MATH 341 EXAM # 3 May 7, 2007

1. [30] The matrix $A = \begin{bmatrix} 2 & 1 & 0 \\ -8 & -2 & 1 \\ 13 & 1 & -3 \end{bmatrix}$ has characteristic polynomial $(\lambda+1)^3$. Find all solutions to $x' = Ax$ and determine the stability of the equilibrium point at 0.

Answer: *We know the generalized eigenspace for* $\lambda = -1$ *is all of* $\mathbb{R}^3$. *Let* $N = A + I = \begin{bmatrix} 3 & 1 & 0 \\ -8 & -1 & 1 \\ 13 & 1 & -2 \end{bmatrix}$. *Then* $N^2 = \begin{bmatrix} 1 & 2 & 1 \\ -3 & -6 & -3 \\ 5 & 10 & 5 \end{bmatrix}$ *and* $N^3 = 0$. *So solutions are* $x(t) = e^{At}v = e^{-t}(I + Nt + N^2t^2/2)v$ *where* $x(0) = v$ *is the initial condition. Writing this out, it is*

$$x(t) = e^{-t}\begin{bmatrix} 1+3t+t^2/2 & t+t^2 & t^2/2 \\ -8t-3t^2/2 & 1-t-3t^2 & t-3t^2/2 \\ 13t+5t^2/2 & t+5t^2 & 1-2t+5t^2/2 \end{bmatrix} v$$

You could also solve this as follows. Pick a basis of eigenvectors, for example the standard basis, but I will use the basis $\{\alpha_1, \alpha_2, \alpha_3\} = \{e_3, Ne^3, N^2e_3\} = \{(0,0,1)^T, (0,1,-2)^T, (1,-3,5)^T\}$ *which will give an answer with fewer* t *terms. One solution is* $e^{-t}(\alpha_1 + Nt\alpha_1 + N^2t^2\alpha_1/2) = e^{-t}(t^2/2, t-3t^2/2, 1-2t+5t^2/2)^T$. *Another is* $e^{-t}(\alpha_2 + Nt\alpha_2 + N^2t^2\alpha_2/2) = e^{-t}(Ne_3 + N^2te_3) = e^{-t}(t, 1-3t, -2+5t)^T$. *The third is* $e^{-t}(\alpha_3 + Nt\alpha_3 + N^2t^2\alpha_3/2) = e^{-t}(N^2e_3) = e^{-t}(1,-3,5)^T$. *So in the end we get the general solution*

$$x(t) = c_1e^{-t}\begin{bmatrix} t^2/2 \\ t-3t^2/2 \\ 1-2t+5t^2/2 \end{bmatrix} + c_2e^{t}\begin{bmatrix} t \\ 1-3t \\ -2+5t \end{bmatrix} + c_3e^{-t}\begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix}$$

The equilibrium point 0 is asymptotically stable since the only eigenvalue of A *is negative.*

2. [30] Find and sketch the solution to the system $x' = 3x - 5y$, $y' = 5x - 3y$, $x(0) = 0$, $y(0) = 1$. Determine the stability of the equilibrium point at 0.

Answer: *The characteristic polynomial of* $\begin{bmatrix} 3 & -5 \\ 5 & -3 \end{bmatrix}$ *is* $\lambda^2 + 16$ *so its eigenvalues are* $\pm 4i$. *An eigenvector for* $4i$ *is the null space of* $\begin{bmatrix} 3-4i & -5 \\ 5 & -3-4i \end{bmatrix}$ *which is* $(5, 3-4i)^T$ *and an eigenvector for* $-4i$ *is the conjugate,* $(5, 3+4i)^T$. *So the general solution is*

$$(x,y) = c_1e^{4it}(5, 3-4i) + c_2e^{-4it}(5, 3+4i)$$

To make all this real though, take the real and imaginary parts of $e^{4it}(5, 3-4i) = (\cos 4t + i\sin 4t)(5, 3-4i) = (5\cos 4t + 5i\sin 4t, 3\cos 4t + 4\sin 4t - 4i\cos 4t + 3i\sin 4t)$ *and get the solutions* $(5\cos 4t, 3\cos 4t + 4\sin 4t)$ *and* $(5\sin 4t, -4\cos 4t + 3\sin 4t)$. *The initial values of these solutions are* $(5,3)$ *and* $(0,-4)$ *so our initial condition is satisfied by the solution*

$$(x,y) = -1/4(5\sin 4t, -4\cos 4t + 3\sin 4t) = (-1.25\sin 4t, \cos 4t - .75\sin 4t)$$

A second way to do the problem is a sort of judicious guessing method. Since the eigenvalues are $\pm 4i$, *we know the solution will have the form* $(x,y) = (a\cos 4t + b\sin 4t, c\cos 4t + d\sin 4t)$ *for some* a, b, c, d. *From our initial* *condition we know* $a = 0$ *and* $c = 1$*. Then* $x' = (b \sin 4t)' = 4b \cos 4t = 3x - 5y = 3b \sin 4t - 5 \cos 4t - 5d \sin 4t$ *from which we see that* $4b = -5$ *and* $0 = 3b - 5d$*. Solving, we get* $b = -5/4$ *and* $d = -3/4$*. As a check,* $y' = -4 \sin 4t - (3/4)4 \cos 4t$ *and* $5x - 3y = -25/4 \sin 4t - 3 \cos 4t + 9/4 \sin 4t = y'$*. The solution curve is an ellipse about the origin and the origin is a stable equilibrium point, but not asymptotically stable.*

3. [40] Find all equilibrium points of the system $x' = x(y - 1)$, $y' = -y + x^2$. Sketch solution curves near each equilibrium point. Determine the stability of each equilibrium point.

Answer: *A equilibrium point satisfies* $x(y-1) = 0$ *and* $-y+x^2 = 0$*. So either* $x = 0$ *or* $y = 1$*. If* $x = 0$ *then* $-y+0^2 = 0$ *so* $y = 0$*. If* $y = 1$ *then* $-1+x^2 = 0$ *so* $x = \pm 1$*. So we get three equilibrium points,* $(0, 0)$*,* $(1, 1)$*, and* $(-1, 1)$*. The linearized system has matrix* $\begin{bmatrix} y-1 & x \\ 2x & -1 \end{bmatrix}$*. At* $(0, 0)$ *this is* $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ *which is diagonal with equal eigenvalues so solutions come into the origin from every direction. This is asymptotically stable. At* $(1, 1)$ *this is* $\begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}$ *which has eigenvalues* $1, -2$ *and eigenvectors* $(1, 1)$ *and* $(1, -2)$*. So it is an unstable saddle, flowing in tangent to* $(1, -2)$ *and flowing out tangent to* $(1, 1)$*. At* $(-1, 1)$ *this is* $\begin{bmatrix} 0 & -1 \\ -2 & -1 \end{bmatrix}$ *which has eigenvalues* $1, -2$ *and eigenvectors* $(1, -1)$ *and* $(1, 2)$*. So it is an unstable saddle, flowing in tangent to* $(1, 2)$ *and flowing out tangent to* $(1, -1)$*.*