3.2 #8

Given A and D are square and $\det(A) \neq 0$

Want to show $\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A) \cdot \det(D - CA^{-1}B)$

The hint tells us to verify:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I & 0 \\ CA^{-1} & I \end{bmatrix}\begin{bmatrix} A & 0 \\ 0 & D-CA^{-1}B \end{bmatrix}\begin{bmatrix} I & A^{-1}B \\ 0 & I \end{bmatrix}$$

$$\begin{bmatrix} I & 0 \\ CA^{-1} & I \end{bmatrix}\begin{bmatrix} A & 0 \\ 0 & D-CA^{-1}B \end{bmatrix} = \begin{bmatrix} A & 0 \\ C & D-CA^{-1}B \end{bmatrix}$$

$$\begin{bmatrix} A & 0 \\ C & D-CA^{-1}B \end{bmatrix}\begin{bmatrix} I & A^{-1}B \\ 0 & I \end{bmatrix} = \begin{bmatrix} A & B \\ C & CA^{-1}B + D - CA^{-1}B \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

Then it says to use Exercises 3 and 4, which we were told we could assume.

We know $\det(ABC) = \det A \cdot \det B \cdot \det C$. So

$$\det\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det\begin{bmatrix} I & 0 \\ CA^{-1} & I \end{bmatrix} \cdot \det\begin{bmatrix} A & 0 \\ 0 & D-CA^{-1}B \end{bmatrix} \cdot \det\begin{bmatrix} I & A^{-1}B \\ 0 & I \end{bmatrix}$$

By exercise 3 and 4, we can see:

$$\det\begin{bmatrix} I & 0 \\ CA^{-1} & I \end{bmatrix} = \det(I) \cdot \det(I) = 1$$

$$\det\begin{bmatrix} I & A^{-1}B \\ 0 & I \end{bmatrix} = \det(I) \cdot \det(I) = 1$$

$$\det\begin{bmatrix} A & 0 \\ 0 & D-CA^{-1}B \end{bmatrix} = \det(A) \cdot \det(D-CA^{-1}B)$$

Therefore $\det\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(A) \cdot \det(D - CA^{-1}B)$.