A short guide to determinents for Math 340

If A is an $n \times n$ matrix with entries in a field $\mathcal{F}$ there is a scalar in $\mathcal{F}$ called the determinent of A, and denoted $\det(A)$, which has the following properties:

a) $\det(AB) = \det(A)\det(B)$.
b) $\det(I) = 1$.
c) $\det(A_{(R_i \leftrightarrow R_j)}) = -\det(A)$.
d) $\det(A_{(kR_i)} = k\det(A)$
e) $\det(A_{(kR_i+R_j)}) = \det(A)$.
f) $\det(A^T) = \det(A)$.
g) $\det(A^{-1}) = 1/\det(A)$.
h) A is nonsingular if and only if $\det(A) \neq 0$.
i) If T is triangular, then $\det(T)$ is the product of its diagonal entries.
j) If $\mathcal{F} = \mathbb{R}$ then $|\det(A)|$ computes the volume of the n dimensional parallelepiped formed by the columns of A, whatever that means.
k) If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $\det(A) = ad - bc$.
l) If $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ then $\det(A) = aei + bfg + cdh - afh - bdi - ceg$.
m) $\det(A)$ is a polynomial function in the entries of A. In fact it can be computed as an expression involving just addition, subtraction and multiplication of the entries of A. There will be $n! = 1 \cdot 2 \cdots (n-1) \cdot n$ terms.

There are ways to calculate determinents of matrices bigger than 3×3, but we will only mention the most efficient, which is to do row operations to change A into row reduced echelon form and use properties c), d), and e) above to calculate the determinent. In fact, by property i) above it is not necessary to go all the way to row reduced echelon form, you only need to make it upper triangular. If you want you can make a matrix upper triangular by just using type I and III row operations. Since type III row operations do not change the determinent you will only need to keep track of the number of row switches you do.

It is messy and not very instructive to show that there is such a function det which has the above properties. I hope you will excuse me for not proving that such a determinent exists, even Cullen declines to show you. If anyone wants a reference to a proof, see for example *Linear Algebra* by Hoffman and Kunze. But many of the above properties can be derived from the others, some fairly easily. In fact Cullen starts with just two properties, a) and a very special case of d) and derives the rest (except for j) which he doesn't mention).

It is not hard to show that there is at most one function det satisfying the above properties. If A is nonsingular then we showed that A can be written as a product of elementary matrices $A = E_1E_2\cdots E_m$. But then by property a) we know that $\det(A) = \det(E_1)\det(E_2)\cdots\det(E_m)$ and we know by properties b)-e) that $\det(E_i) = -1$ if E_i is type I, $\det(E_i) = 1$ if E_i is type III, and $\det(E_i) = k$ if $E_i = I_{(kR_j)}$. So a)-e) completely determine the value of $\det(A)$. The only question is, if you wrote A in a different way as the product of elementary matrices, would you get the same answer. In fact you do get the same answer but I will not even attempt to justify this.