MATH 340 **EXAM # 2** **November 3, 2006**

1. [30] Let $f(x,y,z) = z^2 + x^2 + xy^2 + e^{yz}$

a) Find the gradient of f at $(1,2,0)$.

Answer: $\nabla f = (2x + y^2, 2xy + ze^{yz}, 2z + ye^{yz}) = (6,4,2)$ *at* $(1,2,0)$.

b) In which direction is f increasing most rapidly at the point $(1,2,0)$? (Your answer should be a unit vector with this direction.)

Answer: $(6,4,2)/\sqrt{36+16+4} = (6,4,2)/\sqrt{56}$.

c) Find $\lim_{(x,y,z)\to(1,2,0)} f(x,y,z)$.

Answer: *Since f is continuous, the limit is $f(1,2,0) = 6$.*

d) Find an equation for the tangent plane at $(1,2,0)$ of the level surface $f(x,y,z) = 6$.

Answer: $6(x-1) + 4(y-2) + 2z = 0$

e) The level surface $f(x,y,z) = 6$ defines z implicitly as a function of x and y near $(1,2,0)$. Find $\partial z/\partial y$ at $x=1$, $y=2$, $z=0$.

Answer: $\partial z/\partial y = -\frac{\partial f/\partial y}{\partial f/\partial z} = -4/2 = -2$

2. [10] Find the boundary of each of the following subsets of the plane. Which of the these subsets are open? Which of these subsets are closed?

a) $A = \{(x,y) \mid x^2 + y^2 \le 2\}$.

Answer: *The boundary is the circle with equation $x^2 + y^2 = 2$. A is closed since it contains its boundary. It is not open.*

b) $B = \{(x,y) \mid x^2 + y^2 > 1\}$.

Answer: *The boundary is the circle with equation $x^2 + y^2 = 1$. B is open since it is disjoint from its boundary. It is not closed.*

c) $C = A \cap B = \{(x,y) \mid 1 < x^2 + y^2 \le 2\}$.

Answer: *The boundary is the two circles with equations $x^2 + y^2 = 1$ and $x^2 + y^2 = 2$. C is neither open nor closed since it does not contain all of its boundary, but does contain some points of its boundary.*

3. [30] Let R be the region in the first octant which lies between the cone $z^2 = x^2 + y^2$ and the paraboloid $z = x^2 + y^2$. Set up, but do not evaluate, $\int\int\int_R x + 2y + 3z\, dV$

a) in cartesian coordinates.

Answer: *The cone and paraboloid intersect where $z^2 = z$, so $z = 0, 1$. The intersection at $z = 0$ is just the origin, but at $z = 1$ the intersection is an arc of the circle $1 = x^2 + y^2$. The projection of R to the xy plane is then the quarter disc $x^2 + y^2 \le 1$, $x \ge 0$, $y \ge 0$ in the first quadrant. The cone lies above the paraboloid as can be seen by testing a point, for example if $x = 1/4$, $y = 0$, then $z = \sqrt{1/4} = 1/2$ on the cone, but $z = 1/4$ on the paraboloid. So* $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{x^2+y^2}^{\sqrt{x^2+y^2}} x + 2y + 3z\, dzdydx$. *Some of you used a different order of integration,* *for example $dxdydz$. This requires setting up the integral as the sum or difference of two integrals, for example* $\int_0^1 \int_0^{\sqrt{z}} \int_0^{\sqrt{z-y^2}} x+2y+3z\, dxdydz - \int_0^1 \int_0^z \int_0^{\sqrt{z^2-y^2}} x+2y+3z\, dxdydz$.

b) in cylindrical coordinates.

Answer: *The cone has equation $z = r$ and the paraboloid is $z = r^2$. So in the usual order,* $\int_0^{\pi/2} \int_0^1 \int_{r^2}^r r^2\cos\theta + 2r^2\sin\theta + 3rz\, dzdrd\theta$. *But if you wanted to do another order, here is a possibility,* $\int_0^1 \int_0^{\pi/2} \int_z^{\sqrt{z}} r^2\cos\theta + 2r^2\sin\theta + 3rz\, drd\theta dz$.

c) in spherical coordinates.

Answer: *The equation of the cone $z = r$ is $\rho\cos\phi = \rho\sin\phi$ which reduces to $\phi = \pi/4$. the equation of the paraboloid $z = r^2$ is $\rho\cos\phi = \rho^2\sin^2\phi$ which reduces to $\rho = \cos\phi/\sin^2\phi$. So* $\int_0^{\pi/2} \int_{\pi/4}^{\pi/2} \int_0^{\cos\phi/\sin^2\phi} (\rho\sin\phi\cos\theta + 2\rho\sin\phi\sin\theta + 3\rho\cos\phi)\rho^2\sin\phi\, d\rho d\phi d\theta$.

4. [15] Calculate **one** of the following integrals where D is the region in the first quadrant bounded by the lines $y = x$, $y = x + 1$ and by the ellipses $x^2 + 4y^2 = 4$ and $x^2 + 4y^2 = 16$. You must clearly indicate which one of the three integrals you choose to evaluate. Choose wisely.

a) $\int\int_D e^{y-x}\, dA$.

b) $\int\int_D (x + 4y)e^{y-x}\, dA$.

c) $\int\int_D e^{y-x}/(x + 4y)\, dA$.

Answer: *A good coordinate change is $u = y - x$, $v = x^2 + 4y^2$. Then* $\partial(u, v)/\partial(x, y) = \det\begin{bmatrix} -1 & 1 \\ 2x & 8y \end{bmatrix} = -8y - 2x$. *So $\partial(x, y)/\partial(u, v) = -1/(8y + 2x)$. So b) is a good choice to evaluate.*

$$\int\int_D (x + 4y)e^{y-x}\, dA = \int_0^1 \int_4^{16} (x + 4y)e^u \left|\frac{-1}{8y + 2x}\right| dvdu$$

$$= \int_0^1 \int_4^{16} e^u/2\, dvdu = \int_0^1 ve^u/2\Big]_4^{16} du = \int_0^1 6e^u\, du = 6e^u\Big]_0^1 = 6e - 6$$

5. [15] A region D in space has volume 3 and centroid $(\bar{x}, \bar{y}, \bar{z}) = (1, 2, -1)$. Calculate the following integrals:

a) $\int\int\int_D 4\, dV$.

Answer: $\int\int\int_D 4\, dV = 4\int\int\int_D 1\, dV = 4 \cdot 3 = 12$.

b) $\int\int\int_D x\, dV$.

Answer: $1 = \bar{x} = \int\int\int_D x\, dV/3$ *so* $\int\int\int_D x\, dV = 3$.

c) $\int\int\int_D 3 + 2x - 4y\, dV$.

Answer: $\int\int\int_D y\, dV = 3\bar{y} = 6$ *so* $\int\int\int_D 3 + 2x - 4y\, dV = 3 \cdot 3 + 2 \cdot 3 - 4 \cdot 6 = -9$.