Solution to Cullen, 1.7 problem 7

Temporarily, for this problem only, we will say that the k superdiagonal of a matrix A is all the entries in position (i, j) with $i = j - k$. So the ordinary diagonal is the 0 superdiagonal. The 1 superdiagonal is the entries just above the diagonal and in general, the $k+1$ superdiagonal is just above the k superdiagonal.

We say a square matrix is k super upper triangular if it is all 0 below the k superdiagonal, in other words if $a_{ij} = 0$ whenever $i < j - k$. So an upper triangular matrix is 0 super upper triangular and a strictly upper triangular matrix is 1 super upper triangular.

The crucial observation is that the product of a k super upper triangular matrix with an m super upper triangular matrix is a $k+m$ super upper triangular matrix. I'll show this below, but the consequence is that if T is 1 super upper triangular then T^2 is 2 super upper triangular, T^3 is 3 super upper triangular and so on. (The "and so on" can be made precise using mathematical induction which Cullen introduces at the end of 1.1, but in this course we'll try to avoid induction.) In particular, if T is $n \times n$ then T^n is n super upper triangular and consequently 0, so T is nilpotent.

So it remains to show that the product of a k super upper triangular matrix T with an m super upper triangular matrix S is a $k+m$ super upper triangular matrix. We know $t_{ij} = 0$ for $i < j - k$ and $s_{ij} = 0$ for $i < j - m$. Also $ent_{ij}(TS) = t_{i1}s_{1j} + \cdots + t_{in}s_{nj}$. Note that $t_{i\ell} = 0$ if $i < \ell - k$ and $s_{\ell j} = 0$ if $\ell < j - m$. Consequently, $t_{i\ell}s_{\ell j} = 0$ if $\ell > i + k$ or $\ell < j - m$. But if $i < j - (k + m)$ then $i + k < j - m$ so every ℓ satisfies either $\ell > i + k$ or $\ell < j - m$. Consequently, $ent_{ij}(TS) = 0$ if $i < j - (k + m)$ so TS is $k + m$ super upper triangular.

To show a strictly lower triangular matrix is nilpotent we could make a similar argument with k subdiagonals, but it is slicker to just use the transpose since if L is strictly lower triangular then L^T is strictly upper triangular, so $(L^T)^n = 0$. Then

$$L^n = ((L^n)^T)^T = ((L^T)^n)^T = 0^T = 0$$