Math 341 Exam #2 April 14, 2003

1. (15) Find all solutions of $y\cos(yt)+e^t+t\cos(yt)y'=0$.

This is exact and is $(\sin(yt)+e^t)'=0$ *which means* $\sin(yt)+e^t=c$ *which means* $y=t^{-1}\sin^{-1}(c-e^t)$. *Here* $\sin^{-1}$ *refers to all possible arcsines, not just those with values from* $-\pi/2$ *to* $\pi/2$.

2. (25) Find all solutions of $y'''-3y''+3y'-y=e^t$. Or, for a 10 point penalty, you may find all solutions of $y''-2y'+y=e^t$.

The associated polynomial is $(r-1)^3$ *which means the method of judicious guessing can be applied with a guess* $y=at^3e^t$. *Then* $y'=3at^2e^t+at^3e^t$, $y''=6ate^t+6at^2e^t+at^3e^t$, *and* $y'''=6ae^t+18ate^t+9at^2e^t+at^3e^t$. *Plugging in to* $y'''-3y''+3y'-y=e^t$ *we get* $6ae^t=e^t$ *so* $a=1/6$ *and thus all solutions are*

$$y=t^3e^t/6+c_0e^t+c_1te^t+c_2t^2e^t$$

The second order poblem is similar, except you guess $y=at^2e^t$ *and get the solution* $y=t^2e^t/2+c_0e^t+c_1te^t$. *You could also solve the second order problem using variation of parameters.*

3. (10) Find the Laplace transform of $f(t)=\begin{cases} t^2 & \text{if } t<1 \\ 1 & \text{if } 1\le t\le 3 \\ \sin t & \text{if } t>3 \end{cases}$.

We have $f(t)=t^2+H_1(t)(1-t^2)+H_3(t)(\sin t-1)$ *(except at* $t=3$*) so*

$$f(t)=t^2+H_1(t)(1-((t-1)+1)^2)+H_3(t)(\sin((t-3)+3)-1)$$

$$f(t)=t^2-H_1(t)(2(t-1)+(t-1)^2)+H_3(t)(\sin(t-3)\cos 3+\cos(t-3)\sin 3-1)$$

$$\mathcal{L}(f)=2/s^3-e^{-s}(2/s^2+2/s^3)+e^{-3s}(\frac{\cos 3+s\sin 3}{s^2+1}-\frac{1}{s})$$

4. (20) Solve $y'' + 2y' = \delta(t-2)$, $y(0) = 1$, $y'(0) = 2$.

Taking Laplace transforms we have

$$s^2Y - s - 2 + 2(sY - 1) = e^{-2s}$$

$$Y = (s+4)/(s^2+2s) + e^{-2s}/(s^2+2s)$$

If $1/(s^2+2s) = a/s + b/(s+2)$ *then* $1 = a(s+2) + bs$ *so* $a = 1/2$, $b = -1/2$. *If* $(s+4)/(s^2+2s) = c/s + d/(s+2)$ *then* $s+4 = c(s+2) + ds$ *so* $c = 2$, $d = -1$. *So*

$$Y = 2/s - 1/(s+2) + (1/2)e^{-2s}/s - (1/2)e^{-2s}/(s+2)$$

$$y = 2 - e^{-2t} + H_2(t)(1 - e^{-2(t-2)})/2$$

5. (18) What does the existence and uniqueness theorem say about each of the following IVPs:

a) $ty'' + y' - y^2 = t$, $y(1) = 2$, $y'(1) = 4$.

If we set $w = y'$ *we may write this as* $(y, w)' = (w, 1 - (1/t)w + (1/t)y^2) = F(y, w, t)$. *Note* F *is differentiable at* $t = 1$, $y = 2$, $w = 4$ *so by the theorem we know that there is a unique solution* $y(t)$ *and it is valid for* $t \in (1-\epsilon, 1+\epsilon)$ *for some* $\epsilon > 0$.

b) $ty'' + y' - y^2 = t$, $y(0) = 2$, $y'(0) = 4$.

Since F *is not defined at* $t = 0$ *the existence and uniqueness theorem has nothing to say in this case.*

c) $ty'' + (\sin t)y' - (e^t - 1)y^2 = t$, $y(0) = 2$, $y'(0) = 4$.

Here $(y, w)' = (w, 1 - (\sin t/t)w + ((e^t - 1)/t)y^2) = F(y, w, t)$. *Using L'Hospital's rule we see that* F *is continuous when* $t = 0$ *if we define* $F(y, w, 0) = (w, 1 - w + y^2)$. *Since* $\partial F/\partial y$ *and* $\partial F/\partial w$ *also are continuous* *we know F is Lipschitz in y and w (see the handout). So in fact the theorem applies here and we know that there is a unique solution $y(t)$ and it is valid for $t \in (-\epsilon, \epsilon)$ for some $\epsilon > 0$.*

6. (12) Suppose y_1 is a solution to $ty''' + e^t y' - y = 1$ and y_2 is a solution to $ty''' + e^t y' - y = e^t$ and y_3 is a solution to $ty''' + e^t y' - y = 3$. Which of the following statements must be true?

a) $y_2 = e^t y_1$.

There is no reason this should be true. With sufficient work you could show that y_2 could never equal $e^t y_1$. But it suffices to note that if y_h is any nonzero solution to the homogeneous equation $ty''' + e^t y' - y = 0$ then $y_1 + y_h$ also solves $ty''' + e^t y' - y = 1$. So if this must be true we would have both $y_2 = e^t y_1$ and $y_2 = e^t(y_1 + y_h)$ which contradicts $y_h \neq 0$.

b) $y_3 = 3y_1$.

This is true for certain choices of y_1 and y_3 but not for others. See d which shows that $y_3 = 3y_1 + y_h$ where y_h is any of the many solutions of the homogeneous equation $ty''' + e^t y' - y = 0$.

c) $3y_1 - y_2$ is a solution to $ty''' + e^t y' - y = 3 - e^t$.

True, since it is a linear differential equation.

d) $3y_1 - y_3$ is a solution to $ty''' + e^t y' - y = 0$.

True, since it is a linear differential equation.