Differential Forms

Colley devotes Chapter 8 to differential forms. This note is intended to give you the highlights, and is complementary to chapter 8 rather a summary of chapter 8. Reading chapter 8 can provide you with more details. I will also look at forms slightly differently than in chapter 8, in particular evaluating them without determinents. In fact a very good way to define determinents is to use forms.

The big picture: If $X \subset \mathbb{R}^n$ is an open set, then for every integer $k \geq 0$ there is a vector space $\Omega^k(X)$ of k-forms on X. There is an exterior derivative $d_k : \Omega^k(X) \to \Omega^{k+1}(X)$, usually the subscript k is omitted and this is just written as d. The map d_k is a linear transformation. So if ω is a k-form then $d\omega$ is a $k+1$-form and $d(\omega + \nu) = d\omega + d\nu$. There is an operation wedge which takes a k-form ω and an ℓ-form ν and produces a $k + \ell$-form $\omega \wedge \nu$. If ω is a k-form and M is a parameterized k dimensional manifold in X (whatever that is) then an integral $\int_M \omega$ can be defined, it is a scalar.

In what follows, when we talk about a function f on X we will assume without explicitly saying so that $f\colon X \to \mathbb{R}$ is real valued and is infinitely differentiable, i.e., partial derivatives of f all orders exist. You can get away with less, but it would only obscure the main ideas.

You can not only multiply a form by a scalar (from the vector space axioms) but you can multiply a form by a function on X. So if ω is a k-form and $f\colon X \to \mathbb{R}$ is a function, then $f\omega$ is another k-form.

The derivative d and the wedge satisfy some algebraic rules. Some of these rules are, $d^2 = 0$ and there is a product rule $d(f\omega) = f d\omega + df \wedge \omega$. Also $\wedge$ is bilinear which means $(\omega_1 + \omega_2) \wedge \nu = \omega_1 \wedge \nu + \omega_2 \wedge \nu$ and $\omega \wedge (\nu_1 + \nu_2) = \omega \wedge \nu_1 + \omega \wedge \nu_2$. If ω and ν are k and ℓ-forms then $\omega \wedge \nu = (-1)^{k\ell} \nu \wedge \omega$, in particular if both k and ℓ are odd, then $\omega \wedge \nu = -\nu \wedge \omega$. We will use this extensively below for $k = \ell = 1$.

The nitty gritty: Okay, so what is a form anyway.

0) A 0-form is just a function $f\colon X \to \mathbb{R}$.
1) A 1-form is an expression of the form $f_1 dg_1 + f_2 dg_2 + \cdots + f_m dg_m$ where the f_i and g_i are functions on X. For example, $xy^2 dx + d(xy)$ is a 1-form on $\mathbb{R}^2$.
2) A 2-form is an expression of the form $f_1 dg_1 \wedge dh_1 + f_2 dg_2 \wedge dh_2 + \cdots + f_m dg_m \wedge dh_m$ where the f_i, g_i, and h_i are functions on X. For example $x^2 y\, d(x+y) \wedge d(xz) + 3dx \wedge dz$ is a 2-form on $\mathbb{R}^3$.

k) In general a k-form is a finite sum of expressions of the form $f dg_1 \wedge dg_2 \wedge \cdots \wedge dg_k$ where f and the g_i are functions on X.

There are some rules for manipulating forms.

a) $df = \partial f/\partial x_1\, dx_1 + \partial f/\partial x_2\, dx_2 + \cdots + \partial f/\partial x_n\, dx_n$. So for example $d(xz) = zdx + xdz$ and $d(x + y) = dx + dy$.
b) $\wedge$ distributes over addition and commutes with multiplication, so for example

$$x^2 y\, d(x + y) \wedge d(xz) = x^2 y\, (dx + dy) \wedge (zdx + xdz) = x^2 y\, dx \wedge (zdx + xdz) + x^2 y\, dy \wedge (zdx + xdz)$$

$$= x^2 yz\, dx \wedge dx + x^2 yx\, dx \wedge dz + x^2 yz\, dy \wedge dx + x^2 yx\, dy \wedge dz$$

c) Switching two 1-forms you are wedging will change the sign, $df \wedge dg = -dg \wedge df$. So for example

$$x^2 yz\, dx \wedge dx + x^3 y\, dx \wedge dz + x^2 yz\, dy \wedge dx + x^3 y\, dy \wedge dz = x^2 yz\, dx \wedge dx + x^3 y\, dx \wedge dz - x^2 yz\, dx \wedge dy + x^3 y\, dy \wedge dz$$

d) Wedging a 1-form with itself is always 0, $df \wedge df = 0$. (This is actually equivalent to c, can you see why?) So for example $x^2 yz\, dx \wedge dx = 0$. The equal items need not be adjacent so for example $dx \wedge dy \wedge dx = 0$. Applying all these rules to our example we get

$$x^2 y\, d(x + y) \wedge d(xz) + 3dx \wedge dz = x^2 yx\, dx \wedge dz - x^2 yz\, dx \wedge dy + x^2 yx\, dy \wedge dz + 3dx \wedge dz$$

$$= x^3 y\, dy \wedge dz + (x^3 y + 3)\, dx \wedge dz - x^2 yz\, dx \wedge dy$$

Using these rules we can reduce any k-form to a sum of terms of the form

$$f\, dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$$

where $i_1 < i_2 < \cdots < i_k$ just as we did with our example. So for example any 1-form on $\mathbb{R}^3$ has the form $M\,dx + N\,dy + P\,dz$, any 2-form on $\mathbb{R}^3$ has the form $M\,dy \wedge dz + N\,dx \wedge dz + P\,dx \wedge dy$ and any 3-form in $\mathbb{R}^3$ has the form $f\,dx \wedge dy \wedge dz$. Thus 1-forms and 2-forms on $\mathbb{R}^3$ can be identified with vector fields (M, N, P) and 0-forms and 3-forms on $\mathbb{R}^3$ can be identified with scalar valued functions. In general in dimension n, the 0-forms and n-forms can be identified with scalar functions and the 1-forms and $n-1$ forms can be identified with vector fields. Note that all k-forms with $k > n$ are 0 since we cannot find k distinct integers from 1 to n. Things work out better if we identify the vector field (M, N, P) on $\mathbb{R}^3$ with the 2-form $M\,dy \wedge dz - N\,dx \wedge dz + P\,dx \wedge dy$.

If $f: Y \to X$, then composition (i.e., substitution) gives a linear transformation $f^*: \Omega^k(X) \to \Omega^k(Y)$. For example, suppose $f: \mathbb{R}^2 \to \mathbb{R}^3$ is given by $f(s,t) = (st, s+t, t^2)$. Then

$$f^*(x\,dy \wedge dz + yz\,dx \wedge dy) = st\,d(s+t) \wedge d(t^2) + (s+t)t^2 d(st) \wedge d(s+t)$$

$$= st\,ds \wedge d(t^2) + st\,dt \wedge d(t^2) + (s+t)t^2 d(st) \wedge ds + (s+t)t^2 d(st) \wedge dt$$

$$= 2tst\,ds \wedge dt + 2tst\,dt \wedge dt + (s+t)t^2(sdt + tds) \wedge ds + (s+t)t^2(sdt + tds) \wedge dt$$

$$= 2t^2 s\,ds \wedge dt + (s+t)st^2\,dt \wedge ds + (s+t)t^3 ds \wedge dt$$

$$= (2t^2 s - (s+t)st^2 + (s+t)t^3)\,ds \wedge dt$$

This gives us a way to evaluate $\int_Z \omega$ where $Z \subset X$ is a k dimensional manifold and ω is a k-form on X. Suppose we can parameterize Z by a map $f: D \to \mathbb{R}^n$ where $D \subset \mathbb{R}^k$. Then $f^*\omega$ is a k-form on D which can be thought of as a scalar function which we can then integrate on D. For example, let Z be the cylinder parameterized by $r(s,t) = \cos t\mathbf{i} + \sin t\mathbf{j} + s\mathbf{k}$ for $0 \le t \le 2\pi$ and $0 \le s \le 4$. Let ω be any 2-form on $\mathbb{R}^3$, $\omega = zdy \wedge dz + xdx \wedge dz - dx \wedge dy$. Then

$$r^*\omega = sd\sin t \wedge ds + \cos t d\cos t \wedge ds - d\cos t \wedge d\sin t = s\cos t dt \wedge ds - \cos t \sin t dt \wedge ds + \sin t \cos t dt \wedge dt$$

$$= (\cos t \sin t - s\cos t) ds \wedge dt$$

So $\int_Z \omega = \int_0^{2\pi} \int_0^4 \cos t \sin t - s\cos t ds dt = 0$.

Change of variable: Suppose we have a change of variables $x = x(u,v)$, $y = y(u,v)$. Then

$$dx \wedge dy = (x_u du + x_v dv) \wedge (y_u du + y_v dv) = x_u y_u du \wedge du + x_u y_v du \wedge dv + x_v y_u dv \wedge du + x_v y_v dv \wedge dv$$

$$= x_u y_v du \wedge dv + x_v y_u dv \wedge du = (x_u y_v - x_v y_u) du \wedge dv = \partial(x,y)/\partial(u,v) du \wedge dv$$

Thus we get the change of variables formula for integration (except that we allow signed area, so there is no absolute value of the Jacobian). This all works in n dimensions too.

Work integrals: Let C be a curve (i.e., a 1 manifold) in $\mathbb{R}^3$ parameterized by $r(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ for $a \le t \le b$. Let $F = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ be any vector field on $\mathbb{R}^3$. Then F can be identified with the 1-form $\omega = Mdx + Ndy + Pdz$. Then $r^*\omega = Mdx/dtdt + Ndy/dtdt + Pdz/dtdt$ so $\int_C \omega = \int_a^b Mdx/dt + Ndy/dt + Pdz/dt\,dt = \int_C F \cdot \mathbf{T}\,ds$.

Flux integrals: Let Z be the graph $z = g(x,y)$ for $(x,y) \in D$. Let $F = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ be any vector field on $\mathbb{R}^3$. Then F can be identified with the 2-form $\omega = Mdy \wedge dz - Ndx \wedge dz + Pdx \wedge dy$. Let $r(x,y) = (x, y, g(x,y))$ parameterize Z. Then $dz = g_x dx + g_y dy$ so

$$r^*\omega = Mdy \wedge (g_x dx + g_y dy) - Ndx \wedge (g_x dx + g_y dy) + Pdx \wedge dy$$

$$= Mg_x dy \wedge dx - Ng_y dx \wedge dy + Pdx \wedge dy$$

$$= (M, N, P) \cdot (-g_x, -g_y, 1) dx \wedge dy$$

Thus $\int_Z \omega = \int\int_Z F \cdot n\,dS$. As an excercise you can prove this formula for a general parameterized surface.

Relation of d with grad, curl and div: Recall that the exterior derivative can be calculated by $d(f dg_1 \wedge dg_2 \wedge \cdots \wedge dg_k) = df \wedge dg_1 \wedge dg_2 \wedge \cdots \wedge dg_k$.

If f is a function on $\mathbb{R}^3$, then $df = f_x dx + f_y dy + f_z dz$ is the 1-form identified with the vector field $(f_x, f_y, f_z) = grad(f)$.

If $F = (M, N, P)$ is a vector field on $\mathbb{R}^3$ we may identify it with the 1-form $\omega = M dx + N dy + P dz$. Then

$$d\omega = (M_x dx + M_y dy + M_z dz) \wedge dx + (N_x dx + N_y dy + N_z dz) \wedge dy + (P_x dx + P_y dy + P_z dz) \wedge dz$$

$$= M_y dy \wedge dx + M_z dz \wedge dx + N_x dx \wedge dy + N_z dz \wedge dy + P_x dx \wedge dz + P_y dy \wedge dz$$

$$= (P_y - N_z) dy \wedge dz - (M_z - P_x) dx \wedge dz + (N_x - M_y) dx \wedge dy$$

Thus $d\omega$ is identified with the vector field $(P_y - N_z, M_z - P_x, N_x - M_y) = \text{curl}F$.

If $F = (M, N, P)$ is a vector field on $\mathbb{R}^3$ we may identify it with the 2-form $\nu = M\, dy \wedge dz - N\, dx \wedge dz + P\, dx \wedge dy$. Then

$$d\nu = (M_x dx + M_y dy + M_z dz) \wedge dy \wedge dz - (N_x dx + N_y dy + N_z dz) \wedge dx \wedge dz + (P_x dx + P_y dy + P_z dz) \wedge dx \wedge dy$$

$$= M_x dx \wedge dy \wedge dz - N_y dy \wedge dx \wedge dz + P_z dz \wedge dx \wedge dy = \text{div}F dx \wedge dy \wedge dz$$

Generalized Stokes' theorem: Suppose $r \colon D \to X$ where D is a region in $\mathbb{R}^k$. Let $U = r(D)$ be the image of r. Then we say r parametrizes U. We say ∂U is the image of the boundary of D. For example let $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ and define $r \colon D \to \mathbb{R}^3$ by $r(x, y) = (x, y, x^2 + y^2)$. Then r parameterizes U where U is the portion of the paraboloid $z = x^2 + y^2$ which lies below the plane $z = 1$. We have ∂U is the circle $x^2 + y^2 = 1$, $z = 1$.

Suppose ω is a $k-1$-form. The generalized Stokes' theorem says that

$$\int_{\partial U} \omega = \int_U d\omega$$

This gives us Stokes' theorem, Gauss' theorem, Greens theorem, and many others.

For example, if ω is a 1-form in $\mathbb{R}^3$ corresponding to the vector field F then we saw above that $\int_{\partial U} \omega = \int_{\partial U} F \cdot T\, ds$. Also $d\omega$ corresponds to the vector field curl$F$ so $\int_U d\omega = \int\int_U \text{curl}F \cdot n\, dS$ so we get the classic Stokes' theorem.

For another example, if ν is a 2-form in $\mathbb{R}^3$ corresponding to the vector field F, then we saw above that $d\nu$ corresponds to div$F$. So if U is a solid region in $\mathbb{R}^3$ we have $\int\int\int_U \text{div}F\, dV = \int_U d\nu = \int_{\partial U} \nu = \int\int_{\partial U} F \cdot n\, dS$.