2\theta\, d\theta = -4\pi$$

Thus $\int\int_{S'} \mathbf{F}\cdot \mathrm{d}S = 4\pi$. *A sneakier way to do this is to let* S''' *be the disc* $z = 4$, $r \le 2$, *oriented downwards. Then* $S' \cup S'''$ *is a closed surface so by Gauss' theorem we know the flux through it is 0. Consequently* $\int\int_{S'} \mathbf{F}\cdot \mathrm{d}S = -\int\int_{S'''} \mathbf{F}\cdot \mathrm{d}S$. *But*

$$\int\int_{S'''} \mathbf{F}\cdot \mathrm{d}S = \int_0^{2\pi}\int_0^2 (4, -x, y^2)\cdot(0, 0, -1)\, r dr d\theta$$

$$= \int_0^{2\pi}\int_0^2 -r^3\sin^2\theta\, dr d\theta = -4\pi$$

You could also evaluate directly by decomposing S' *into two pieces* S_1 *and* S_2 *where* S_1 *is the top* $z = 10 + x^2 - 3y^2 + y$, $r \le 2$ *and* S_2 *is the side* $r = 2$, $4 \le z \le 10 + x^2 - 3y^2 + y$. *Then*

$$\int\int_{S_1} \mathbf{F}\cdot \mathrm{d}S = \int_0^{2\pi}\int_0^2 (10 + x^2 - 3y^2 + y, -x, y^2)\cdot(-2x, 6y - 1, 1)\, r dr d\theta$$

$$= \int_0^{2\pi} \int_0^2 -20x - 2x^3 + 6xy^2 - 2xy - 6xy + x + y^2 \, r dr d\theta$$

By symmetry the integrals of the first terms are zero (since they are odd functions with respect to x) and we are left with $\int_0^{2\pi} \int_0^2 y^2 \, r dr d\theta = 4\pi$. We can parameterize S_2 by $X(z,\theta) = (2\cos\theta, 2\sin\theta, z)$ $0 \le \theta \le 2\pi$ and $4 \le z \le 10 + 4\cos^2\theta - 12\sin^2\theta + 2\sin\theta$. Then $X_z \times X_\theta = -2\cos\theta\mathbf{i} - 2\sin\theta\mathbf{j}$ which is pointed inwards so we negate it for the correct orientation. Then

$$\int\int_{S_2} \mathbf{F} \cdot \mathrm{d}S = \int_0^{2\pi} \int_4^{10+4\cos^2\theta - 12\sin^2\theta + 2\sin\theta} (z, -x, y^2) \cdot (2\cos\theta, 2\sin\theta, 0) \, dz d\theta$$

$$= \int_0^{2\pi} \int_4^{10+4\cos^2\theta - 12\sin^2\theta + 2\sin\theta} 2z\cos\theta - 4\cos\theta\sin\theta \, dz d\theta$$

$$= \int_0^{2\pi} z^2\cos\theta - 4z\cos\theta\sin\theta]_4^{10+4\cos^2\theta - 12\sin^2\theta + 2\sin\theta} \, d\theta$$

This is 0 after laborious computation. Or you can use the change of variables $u = \sin\theta$, $du = \cos\theta d\theta$ and this is:

$$\int_0^{2\pi} \int_4^{10+4\cos^2\theta - 12\sin^2\theta + 2\sin\theta} 2z\cos\theta - 4\cos\theta\sin\theta \, dz d\theta$$

$$= \int_0^0 \int_4^{10+4-4u^2-12u^2+2u} 2z - 4u \, dz du = 0$$

since u goes from $\sin 0$ to $\sin 2\pi$ in other words 0 to 0. A sneaky other way to do it is to find a vector field G so $\mathbf{F} = \mathrm{curl}G$, we have some hope for this since $\mathrm{div}\mathbf{F} = 0$. When doing this you can simplify your search for G by assuming it is parallel to the xy plane, in other words $G(x,y,z) = M\mathbf{i} + N\mathbf{j}$. Then $\mathrm{curl}G = -N_z\mathbf{i} + M_z\mathbf{j} + (N_x - M_y)\mathbf{k}$ So we need to solve $-N_z = z$, $M_z = -x$, $N_x - M_y = y^2$. From the first two equations we get $N(x,y,z) = -z^2/2 + C(x,y)$ and $M(x,y,z) = -xz + D(x,y)$. Then the third equation becomes $C_x - D_y = y^2$ which we can solve by $C = xy^2$, $D = 0$. So we can let $G = (xy^2 - z^2/2)\mathbf{i} - xz\mathbf{j}$. Now by Stokes' theorem $\int\int_{S'} \mathbf{F} \cdot \mathrm{d}S = \int_C G \cdot \mathrm{ds} = \int_C (xy^2 - z^2/2)dy - xzdx$ where C is the boundary of S' which is just the circle $z = 4$, $r = 2$. Parameterize C by $x(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + 4\mathbf{k}$ and we get

$$\int_C (xy^2 - z^2/2)dy - xzdx = \int_0^{2\pi} (8\cos t\sin^2 t - 8)(2\cos t) - 8\cos t(-2\sin t) \, dt$$

$$= \int_0^{2\pi} 16\cos^2 t\sin^2 t - 16\cos t + 16\cos t\sin t \, dt = \int_0^{2\pi} 4\sin^2(2t) - 16\cos t + 16\cos t\sin t \, dt$$

$$= \int_0^{2\pi} 2(1 - \cos(4t)) - 16\cos t + 16\cos t\sin t \, dt = 4\pi$$

7. (20) Let D be the portion of the ellipse $(2x+y)^2+(x-y)^2 \leq 4$ above the line $y=x$. Find $\int\int_D x-y\,dA$. (Hint: use a change of variables $u=2x+y$, $v=x-y$.)

$\partial(u,v)/\partial(x,y) = \det\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} = -3$ *also* $y \geq x$ *if and only if* $v \geq 0$ *so the integral is*

$$\int_{-2}^{2}\int_{0}^{\sqrt{4-u^2}} v|\frac{1}{-3}|\,dvdu = \int_{-2}^{2}\int_{0}^{\sqrt{4-u^2}} v/3\,dvdu$$

$$= \int_{-2}^{2} v^2/6]_0^{\sqrt{4-u^2}}\,du$$

$$= \int_{-2}^{2}(4-u^2)/6\,du = 16/9$$

You could also evaluate readily by switching to polar coordinates or using dudv order.

8. (30) Let A and B be matrices so that $AB=0$. For each of the following statements, either show the statement is true or give a counterexample to show it is false.

a) Either A or B is zero.

False, for example $A = \begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix}$ *and* $B = \begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix}$.

b) The column space of B is a subspace of the null space of A.

True. First of all we know that the column space is a vector space, the only issue is whether the column space of B is contained in the null space of A. But any vector in the column space of B is of the form Bx for some vector x. Then Bx is in the null space of A since $A(Bx)=(AB)x=0x=0$.

c) $\mathrm{rank}A + \mathrm{rank}B \leq 7$ if A is a 6×7 matrix.

We know $\mathrm{rank}A + \dim(\text{null space of } A) = 7$. *But* $\mathrm{rank}B = \dim(\text{column space of } B) \leq \dim(\text{null space of } A)$ *by part b). So the result is true.*

d) Suppose $g:\mathbb{R}^n \to \mathbb{R}^k$ is differentiable and C is a curve in the level set $g^{-1}(0)$. Let T be a tangent vector to C at a point p of C. Then T is in the null space of $Dg(p)$.

This is true. Let $x(t)$ parameterize C and suppose $x(a)=p$. Then $x'(a)$ is tangent to C at p. We have $g(x(t))=0$ so $0=d/dt(g(x(t)))=Dg\,x'(t)$. Letting $A=Dg(p)$ and $B=x'(a)$ we see by part b) that $x'(a)$ is in the null space of $Dg(p)$. But T must be some multiple of $x'(a)$ so the result follows.

TURN OVER

9. (30) Short answer, true or false. (no justification required). A and B are nonsingular 7×7 matrices.

a) $(AB)^{-1} =$_______________.

$B^{-1}A^{-1}$

b) $((A^T - 2B)(I + B))^T =$_______________. (Multiply out and simplify as much as possible,)

$((A^T - 2B)(I + B))^T = (A^T - 2B + A^T B - 2B^2)^T = A - 2B^T + B^T A - 2(B^T)^2$

c) $\det(AB) =$_______________.

$\det(AB) = \det(A)\det(B)$

d) Adding twice the second row to the third row of a 3×4 matrix is the same as multiplying it on the _______________by the matrix _______________.

multiply on the left by $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$

e) If $\mathbf{v_1}, \ldots, \mathbf{v_p}$ is a linearly independent set of vectors in $\mathbb{R}^m$, then we always have $p \leq m$.

true

f) If T is a linear transformation, then $T(2\mathbf{v} - \mathbf{w}) = 2T(\mathbf{v}) - T(\mathbf{w})$.

true

g) Any four vectors which span a four dimensional vector space V form a basis for V.

true

h) If v_1, v_2, v_3 are linearly independent vectors in a four dimensional vector space V, then there is a vector v_4 so that v_1, v_2, v_3, v_4 is a basis for V.

true

i) Any two bases of a vector space V have the same number of elements.

true

j) Every vector space has a finite basis.

false, for example the polynomials