Math 341 Final Exam May 20, 2003

1. (30) Find all solutions of $y''' - 2y'' + 5y' = t + e^t$.

2. (30) Find the maximum and minimum of the function $f(x, y, z) = x^3 + 3x^2 + 2y^2 + z^2$ in the oval region $2x^2 + y^2 + z^2 \leq 1$.

3. (30) Solve $y'' + 2y' + y = \delta(t - 3)$, $y(0) = 1$, $y'(0) = -1$.

4. (20) Solve $(1 + t^2)y' + 2ty = e^t$, $y(0) = 0$.

5. (30) Let $A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & 2 & 0 \end{pmatrix}$ which has characteristic polynomial $\lambda^3 - 3\lambda^2 + 2\lambda$. Solve $y' = Ay$, $y(0) = (1, 2, 3)^T$.

6. (30) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a thrice differentiable function. Suppose that at each critical point of f the Hessian $(\partial^2 f / \partial x_i \partial x_j)$ is nonsingular. (Such an f is called a Morse function.) Let F be the gradient of f.

a) Show that every critical point of f is an equilibrium point of the differential equation $x' = F(x)$ and vice versa.

b) Find a relation between the linearization of F and the Hessian of f at a critical point.

c) Show that every stable equilibrium point of $x' = F(x)$ is assymptotically stable and is also a local maximum of f. Is every local maximum of f a stable equilibrium point?

d) If $n = 2$, show that every saddle critical point of f is a saddle equilibrium point of $x' = F(x)$ and vice versa.

7. (30) If A is any $k \times n$ matrix and v is an eigenvector of $A^T A$ of length 1 with eigenvalue λ, show that $||Av|| = \sqrt{\lambda}$.

8. (20) (Extra Credit) (Doroslovaki's Theorem) Let y_0 be a stable equilibrium point of the differential equation $y' = F(y)$. Show that $\mathrm{div} F(y_0) \leq 0$. Show by example that the converse is not true, and in fact there is a vector field F with $\mathrm{div} F(0) < 0$ and an unstable equilibrium point at 0.