Math 341 Exam #2 April 13, 2007

1. (15) Solve $y' = e^{t+2y-1}$. $y(1) = 0$.

Answer: *This is similar to a homework problem. Rewrite as* $e^{-2y}y' = e^{t-1}$. *Integrate to get* $(-1/2)e^{-2y} = e^{t-1} + C$. *Plug in* $t = 1$ *and* $y = 0$ *and get* $-1/2 = 1+C$ *so* $C = -3/2$. *So* $-e^{-2y}/2 = e^{t-1}-3/2$ *so* $e^{-2y} = 3-2e^{t-1}$ *so* $y = (-1/2)\ln(3 - 2e^{t-1})$.

2. (15) Suppose y satisfies the initial value problem $y' = 1-y+t$, $y(0) = 2$.

a) Find the exact value $y(2)$. In case you need them, here are a few values for powers of e, $e^2 \approx 7.3891$, $e \approx 2.7183$, $e^{-1} \approx .3679$, $e^{-2} \approx .1353$.

Answer: *We have* $y' + y = 1 + t$. *Judicious guessing gives a particular solution* $y = at + b$ *so* $a + at + b = 1 + t$ *so* $a = 1$, $b = 0$, *so* $y = t$. *The general solution is thus* $y = t + ce^{-t}$. *Since* $y(0) = 2$ *we have* $c = 2$ *and* $y = t + 2e^{-t}$. *So* $y(2) = 2 + 2e^{-2} \approx 2.2706$.

b) Use the improved Euler method with a step size $h = 1$ to estimate $y(2)$.

Answer: $k_1 = 1 - 2 + 0 = -1$, $\hat{y} = 2 + 1(-1) = 1$, $k_2 = 1 - \hat{y} + 1 = 1$, *so our estimate for* $y(1)$ *is* $2 + (k_1 + k_2)h/2 = 2$. *For the next step,* $k_1 = 1-2+1 = 0$, $\hat{y} = 2+1(0) = 2$, $k_2 = 1-2+2 = 1$, *so our estimate for* $y(2)$ *is* $2 + (0 + 1)/2 = 2.5$.

c) What is the error in your estimate of $y(2)$?

Answer: *the error is* $|2.5 - 2.2706| \approx .23$.

d) If you used the improved Euler method with a step size $h = .1$, what is your best guess for the error in the resulting estimate of $y(2)$? Give an intelligent reason for your guess.

Answer: *The error in improved Euler is roughly proportional to* h^2 *so the best guess is that the error would go down by a factor of* 10^2 *so we could guess the error would be* .0023. *Note though that since our initial*

step size was pretty large this is probably way off. In fact as you can easily verify by making a spreadsheet, running improved Euler with $h = .1$ *gives* $y(2) \approx 2.2716$ *for an error of* .001.

3. (25) Find all solutions of $y''' - 2y'' + y' = 1 + 8e^t + e^{-t}$.

Answer: *Since* $r^3 - 2r^2 + r = r(r-1)^2$ *the homogeneous solutions are* $c_1 + c_2e^t + c_3te^t$. *Use judicious guessing. To get* 1, *try* $y_1 = at$, *then* $y''' - 2y'' + y' = a$ *and we let* $a = 1$, *so we get* $y_1 = t$. *To get* $8e^t$ *we try* $y_2 = at^2e^t$. *Then* $y_2' = 2ate^t + at^2e^t$, $y_2'' = 2ae^t + 4ate^t + at^2e^t$, $y_2''' = 6ae^t + 6ate^t + at^2e^t$. *Then* $y_2''' - 2y_2'' + y_2' = 2ae^t$ *so* $a = 4$ *and* $y_2 = 4t^2e^t$. *For* e^{-t} *try* $y_3 = ae^{-t}$ *then* $y_3''' - 2y_3'' + y_3' = -4ae^{-t}$ *so* $a = -1/4$. *So the general solution is* $y = t + 4t^2e^t - e^{-t}/4 + c_1 + c_2e^t + c_3te^t$.

4. (20) Solve using Laplace transforms $y'' - y = 4\delta(t-2)$, $y(0) = 0$, $y'(0) = 2$.

Answer: $s^2Y - 2 - Y = 4e^{-2s}$ *so* $Y = 2/(s^2-1) - 4e^{-2s}/(s^2-1)$. *Write* $2/(s^2-1) = a/(s-1) + b/(s+1)$ *then* $2 = a(s+1) + b(s-1)$ *so plugging in* $s = 1$ *we get* $a = 1$ *and from* $s = -1$ *we get* $b = -1$. *So* $Y = 1/(s-1) - 1/(s+1) + 2e^{-2s}(1/(s-1) - 1/(s+1))$. *So* $y = e^t - e^{-t} + 2H_2(t)(e^{t-2} - e^{2-t})$.

5. (15) Consider the differential operator $L(y) = y''' - ty' + 2y$. Suppose $L(y_1) = t$, $L(y_2) = 7t$, and $L(y_3) = 0$.

a) What is the dimension of the null space of L?

Answer: *Since L is order 3 its null space has dimension 3.*

b) Show that $y_2 - 3y_1 + 4y_3$ is a solution to the differential equation $y''' - ty' + 2y = 4t$.

Answer: $L(y_2 - 3y_1 + 4y_3) = L(y_2) - 3L(y_1) + 4L(y_3) = 7t - 3t + 4(0) = 4t$. *So* $y = y_2 - 3y_1 + 4y_3$ *satisfies* $y''' - ty' + 2y = 4t$.

c) Does $y_2 = 7y_1$? Your answer should be one of the following:

i) y_2 must equal $7y_1$.

ii) y_2 might or might not equal $7y_1$.

iii) y_2 could not equal $7y_1$.

Answer: *ii) since all we know is that $7y_1 - y_2$ is in the null space of L. So we might or might not have $7y_1 - y_2 = 0$.*

6. (10) For each of the differential equations given in questions 1-5 determine whether the differential equation is linear or not and determine its order.

Answer: *1 is not linear but 2-5 are. 1 and 2 are order 1, 4 is order 2, and 3 and 5 are order 3,*