Math 340 Final Exam December 16, 2006

1. (20) Suppose $A = \begin{pmatrix} 1 & 2 & 3 & 0 \\ 1 & 0 & 1 & 4 \\ 0 & 1 & 1 & -2 \end{pmatrix}$.

a) Find the row-reduced echelon form of A.

Answer: $\begin{pmatrix} 1 & 2 & 3 & 0 \\ 1 & 0 & 1 & 4 \\ 0 & 1 & 1 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & -2 & -2 & 4 \\ 0 & 1 & 1 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & 4 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

so the row reduced echelon form is $\begin{pmatrix} 1 & 0 & 1 & 4 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

b) What is rank(A)?

Answer: The rank is two since there are two pivots.

c) Find a basis for the null space of A.

Answer: The vectors $(x, y, z, w)^T$ *in the null space satisfy* $x+z+4w = 0$ *and* $y+z-2w = 0$ *so* $(x, y, z, w) = z(-1, -1, 1, 0) + w(-4, 2, 0, 1)$. *So the two linearly independent vectors* $(-1, -1, 1, 0)^T, (-4, 2, 0, 1)^T$ *form a basis of the null space of* A.

d) Find a basis for the column space of A.

Answer: The pivot columns of A *form a basis, so* $(1, 1, 0)^T, (2, 0, 1)^T$ *forms a basis of the column space of* A.

e) Find all solutions X of $AX = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

Answer: By either inspection or Gaussian elimination we see that one solution is $(1, 0, 0, 0)^T$ *so all solutions are of the form* $(1, 0, 0, 0)^T + z(-1, -1, 1, 0)^T + w(-4, 2, 0, 1)^T$.

2. (10) Suppose A and B are 3×3 matrices and I is the 3×3 identity matrix. Find $\begin{pmatrix} A & I \\ 0 & B \end{pmatrix}^2$ and $\begin{pmatrix} A & I \\ 0 & B \end{pmatrix}^3$.

Answer: $\begin{pmatrix} A & I \\ 0 & B \end{pmatrix}^2 = \begin{pmatrix} A^2 & A+B \\ 0 & B^2 \end{pmatrix}$ *and* $\begin{pmatrix} A & I \\ 0 & B \end{pmatrix}^3 = \begin{pmatrix} A^3 & A^2+AB+B^2 \\ 0 & B^3 \end{pmatrix}$.

3. (20) Recall a square matrix A is skew symmetric if $A^T = -A$. Show that the skew symmetric matrices are a subspace of $\mathbb{R}_{n \times n}$. Find a basis for the 3×3 skew symmetric matrices.

Answer: $0^T = -0$ *so* 0 *is skew symmetric. If* A *and* B *are skew symmetric then* $(A+B)^T = A^T + B^T = -A + (-B) = -(A+B)$ *so* $A+B$ *is skew symmetric. If* c *is a scalar and* A *is skew symmetric, then* $(cA)^T = cA^T = c(-A) = -(cA)$ *so* cA *is skew symmetric. So the* *skew symmetric matrices form a subspace of* $\mathbb{R}_{n\times n}$. *Many of you also correctly showed this by using the formula* $ent_{ji}(A) = -ent_{ij}(A)$. *Although nobody stated it this way, what these formulae give you are* $n(n+1)/2$ *homogeneous linear equations in the entries of the matrix, so their solutions form a subspace.*

4. (35) Consider the curve C parameterized by $\mathbf{r}(t) = 4t\mathbf{i} + 3\sin t\mathbf{j} + 3\cos t\mathbf{k}$, $0 \le t \le \pi$.

a) Find the curvature κ of C as a function of t.

Answer: $v(t) = 4\mathbf{i} + 3\cos t\mathbf{j} - 3\sin t\mathbf{k}$ *and* $a(t) = -3\sin t\mathbf{j} - 3\cos t\mathbf{k}$. *The speed* $||v||$ *is a constant, 5. So* $a_T = d(speed)/dt = 0$. *Then* $a_N = \sqrt{||a||^2 - a_T^2} = ||a|| = 3$. $\kappa = a_N/||v||^2 = 3/25$.

b) Find an equation of the line tangent to the curve C at the point $\mathbf{r}(\pi/2)$.

Answer: One point on the curve is $r(\pi/2) = (2\pi, 3, 0)$ *and a vector in the direction of the curve is* $r'(\pi/2) = (4, 0, -3)$. *So an equation is* $(x, y, z) = (2\pi + 4t, 3, -3t)$.

c) Find $\int_C 3x\,ds$.

Answer: $\int_C 3x\,ds = \int_0^\pi 3(4t)||v(t)||\,dt = \int_0^\pi 60t\,dt = 30\pi^2$.

d) Find $\int_C ydx - zdy$

Answer:

$$\int_C ydx - zdy = \int_C (y, -z, 0)\cdot \mathrm{ds} = \int_0^\pi (3\sin t, -3\cos t, 0)\cdot(4, 3\cos t, -3\sin t)\,dt$$

$$= \int_0^\pi 12\sin t - 9\cos^2 t\,dt = \int_0^\pi 12\sin t - 4.5(1+\cos(2t))\,dt$$

$$= -12\cos t - 4.5t + 2.25\sin(2t)\Big]_0^\pi = 12 - 4.5\pi + 12 = 24 - 9\pi/2$$

5. (25) Let D be the triangular region in the xy plane bounded by the lines $x = 0$, $y = 0$, and $x + y = 1$. Let S be the portion of the surface $z = x^2 + y^2$ lying above D. Let C be the boundary of S, oriented counterclockwise when viewed from above. Let $\mathbf{F}(x.y, z) = 3xy\mathbf{i} - z\mathbf{j}$.

a) Set up completely, but do not evaluate, an integral giving the area of S.

Answer: $dS = \sqrt{f_x^2 + f_y^2 + 1}\,dA = \sqrt{4x^2 + 4y^2 + 1}\,dA$. *So the area is*

$$\int_0^1 \int_0^{1-x} \sqrt{4x^2 + 4y^2 + 1}\,dydx$$

b) Use Stokes' theorem to find $\int_C \mathbf{F}\cdot\mathbf{T}\,ds$.

Answer: curl$\mathbf{F} = (1, 0, -3x)$ *so by Stokes' theorem,*

$$\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int\int_S \text{curl}\mathbf{F}\cdot n\,dS = \int_0^1\int_0^{1-x}(1,0,-3x)\cdot(-2x,-2y,1)\,dydx$$

$$= \int_0^1\int_0^{1-x} -5x\,dydx = \int_0^1 -5x(1-x)\,dx = -5/2 + 5/3 = -5/6$$

6. (30) Let D be the solid region above the surface $z = x^2 + y^2 - 1$, and below the surface $z = 1 - x^2 - y^2$. Let S be the boundary of D oriented pointing outward from D. Let $\mathbf{F}(x, y, z) = x\mathbf{i} - 2y\mathbf{j} + z\mathbf{k}$.

a) Find $\int\int_S \mathbf{F}\cdot \mathrm{d}S$.

Answer: div$\mathbf{F} = 0$ *so* $\int\int_S \mathbf{F}\cdot\mathrm{d}S = \int\int\int_D 0\,dV = 0$ *by Gauss' theorem.*

b) Set up completely, but do not evaluate, integrals giving the volume of D in cartesian, cylindrical, and spherical coordinates.

Answer: These surfaces intersect where $x^2 + y^2 = 1$ *and the shadow of* D *in the* xy *plane is the disc* $x^2 + y^2 \le 1$. *So in cartesian and cylindrical coordinates:*

$$\text{volume} = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{x^2+y^2-1}^{1-x^2-y^2} dzdydx$$

$$= \int_0^{2\pi}\int_0^1\int_{r^2-1}^{1-r^2} r\,dzdrd\theta$$

For spherical coordinates there are different limits for ρ *above and below the* xy *plane. Above we have* $z = 1 - x^2 - y^2$ *so* $\rho\cos\phi = 1 - \rho^2\sin^2\phi$, *so* $\rho^2\sin^2\phi + \rho\cos\phi - 1 = 0$ *and by the quadratic formula,* $\rho = \frac{-\cos\phi+\sqrt{\cos^2\phi+4\sin^2\phi}}{2\sin^2\phi} = \frac{-\cos\phi+\sqrt{1+3\sin^2\phi}}{2\sin^2\phi}$. *Similarly for* $\phi \ge \pi/2$ *we have* $\rho^2\sin^2\phi - \rho\cos\phi - 1 = 0$ *so* $\rho = \frac{\cos\phi+\sqrt{1+3\sin^2\phi}}{2\sin^2\phi}$. *So here are a few ways to compute the volume in spherical coordinates*

$$\int_0^{2\pi}\int_0^{\pi/2}\int_0^{\frac{-\cos\phi+\sqrt{1+3\sin^2\phi}}{2\sin^2\phi}} \rho^2\sin\phi\,d\rho d\phi d\theta + \int_0^{2\pi}\int_{\pi/2}^{\pi}\int_0^{\frac{\cos\phi+\sqrt{1+3\sin^2\phi}}{2\sin^2\phi}} \rho^2\sin\phi\,d\rho d\phi d\theta$$

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{\frac{-|\cos\phi|+\sqrt{1+3\sin^2\phi}}{2\sin^2\phi}} \rho^2\sin\phi\,d\rho d\phi d\theta$$

$$2\int_0^{2\pi}\int_0^{\pi/2}\int_0^{\frac{-\cos\phi+\sqrt{1+3\sin^2\phi}}{2\sin^2\phi}} \rho^2\sin\phi\,d\rho d\phi d\theta$$

The last expression comes from the observation that D is symmetric about the xy plane so the volume of D is twice the volume of the part of D above the xy plane.

7. (20) Let D be the region in the first quadrant bounded by $xy = 1$, $xy = 2$, $x = y$, and $x - y = 3 - xy$. Find $\int\int_D x + y\, dA$.

Answer: Let $u = xy$ and $v = x - y$. Then in the uv plane the region is bounded by $u = 1$, $u = 2$, $v = 0$, and $v = 3 - u$. $\partial(u,v)/\partial(x,y) = \det\begin{pmatrix} y & x \\ 1 & -1 \end{pmatrix} = -x - y$ so the integral is

$$\int_1^2 \int_0^{3-u} (x+y)\left|\frac{1}{-x-y}\right| dv du = \int_1^2 \int_0^{3-u} dv du$$

$$= \int_1^2 3 - u\, du = 3u - u^2/2\Big]_1^2 = 6 - 2 - 3 + 1/2 = 3/2$$

8. (20) Let $\mathbf{F}(x,y,z) = (3x^2 + y\sin(xy))\mathbf{i} + (z + x\sin(xy))\mathbf{j} + (y + 4z)\mathbf{k}$. Let C be the curve parameterized by $\mathbf{r}(t) = t^8\mathbf{i} + t^9(t+1)\mathbf{j} - e^{\sin(\pi t)}\mathbf{k}$ for $0 \le t \le 1$. Find $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$.

Answer: Note $\mathbf{F}$ is conservative. Trying to solve $\mathbf{F} = grad g$ we get

$$\partial g/\partial x = 3x^2 + y\sin(xy) \;\; => \;\; g(x,y,z) = x^3 - \cos(xy) + C(y,z)$$

$$z + x\sin(xy) = \partial g/\partial y = 0 + x\sin(xy) + \partial C(y,z)/\partial y \;\; => C(y,z) = yz + D(z)$$

$$y + 4z = \partial g/\partial z = y + D'(z) \;\; => \;\; D(z) = 2z^2 + E$$

so we may let $g(x,y,z) = x^3 - \cos(xy) + yz + 2z^2$. C begins at $(0,0,-1)$ and ends at $(1,2,-1)$ so

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = g(1,2,-1) - g(0,0,-1) = 1 - \cos 2 - 2 + 2 - (0 - 1 + 0 + 2) = -\cos 2$$

9. (20) True or false. (no justification required).

a) If A is a square matrix and $NS(A) = \{0\}$ then A^{-1} exists.

Answer: True, see page 55 or 89-90 of Cullen

b) $(AB)^T = A^T B^T$.

Answer: False, $(AB)^T = B^T A^T$.

c) $\det(AB) = \det(A)\det(B)$.

Answer: True, see page 105 of Cullen

d) Any 5 vectors in $\mathbb{R}^4$ are linearly dependent.

Answer: True, see Thm 2.7 page 84 of Cullen

e) Any 5 vectors in $\mathbb{R}^4$ span $\mathbb{R}^4$.

Answer: False, for example $(1,1,1,1),(2,2,2,2),(3,3,3,3),(4,4,4,4),(5,5,5,5)$ *span a one dimensional subspace.*

f) Any 3 vectors in $\mathbb{R}^4$ span a subspace of $\mathbb{R}^4$.

Answer: True, the span of any collection of vectors is a subspace. You even know in this case that they span a proper subspace of $\mathbb{R}^4$*, i.e., a subspace which is definitely smaller than* $\mathbb{R}^4$*, since the dimension of the span is at most 3.*

g) If A is a 5×7 matrix then the dimension of $NS(A)$ plus the rank of A is 5.

Answer: False, it is 7, see page 89 of Cullen