MATH 340 EXAM # 3 December 1, 2006

For two integrals of your choice you may set them up but not evaluate them. Just set them up and write "= free pass #1" or "= free pass #2".

1. [35] The position of a particle at time t is $t^3/3\mathbf{i} + t^2\mathbf{j} + 2t\mathbf{k}$. Let C be the curve which the particle travels for $1 \leq t \leq 3$.

a) Find the velocity, speed and acceleration of the particle.

Answer: *The velocity is* $v(t) = t^2\mathbf{i} + 2t\mathbf{j} + 2\mathbf{k}$ *and acceleration is* $a(t) = 2t\mathbf{i} + 2\mathbf{j}$. *The speed is* $||v(t)|| = \sqrt{t^4 + 4t^2 + 4} = t^2 + 2$.

b) Find the tangential and normal components of acceleration.

Answer: $a_{tang}(t) = 2t$, *the derivative of the speed. The normal component is* $a_{norm}(t) = \sqrt{||a(t)||^2 - a_{tang}(t)^2} = \sqrt{4t^2 + 4 - 4t^2} = 2$.

c) Find the curvature of the curve C at time $t = 2$.

Answer: *The curvature is* $\kappa(t) = a_{norm}(t)/||v(t)||^2 = 2/(t^2+2)^2$. *At* $t = 2$ *this is* $1/18$.

d) Find $\int_C yz/x\, ds$.

Answer: $\int_C yz/x\, ds = \int_1^3 t^2 2t/(t^3/3)\,(t^2+2)\, dt = \int_1^3 6(t^2+2)\, dt = 2t^3 + 12t\Big]_1^3 = 54 + 36 - 2 - 12 = 76$.

2. [25] Two vector fields are $\mathbf{F}(x,y,z) = (4x+y)\mathbf{i} + (x + 3y^2z^2)\mathbf{j} + 2y^3z\mathbf{k}$ and $\mathbf{G}(x,y,z) = (4x-y)\mathbf{i} + (x+3z)\mathbf{j} - 3y\mathbf{k}$.

a) Is $\mathbf{F}$ conservative? If so, find a potential function f for $\mathbf{F}$.

Answer: *Solving for* f, *we have* $f_x = 4x + y$ *so* $f(x,y,z) = 2x^2 + xy + C(y,z)$. *Then* $f_y = x + 3y^2z^2$ *means* $x + C_y = x + 3y^2z^2$ *so* $C(y,z) = y^3z^2 + D(z)$. *Then* $f_z = 2y^3z$ *implies* $2y^3z + D' = 2y^3z$ *so* D *is constant. So we may take* $f(x,y,z) = 2x^2 + xy + y^3z^2$ *and* $\mathbf{F}$ *is conservative.*

b) Is $\mathbf{G}$ conservative? If so, find a potential function g for $\mathbf{G}$.

Answer: *Solving for g, we have* $g_x = 4x - y$ *so* $g(x, y, z) = 2x^2 - xy + C(y, z)$. *Then* $g_y = x + 3z$ *means* $-x + C_y = x + 3z$ *so* $C_y(y, z) = 2x + 3z$ *which involves x, so no such g exists and* $\mathbf{G}$ *is not conservative.*

c) Let C be the curve which starts at $(1, 0, 0)$, goes twice around the circle $x^2 + z^2 = 1$ in the xz plane (clockwise when viewed from $(0, 1, 0)$), then follows the parabola $y = 1 - x^2$ in the xy plane to $(2, -3, 0)$, goes on a line segment to $(1, -3, 0)$, and finally returns on a line segment to $(1, 0, 0)$. Let D be the line segment from $(1, 0, -1)$ to $(4, 2, 0)$. Compute three of the following: $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$, $\int_C \mathbf{G} \cdot \mathbf{T}\, ds$, $\int_D \mathbf{F} \cdot \mathbf{T}\, ds$, $\int_D \mathbf{G} \cdot \mathbf{T}\, ds$.

Answer: *The easiest to compute is* $\int_C \mathbf{F} \cdot \mathbf{T}\, ds = 0$ *since C is closed and* $\mathbf{F}$ *is conservative. Next easiest is* $\int_D \mathbf{F} \cdot \mathbf{T}\, ds = f(4, 2, 0) - f(1, 0, -1) = 32 + 8 + 0 - (2 + 0 + 0) = 38$. *Next parameterize D by* $r(t) = (1 + 3t, 2t, t - 1)$ *for* $0 \le t \le 1$ *and then* $\int_D \mathbf{G} \cdot \mathbf{T}\, ds = \int_0^1 (4 + 12t - 2t, 1 + 3t + 3t - 3, -6t) \cdot (3, 2, 1)\, dt = \int_0^1 12 + 30t - 4 + 12t - 6t\, dt = \int_0^1 8 + 36t\, dt = 8t + 18t^2\big]_0^1 = 26$. *Hardest to do is* $\int_D \mathbf{G} \cdot \mathbf{T}\, ds$. *It would be very time consuming to do it by parameterizing the four segments, but we can use Green's theorem to help out. Let* C_1 *be the circle* $x^2 + z^2 = 1$ *which bounds a disc* D_1 *in the xz plane. We want to find* $2\int_{C_1} \mathbf{G} \cdot T\, ds$ *but T is in the xz plane so we can ignore the* $\mathbf{j}$ *component of* $\mathbf{G}$ *and set* $y = 0$ *so* $2\int_{C_1} \mathbf{G} \cdot T\, ds = 2\int_{C_1} 4x dx = 2\int_{D_1} 0\, dA$ *by Green's theorem. Let* C_2 *be the rest of C which is the boundary of the region* D_2 *given by* $1 \le x \le 2$, $-3 \le y \le 1 - x^2$, $z = 0$ *and is oriented clockwise. Then* $\int_{C_2} \mathbf{G} \cdot T\, ds = \int_{C_2} (4x - y)dx + xdy = -\int_1^2 \int_{-3}^{1-x^2} 2\, dydx = -2\int_1^2 4 - x^2\, dx = -8x + 2x^3/3\big]_1^2 = -16 + 16/3 + 8 - 2/3 = -10/3$.

3. [25] Compute the flux integral $\int\int_\Sigma \mathbf{F} \cdot \mathbf{n}\, dS$ where Σ is the portion of the surface $z = x^2 - y^2$ inside the cylinder $x^2 + y^2 = 4$, oriented downwards, and where $\mathbf{F}(x, y, z) = y\mathbf{i} + x\mathbf{j} + (1 + xy)\mathbf{k}$.

Answer: $\int\int_\Sigma \mathbf{F} \cdot \mathbf{n}\, dS = \int\int_R (y\mathbf{i} + x\mathbf{j} + (1 + xy)\mathbf{k}) \cdot (2x, -2y, -1)\, dA = \int\int_R -1 - xy\, dA$ *where R is the disc of radius 2 centered at the origin. By symmetry,* $\int\int_R -xy\, dA = 0$ *so* $\int\int_R -1 - xy\, dA = -\int\int_R 1\, dA = -4\pi$. *You* *could also use polar coordinates*

$$\int\int_R -1 - xy\, dA = \int_0^{2\pi}\int_0^2 -r - r^3 \sin\theta\cos\theta\, dr d\theta = -4\pi$$

5. [15] Let R be the annular region $1 \le r \le 2$ in the plane.

a) Describe the boundary C of R.

Answer: *The boundary is the two circles $r = 1$ and $r = 2$.*

b) Describe the orientation you would use for C when applying Green's theorem.

Answer: *Orient $r = 1$ clockwise and $r = 2$ counterclockwise.*

c) What does Green's theorem say about $\int_C x^2 y dx + e^x dy$? (You need not set up or evaluate any integrals).

Answer: $\int_C x^2 y dx + e^x dy = \int\int_R e^x - x^2\, dA$ *where we orient C as in b) above.*