MATH 341 **EXAM # 1** **March 2, 2007**

1. [20] Find and classify all critical points of the function $f(x, y, z) = x^2 + 5y^2 + 4xy + 3z^2 - z^3$.

Answer: *Critical points are where* $0 = \nabla f = (2x + 4y, 10y + 4x, 6z - 3z^2)$, *so* $x = -2y$ *and* $x = -2.5y$ *which implies* $x = y = 0$. *Also* $0 = 6z - 3z^2 = 3z(2 - z)$ *so* $z = 0$ *or 2. So the two critical points are* $(0, 0, 0)$ *and* $(0, 0, 2)$. *The Hessian is* $H = \begin{bmatrix} 2 & 4 & 0 \\ 4 & 10 & 0 \\ 0 & 0 & 6-6z \end{bmatrix}$. *At* $(0, 0, 0)$ *the characteristic polynomial of* H *is* $(\lambda - 6)(\lambda^2 - 12\lambda + 4)$ *which has roots* $6, \frac{12\pm\sqrt{144-16}}{2}$ *which are all positive. So* $(0, 0, 0)$ *is a local minimum. At* $(0, 0, 2)$ *the characteristic polynomial of* H *is* $(\lambda + 6)(\lambda^2 - 12\lambda + 4)$ *which has roots* $-6, \frac{12\pm\sqrt{144-16}}{2}$ *which are both negative and positive. So* $(0, 0, 2)$ *is a saddle point.*

Comments: Very few of you actually knew how to do this problem. Some of you tried to use the test for two variables in this three variable problem. Some thought the characteristic values were the diagonal entries. While by accident, these incorrect methods gave the correct answer in this problem, they would not do so in other problems.

2. [20] Use Lagrange multipliers to find the critical points of $f(x, y, z) = x + 2y + 3z$ subject to the constraints $z^2 - y^2 = 3$ and $y + 2x = 1$.

Answer: *The gradients of the constraints are* $(0, -2y, 2z)$ *and* $(2, 1, 0)$ *so they are linearly independent unless* $y = z = 0$ *but this does not satisfy the first constraint. So the gradients of the constraints are linearly independent. So the Lagrange multiplier equations are:*

$$1 = 2\lambda_2$$

$$2 = -2y\lambda_1 + \lambda_2$$

$$3 = 2z\lambda_1$$

$$z^2 - y^2 = 3$$

$$y + 2x = 1$$

So we get $\lambda_2 = 1/2$ *and plugging this into the second equation gives* $\lambda_1 = -3/(4y)$. *Plugging this into the third equation we get* $3 = -6z/(4y)$ *so we get* $z = -2y$. *Plugging into the first constraint gives* $4y^2 - y^2 = 3$ *so* $y = \pm 1$. *If* $y = 1$ *the second constraint implies* $x = 0$ *and if* $y = -1$ *then* $x = 1$. *So the critical points are* $(0, 1, -2)$ *and* $(1, -1, 2)$. *For those who are interested, it turns out that these are neither maxima nor minima of* f. *The constraint is two curves, one with* $z < 0$ *and the other with* $z > 0$. *Then* $-4 = f(0, 1, -2)$ *is the maximum of* f *on the first curve and* $5 = f(1, -1, 2)$ *is the minimum of* f *on the second.*

Comments: Many of you found too many critical points. For example you found correctly that $\lambda_1 = \pm 3/4$ and $y = -3/(4\lambda_1)$ and $z = 3/(2\lambda_1)$. Then you said this implies $y = \pm 1$ and $z = \pm 2$ which is true, but the $\pm$ signs are not independent. If $\lambda_1 = 3/4$ then we get $y = -1$ and $z = 2$. If $\lambda_1 = -3/4$ then we get $y = 1$ and $z = -2$. In particular, $(0, 1, 2)$ and $(1, -1, -2)$ are not critical points.

3. [20] Find an orthonormal basis of $\mathbb{R}^3$ by applying the Gram-Schmidt process to the basis $\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Answer: *Lets first find an orthogonal basis. For typographical convenience change our vectors to row vectors.*

$$\beta_1 = (2, 2, -1), \quad \beta_1 \cdot \beta_1 = 9$$

$$\beta_2 = (0, 1, -1) - ((0, 1, -1) \cdot (2, 2, -1)/9)(2, 2, -1) = (0, 1, -1) - (2/3, 2/3, -1/3) = (-2/3, 1/3, -2/3)$$

$$\beta_2 \cdot \beta_2 = 1$$

$$\beta_3 = (0, 0, 1) - ((0, 0, 1) \cdot (2, 2, -1)/9)(2, 2, -1) - ((0, 0, 1) \cdot (-2/3, 1/3, -2/3))(-2/3, 1/3, -2/3)$$

$$= (0, 0, 1) + (2/9, 2/9, -1/9) + (-4/9, 2/9, -4/9) = (-2/9, 4/9, 4/9)$$

$$\beta_3 \cdot \beta_3 = 4/9$$

After normalizing by dividing each vector by its length we get the orthonormal basis

$$(2/3, 2/3, -1/3),\ (-2/3, 1/3, -2/3),\ (-1/3, 2/3, 2/3)$$

Comments: Part of the Gram-Schmidt process is that the span of the first k vectors of each basis is the same. So you can not mess with the order of the vectors in the basis. However, Cullen does not seem to say this. So I ended up regrading the problem and giving full credit if you switched the order of the basis.

4. [20] Find all characteristic values and characteristic vectors of $A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 2 & 6 \\ 0 & 0 & 0 \end{bmatrix}$. Determine whether or not A is similar to a diagonal matrix.

Answer: *This matrix is triangular, so the characteristic values are the diagonal entries* 0 *and* 2*. The characteristic vectors for $\lambda = 0$ are nonzero vectors in $NS(A - 0I) = NS(A) = Span(1, 0, 0)^T$. So characteristic vectors for $\lambda = 0$ have the form $(c, 0, 0)^T$ with $c \neq 0$. The characteristic vectors for $\lambda = 2$ are nonzero vectors in $NS(A - 2I) = Span(1, 2, 0)^T$. So characteristic vectors for $\lambda = 2$ have the form $(c, 2c, 0)^T$ with $c \neq 0$. This matrix is not similar to a diagonal matrix since we cannot find a basis of characteristic vectors. The characteristic vectors only span a two dimensional subspace, the xy plane.*

5. [20] Suppose B is a singular symmetric 3×3 matrix, $B \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$, and $B \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$.

a) What are the characteristic values and characteristic vectors of B?

b) Find an orthogonal matrix P so that $P^T BP$ is diagonal.

Answer: *Since B is singular, 0 is a characteristic value of B. From the examples we see that 2 is also a characteristic value and in fact $NS(B - 2I)$ is two dimensional. Since B is symmetric, the characteristic vectors for $\lambda = 0$ are perpendicular to those for $\lambda = 2$. So $(1, 0, 1) \times (0, 1, 1) = (-1, -1, 1)$ is characteristic for $\lambda = 0$. So the characteristic vectors for $\lambda = 0$ are $c(-1, -1, 1)$ with $c \neq 0$. The characteristic vectors for $\lambda = 2$ are $c(1, 0, 1) + d(0, 1, 1)$ with $c \neq 0$ or $d \neq 0$ (or both nonzero). We can find the first two columns of P by applying Gram-Schmidt to $(1, 0, 1), (0, 1, 1)$. $(0, 1, 1) - ((1, 0, 1) \cdot (0, 1, 1)/2)(1, 0, 1) = (-1/2, 1, 1/2)$ and normalize to $(1, 0, 1)/\sqrt{2}$ and $(-1, 2, 1)/\sqrt{6}$. Thus we can take $P = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{6} & -1/\sqrt{3} \\ 0 & 2/\sqrt{6} & -1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}$.*