Math 341 Final Exam May 20, 2003

1. (30) Find all solutions of $y''' - 2y'' + 5y' = t + e^t$.

Since $r^3 - 2r^2 + 5r = r(r^2 - 2r + 5)$ *we have roots* $r = 0$ *and* $r = 1 \pm 2i$ *so solutions to the homogeneous equation are* $c_1 + e^t(c_2 \cos(2t) + c_3 \sin(2t))$. *By judicious guessing we know a particular solution has the form* $at^2 + bt + ce^t$. *Plugging in, we get*

$$ce^t - 2(2a + ce^t) + 5(2at + b + ce^t) = 10at + 5b - 4a + 4ce^t = t + e^t$$

so $a = 1/10$, $b = 4/50$, *and* $c = 1/4$. *So the solutions are:*

$$y = c_1 + e^t(c_2 \cos(2t) + c_3 \sin(2t)) + t^2/10 + 4t/50 + e^t/4$$

2. (30) Find the maximum and minimum of the function $f(x, y, z) = x^3 + 3x^2 + 2y^2 + z^2$ in the oval region $2x^2 + y^2 + z^2 \leq 1$.

The critical points of f *are where* $3x^2 + 6x = 0$, $4y = 0$, $2z = 0$, *so they are at* $(0, 0, 0)$ *and* $(-2, 0, 0)$. *But only* $(0, 0, 0)$ *is in the oval region. Possible max/min on the boundary are where* $3x^2 + 6x = \lambda 4x$, $4y = \lambda 2y$, $2z = \lambda 2z$. *From the last equation,* $\lambda = 1$ *or* $z = 0$. *In case* $\lambda = 1$ *we get* $4y = 2y$ *so* $y = 0$ *and* $3x^2 + 6x = 4x$ *so* $x = 0$ *or* $-2/3$. *Solving for* z *we then get the points* $(0, 0, \pm 1)$ *and* $(-2/3, 0, \pm 1/3)$. *In the case* $z = 0$ *the second equation gives* $y = 0$ *or* $\lambda = 2$. *In the* $\lambda = 2$ *case we get* $3x^2 + 6x = 8x$ *so* $x = 0$ *or* $x = 2/3$ *so we get the points* $(0, \pm 1, 0)$ *and* $(2/3, \pm 1/3, 0)$. *In the* $y = 0$ *case we solve for* x *and get the points* $(\pm 1/\sqrt{2}, 0, 0)$. *From the table below we see that the minimum is* 0 *at the origin and the maximum is* 2 *at* $(0, \pm 1, 0)$.

Possible extreme point	Value of f at the point
$(0, 0, 0)$	0

$(0,0,\pm 1)$	1
$(-2/3,0,\pm 1/3)$	$31/27$
$(0,\pm 1,0)$	2
$(2/3,\pm 1/3,0)$	$50/27$
$(1/\sqrt{2},0,0)$	$\sqrt{2}/4+3/2 \approx 1.85$
$(-1/\sqrt{2},0,0)$	$-\sqrt{2}/4+3/2 \approx 1.15$

3. (30) Solve $y''+2y'+y=\delta(t-3)$, $y(0)=1$, $y'(0)=-1$.

The Laplace transform is $s^2Y-s+1+2(sY-1)+Y=e^{-3s}$ *so* $Y=1/(s+1)+e^{-3s}/(s+1)^2$. *Since the inverse Laplace transform of* $1/(s+1)^2$ *is* te^{-t} *we then get* $y=e^{-t}+H_3(t)(t-3)e^{-t+3}$.

4. (20) Solve $(1+t^2)y'+2ty=e^t$, $y(0)=0$.

This is $((1+t^2)y)'=e^t$ *so* $(1+t^2)y=e^t+C$. *Plugging in* $t=0=y$ *we get* $C=-1$. *So* $y=(e^t-1)/(1+t^2)$.

5. (30) Let $A=\begin{pmatrix}1 & -1 & 1\\ 0 & 2 & 0\\ 0 & 2 & 0\end{pmatrix}$ which has characteristic polynomial $\lambda^3-3\lambda^2+2\lambda$. Solve $y'=Ay$, $y(0)=(1,2,3)^T$.

The eigenvalues are $0,1,2$. *An eigenvector for* $\lambda=0$ *is* $(1,0,-1)^T$. *An eigenvector for* $\lambda=1$ *is* $(1,0,0)^T$. *An eigenvector for* $\lambda=2$ *is* $(0,1,1)^T$. *We have* $(1,2,3)^T=-(1,0,-1)^T+2(1,0,0)^T+2(0,1,1)^T$ *so the solution is* $y=-(1,0,-1)^T+2e^t(1,0,0)^T+2e^{2t}(0,1,1)^T$.

6. (30) Let $f:\mathbb{R}^n\to\mathbb{R}$ be a thrice differentiable function. Suppose that at each critical point of f the Hessian $(\partial^2 f/\partial x_i\partial x_j)$ is nonsingular. (Such an f is called a Morse function.) Let F be the gradient of f.

a) Show that every critical point of f is an equilibrium point of the differential equation $x'=F(x)$ and vice versa.

A critical point of f is a point x_0 where the gradient of f is 0, i.e., $F(x_0) = 0$. Likewise, an equilibrium point is a point where $F(x_0) = 0$.

b) Find a relation between the linearization of F and the Hessian of f at a critical point.

The linearization of F is the derivative DF whose i, j-th entry is $\partial/\partial x_j$ of the i-th entry of F. But the i-th entry of F is $\partial f/\partial x_i$. So the i, j-th entry of DF is $\partial^2 f/\partial x_i \partial x_j$. In other words, DF is equal to the Hessian.

c) Show that every stable equilibrium point of $x' = F(x)$ is assymptotically stable and is also a local maximum of f. Is every local maximum of f a stable equilibrium point?

If x_0 is a stable equilibrium point then all eigenvalues of $DF(x_0)$ must have nonpositive real part. But $DF(x_0)$ is symmetric by b), so all its eigenvalues are real. It has no zero eigenvalues since it is nonsingular. So all its eigenvalues are strictly negative and thus the point is assymptotically stable. Since all eigenvalues of the Hessian are negative the critical point is a local maximum. Conversely, every local maximum of f is an assymptotically stable equilibrium point since all the eigenvalues of DF are negative.

d) If $n = 2$, show that every saddle critical point of f is a saddle equilibrium point of $x' = F(x)$ and vice versa.

A saddle critical point is where the Hessian has one negative and one positive eigenvalue. A saddle equilibrium point is where DF has one negative and one positive eigenvalue. So by b), these are the same points.

7. (30) If A is any $k \times n$ matrix and v is an eigenvector of $A^T A$ of length 1 with eigenvalue λ, show that $||Av|| = \sqrt{\lambda}$.

Since v has length 1, we know $v^T v = 1$. $||Av||^2 = (Av)^T Av = v^T A^T Av = v^T \lambda v = \lambda v^T v = \lambda$. So $||Av|| = \sqrt{\lambda}$.

8. (20) (Extra Credit) (Doroslovaki's Theorem) Let y_0 be a stable equilibrium point of the differential equation $y' = F(y)$. Show that $\mathrm{div}F(y_0) \leq 0$. Show by example that the converse is not true, and in fact there is a vector field F with $\mathrm{div}F(0) < 0$ and an unstable equilibrium point at 0.

If F_i *is the* i*-th coordinate of* F *we have* $\mathrm{div}F = \partial F_1/\partial x_1 + \cdots + \partial F_n/\partial x_n$. *So* $\mathrm{div}F$ *is the trace of the derivative matrix* DF. *The trace of a matrix is equal to the sum of its eigenvalues. But since* y_0 *is stable, all its eigenvalues must have nonpositive real part. So the sum of all the eigenvalues has nonpositive real part. Since the sum is real, the sum of the eigenvalues is nonpositive. So* $\mathrm{div}F(y_0) \leq 0$. *Consider the example* $F(y) = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} y$ *for* $y \in \mathbb{R}^2$. *Then* $\mathrm{div}F = -1$ *but the equilibrium point* $(0, 0)$ *is an unstable saddle.*