Math 341 Exam #1 March 7, 2003

1. (15) Let $A = \begin{pmatrix} 2 & 1 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{pmatrix}$

a) Find all eigenvalues and eigenvectors of A. Some calculators can do this for you, so show enough work to let me know you can do it by hand.

Since A is upper triangular, the eigenvalues are 0 and 2. The eigenvectors for 0 are nonzero elements in the null space of A, which by inspection (or if you want by putting in echelon form) are all nonzero multiples of $(1,-2,0)^T$. Eigenvectors for the eigenvalue 2 are are all nonzero vectors in the null space of $2I - A = \begin{pmatrix} 0 & -1 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{pmatrix}$ which has echelon form $\begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. So a basis of the null space is $(1,0,0)^T, (0,1,1)^T$. So the eigenvectors are of the form $(a,b,b)^T$ where either a or b is 0.

b) If possible, find a matrix P so $P^{-1}AP$ is diagonal. If this is not possible, say why not.

There is a basis of eigenvectors, which will make up the columns of P. So we may let $P = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 1 & 0 \end{pmatrix}$.

2. (15) Let S be the subspace of $\mathbb{R}^4$ spanned by $(1,1,1,1)^T$, $(1,0,3,0)^T$, and $(1,-1,0,0)^T$. Find an orthogonal basis for S.

Let $u_1 = (1,1,1,1)^T$. We may let

$$u_2 = (1,0,3,0)^T - ((1,0,3,0)^T \cdot u_1/u_1 \cdot u_1)u_1 = (1,0,3,0)^T - u_1 = (0,-1,2,-1)$$

Now to get u_3 continue with Gram-Schmidt

$$u_3 = (1,-1,0,0)^T - ((1,-1,0,0)^T \cdot u_1/u_1 \cdot u_1)u_1 - ((1,-1,0,0)^T \cdot u_2/u_2 \cdot u_2)u_2$$

$$= (1,-1,0,0)^T - u_2/6 = (1,-5/6,-1/3,1/6)$$

If we wanted to pretty it up we could multiply u_3 by 6.

3. (20) Find and classify (local mx, local min, saddle, or degenerate) all critical points of $f(x,y,z) = 2x^3 - 3x^2 + y^5 - 20y + y^3z^2$.

To find the critical points we need to solve $6x^2 - 6x = 0$, $5y^4 - 20 + 3y^2z^2 = 0$, *and* $2y^3z = 0$. *From the first equation we get* $x = 0, 1$. *From the third equation we get* $y = 0$ *or* $z = 0$. *But* $y = 0$ *violates the second equation so we must have* $z = 0$. *Plugging* $z = 0$ *into the second equation gives* $y = \pm\sqrt{2}$. *Note that when* $z = 0$ *the Hessian is diagonal with diagonal entries* $12x - 6$, $20y^3$, $2y^3$. *These diagonal entries are the eigenvalues of the Hessian. So we may summarize the results as follows*

critical point	eigenvalues	type
$(0, \sqrt{2}, 0)$	$-6, 40\sqrt{2}, 4\sqrt{2}$	saddle
$(0, -\sqrt{2}, 0)$	$-6, -40\sqrt{2}, -4\sqrt{2}$	local max
$(1, \sqrt{2}, 0)$	$6, 40\sqrt{2}, 4\sqrt{2}$	local min
$(1, -\sqrt{2}, 0)$	$6, -40\sqrt{2}, -4\sqrt{2}$	saddle

Of course you could also use the derivative test given in Colley.

4. (20) Use Lagrange multipliers to do one of the following:

a) Find the points on the surface $xy^2 + 4z^2 = 16$ which are closest and furthest from the origin.

The constraint is $g(x, y, z) = xy^2 + 4z^2$ *and we wish to find the max/min of* $f(x, y, z) = x^2 + y^2 + z^2$. *So we need to solve* $2x = \lambda y^2$, $2y = \lambda 2xy$, $2z = \lambda 8z$, *and* $xy^2 + 4z^2 = 16$. *From the second equation, either* $y = 0$ *or* $\lambda = 1/x$. *If* $y = 0$ *then the first equation gives* $x = 0$ *and the last equation gives* $z = \pm 4$. *If* $\lambda = 1/x$, *the first equation gives* $y = \pm\sqrt{2}x$, *and the third equation gives either* $z = 0$ *or* $\lambda = 1/4$. *But the* $\lambda = 1/4$ *implies* $x = 1/\lambda = 4$ *and* $y = \pm 4\sqrt{2}$ *and plugging into the last equation gives* $16 = 4(\pm 4\sqrt{2})^2 + 4z^2 = 128 + 4z^2$ *which has no (real) solutions. So we must have* $z = 0$. *Plugging this into the last equation we get* $16 = xy^2 = 2x^3$ *so* $x = 2$ *which means* $y = \pm 2\sqrt{2}$. *So we summarize our results as follows:*

Possible extreme point	$f(x, y, z)$
$(0, 0, \pm 4)$	16
$(2, \pm 2\sqrt{2}, 0)$	12

At first glance we might say the minimum distance is at the points $(2, \pm 2\sqrt{2}, 0)$ *at distance* $\sqrt{12}$ *and the maximum is at* $(0, 0, \pm 4)$ *at distance 4. But what about the point* $(16, 1, 0)$ *which is on the level surface* $g = 16$ *and is much further away than 4? So in fact, there is no maximum distance from the origin. The surface* $g = 16$ *extends infinitely far out. (If there were a maximum distance it would have to be at one of the points we found,* *but it is not as the point* $(16, 1, 0)$ *shows.) So the minimum is at the points* $(2, \pm 2\sqrt{2}, 0)$ *but there is no maximum distance.*

b) Let a and b be perpendicular unit vectors in $\mathbb{R}^n$. Let T be the set of points x on the sphere $||x|| = \sqrt{2}$ where $a \cdot x = 1$. Find the maximum and minimum of $b \cdot x$ for $x \in T$.

We have two constraints $g_1(x) = ||x||^2 = 2$ *and* $g_2(x) = a \cdot x = 1$. *Let* $f(x) = b \cdot x$. *We need to solve the equations* $\nabla f = \lambda \nabla g_1 + \mu \nabla g_2$ *and* $||x||^2 = 2$ *and* $a \cdot x = 1$. *Note* $\nabla f = b$ *and* $\nabla g_1 = 2x$ *and* $\nabla g_2 = a$. *So our first equation is* $b = 2\lambda x + \mu a$. *Note* $\lambda \neq 0$ *since* b *and* a *are linearly independent. So* $x = b/(2\lambda) - \mu a/(2\lambda)$. *Dot this equation with* a *and we get* $1 = x \cdot a = 0 - \mu/(2\lambda)$ *so* $\mu = -2\lambda$. *So* $x = b/(2\lambda) + a$. *We have* $2 = ||x||^2 = 1/(2\lambda)^2 + 1^2$ *so* $\lambda = \pm 1/2$. *So* $x = a \pm b$. *The maximum occurs when* $x = a + b$ *and* $b \cdot x = 1$. *The minimum occurs when* $x = a - b$ *and* $b \cdot x = -1$.

There are other ways to solve this problem without lagrange multipliers. For example since $x \cdot a = 1$ *we know* $x = a + y$ *where* $y \cdot a = 0$. *Then* $||x||^2 = 1 + ||y||^2$ *so* $||y|| = 1$. *Then* $b \cdot x = b \cdot a + b \cdot y = 0 + \cos\theta$ *where* θ *is the angle between* b *and* y. *So the max is* $\theta = 0$ *where* $y = b$ *and the minimum is where* $y = -b$. *You could also solve this by letting* Q *be an orthogonal matrix with first two columns* a *and* b. *Change variables to* z *where* $z = Q^T x$. *Then* T *is where* $||z||^2 = 2$ *and* $z_1 = 1$, *and we wish to maximize* z_2 *on* T. *Of course* z_2 *ranges from* -1 *to* 1.

5. (15) Prove one of the following:

a) Suppose B is orthogonally diagonalizable, that is, there is an orthogonal matrix Q so that $Q^{-1}BQ$ is diagonal. Show that B is symmetric.

Suppose $Q^{-1}BQ = D$ *where* D *is diagonal. Note* $Q^{-1} = Q^T$ *so* $B = QDQ^T$. *Then* $B^T = (QDQ^T)^T = Q^{TT}D^TQ^T = QDQ^T = B$. *So* B *is symmetric.*

b) Recall a square matrix P is a projection matrix if $P^2 = P$. Show that a projection matrix Phas at most two different eigenvalues. (Hint, take an eigenvector v and write Pv in two ways.)

If the eigenvector v *has eigenvalue* λ, *then* $Pv = \lambda v$. *On the other hand* $Pv = P^2 v = P(Pv) = P(\lambda v) = \lambda P v = \lambda^2 v$. *So* $\lambda = \lambda^2$ *since* $v \neq 0$. *So the only possible* λ *are* $\lambda = 0$ *and* $\lambda = 1$. *In fact, unless* P *is 0 or the identity there will be exactly two eigenvalues, 0 and 1. (To prove this, use Schur's theorem to reduce to the case where* P *is upper triangular then prove it directly for upper triangular projections.)*

6. (15) Prove one of the following:

a) If A is any $k \times n$ matrix then all eigenvalues of $A^T A$ are real and nonnegative. (Hint: If v is an eigenvector of $A^T A$, consider the dot product of v and $A^T Av$. Write this as a product of matrices and simplify in two ways.)

Since $A^T A$ is symmetric, its eigenvalues are real. Now $v^T A^T Av = v^T \lambda v = \lambda v^T v = \lambda ||v||^2$. But also $v^T A^T Av = (Av)^T Av = ||Av||^2$. So $\lambda = ||Av||^2/||v||^2 \geq 0$ since it is the quotient of a nonegative by a positive number.

b) If C is any square matrix show that C and C^T have the same eigenvalues. (Hint, compare their characteristic polynomials). Also, prove or disprove whether or not they have the same eigenvectors.

The characteristic polynomial of C is $c(\lambda) = \det(\lambda I - C)$. The characteristic polynomial of C^T is $c'(\lambda) = \det(\lambda I - C^T)$. But $\det(\lambda I - C) = \det((\lambda I - C)^T) = \det((\lambda I)^T - C^T) = \det(\lambda I - C^T)$. So C and C^T have the same characteristic polynomial. So they have the same eigenvalues since the eigenvalues are exactly the roots of the characteristic polynomial. But any random example will show the eigenvectors differ. For example $C = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ which has eigenvectors $(a, 0)^T$ but C^T has eigenvectors $(0, b)^T$.