Math 341 Exam #3 May 12, 2003

1. (25) Let $A = \begin{pmatrix} 3 & 1 & -1 \\ 0 & 2 & 4 \\ -1 & -1 & -1 \end{pmatrix}$ which has characteristic polynomial $\lambda(\lambda-2)^2$. Find all solutions to $y' = Ay$.

Answer: *The eigenspaces are the span of* $(1\ -2\ 1)^T$ *for* $\lambda = 0$ *and the span of* $(1\ -1\ 0)^T$ *for* $\lambda = 2$. *So we need to use generalized eigenvectors.* $(A-2I)^2 = \begin{pmatrix} 2 & 2 & 6 \\ -4 & -4 & -12 \\ 2 & 2 & 6 \end{pmatrix}$ *so for example,* $(0\ 3\ -1)^T$ *is in the Null space of* $(A-2I)^2$ *but not in the Null space of* $A-2I$. *We have* $(A-2I)(0\ 3\ -1)^T = (4\ -4\ 0)^T$. *So the general solution is:*

$$y(t) = c_1(1\ -2\ 1)^T + c_2e^{2t}(1\ -1\ 0)^T + c_3e^{2t}((0\ 3\ -1)^T + (4t\ -4t\ 0)^T)$$

2. (25) Solve the system $x' = 3x + 2y$, $y' = -x + y$, $x(0) = 1$, $y(0) = 0$. Give a rough sketch of the resulting orbit.

Answer: *Let* $A = \begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix}$. *Then* A *has eigenvalues* $2 \pm i$. *An eigenvector for* $2+i$ *is* $(1+i\ -1)^T$. *So if we set* $c = \cos t$ *and* $s = \sin t$ *then a complex solution is*

$$e^{2t}(\cos t + i\sin t)(1+i\ -1)^T = e^{2t}(c-s, -c)^T + ie^{2t}(s+c, -s)^T$$

Taking real and imaginary parts, we see that the general solution is

$$(x\ y)^T = c_1e^{2t}(c-s, -c)^T + c_2e^{2t}(s+c, -s)^T$$

Setting $t = 0$ *we see that* $(1\ 0)^T = c_1(1\ -1)^T + c_2(1\ 0)^T$ *so* $c_1 = 0$ *and* $c_2 = 1$. *So the solution is* $x = e^{2t}(\cos t + \sin t)$, $y = -e^{2t}\sin t$. *The orbit is a spiral, spiralling outward. Since* $x'(0) = 3$ *and* $y'(0) = -1$ *it is spiraling clockwise.*

3. (25) Find all stationary points of $x' = x - x^3 + xy(x-1)$, $y' = -2y$ and sketch the orbits near each stationary point. For full credit your sketches should account for the eigenvectors. Comment on the stability near each stationary point. For 5 points extra credit, draw a plausible global sketch of the orbits in a region of the xy plane including all your stationary points.

Answer: *$y' = 0$ implies $y = 0$ and then $x' = 0$ implies $x - x^3 = 0$ so $x = 0, \pm 1$. So the three stationary points are $(0,0)$, $(1,0)$ and $(-1,0)$. The linearization is $\begin{pmatrix} 1 - 3x^2 + 2xy - y & x(x-1) \\ 0 & -2 \end{pmatrix}$. At $(0,0)$ this is $\begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$ so it is a saddle which flows outward in the x direction (since e_1 is the eigenvector for 1) and flows inward in the y direction. Of course $(0,0)$ is unstable. At $(1,0)$ this is $\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$ so this is assymptoticly stable and for each direction you have an orbit approaching $(1,0)$ with that limiting tangent. At $(-1,0)$ this is $\begin{pmatrix} -2 & 2 \\ 0 & -2 \end{pmatrix}$ so this is assymptoticly stable. But here the only eigenvector is e_1 so all orbits approach $(-1,0)$ tangent to the x axis. Note in fact that the x and y axes are unions of orbits.*

4. (25) Find all solutions to $x' = x + y + 1$, $y' = 2x + 2y + e^t$.

Answer: *Solutions to the homogeneous system are $c_1(1 \quad -1) + c_2 e^{3t}(1\ 2)$. The method of judicious guessing implies a guess $x = at + b + ce^t$ and $y = dt + f + ge^t$. We need the t terms because 0 is an eigenvalue for the homogeneous system. Plugging in we get $a + ce^t = (a+d)t + b + f + 1 + (c+g)e^t$ and $d + ge^t = (2a+2d)t + 2b + 2f + (2c + 2g + 1)e^t$ which has solutions $a = 2/3$, $d = -2/3$, $f = -1/3 - b$, $c = -1/2$, $g = 0$. So the general solution is: $(x,y) = c_1(1,-1) + c_2 e^{3t}(1,2) + 1/3(2t-1, -2t) - e^t/2(1,0)$*