Math 405 Final Exam December 15, 2006

Give sufficient reason for your answers.

1. (50) Suppose $\mathcal{B} = \{\alpha_1, \alpha_2, \alpha_3\}$ is an orthonormal basis for an inner product space V over $\mathbb{C}$ with inner product $(\ |\ )$. Suppose $T: V \to V$ is the linear operator with $[T]_\mathcal{B} = \begin{bmatrix} 1 & i & 0 \\ 2 & 1 & 2i \\ 0 & 1 & 0 \end{bmatrix}$.

a) Find $(2\alpha_1 + i\alpha_2 \mid i\alpha_1 + 3\alpha_2 + \alpha_3)$.

Answer: $(2\alpha_1 + i\alpha_2 \mid i\alpha_1 + 3\alpha_2 + \alpha_3) = 2\bar{i} + i\bar{3} + 0\bar{1} = i$.

b) Find the adjoint T^* of T. (i.e., calculate $[T^*]_\mathcal{B}$).

Answer: Since $\mathcal{B}$ is orthonormal, this is just the conjugate transpose $\begin{bmatrix} 1 & 2 & 0 \\ -i & 1 & 1 \\ 0 & -2i & 0 \end{bmatrix}$

c) For every $\alpha \in V$ there is a functional $\phi_\alpha: V \to \mathbb{C}$ given by $\phi_\alpha(\beta) = (\beta \mid \alpha)$. This gives a map $S: V \to V^*$ where $S\alpha = \phi_\alpha$. Prove or disprove each of the following:

i) S is a linear transformation.

Answer: False. $\phi_{c\alpha+\beta}(\gamma) = (\gamma \mid c\alpha+\beta) = \bar{c}(\gamma \mid \alpha) + (\gamma \mid \beta) = (\bar{c}\phi_\alpha + \phi_\beta)(\gamma)$. *Thus* $S(c\alpha+\beta) = \bar{c}S(\alpha) + S(\beta)$ *so S is not quite linear. For example,* $S(i\alpha) = -iS(\alpha)$.

ii) True. $\{S\alpha_1, S\alpha_2, S\alpha_3\}$ is the dual basis of $\mathcal{B}$.

Answer: $S\alpha_i(\alpha_j) = (\alpha_j \mid \alpha_i) = 0 = \alpha_i^*(\alpha_j)$ *if* $i \neq j$ *and* $= 1 = \alpha_i^*(\alpha_j)$ *if* $i = j$. *So* $S\alpha_i = \alpha_i^*$ *the dual basis vector.*

iii) S is one to one.

Answer: If $S\alpha = S\beta$ *then* $(\gamma \mid \alpha) = (\gamma \mid \beta)$ *for all* γ *so* $(\gamma \mid \alpha - \beta) = 0$ *for all* γ. *Set* $\gamma = \alpha - \beta$ *and we get* $0 = (\alpha - \beta \mid \alpha - \beta) = ||\alpha - \beta||$ *so* $\alpha = \beta$.

iv) S is onto.

Answer: Since S is not linear we do not automatically know that one to one implies onto. But $S(\bar{c}_1\alpha_1 + \bar{c}_2\alpha_2 + \bar{c}_3\alpha_3) = c_1\alpha_1^* + c_2\alpha_2^* + c_3\alpha_3^*$ *so S is onto since* $\alpha_1^*, \alpha_2^*, \alpha_3^*$ *span* V^*.

2. (60) Do three of the following, be sure to clearly indicate which three.

a) Let $A = \begin{bmatrix} 4 & 4 \\ -1 & 0 \end{bmatrix}$. Find a real matrix P so that $P^{-1}AP$ is in Jordan form.

Answer: The characteristic value of A is 2. The columns of P are then α and $(A - 2I)\alpha$ for any vector α which is not a characteristic vector. For example if $\alpha = (1, 0)^T$ *then* $P = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$.

b) Find an orthonormal basis for the space of complex polynomials p of degree 3 or less with $p(0) = 0$. Use the inner product $(p \mid q) = \int_{-1}^{1} p(x)\overline{q(x)}\, dx$.

Answer: A basis is $\{x, x^2, x^3\}$. *Do G-S to get an orthogonal basis. Since* $(x \mid x^2) = 0$ *and* $(x^3 \mid x^2) = 0$ *we get* $\{x, x^2, x^3 - \frac{(x^3|x)}{||x||^2}x\} = \{x, x^2, x^3 - 3x/5\}$. *Divide by the lengths to make these orthonormal and get* $\{\sqrt{1.5}x, \sqrt{2.5}x^2, \sqrt{.875}(5x^3 - 3x)\}$. *(actual numbers not guaranteed)*

c) Let T be the operator on $\mathbb{C}^2$ with standard inner product which is represented in the standard ordered basis by the matrix $A = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$. Show that T is normal, and find a unitary matrix P so that P^*AP is diagonal.

Answer: Since $T^* = -T$ *we know* $TT^* = -T^2 = T^*T$ *so T is normal, as is any skew hermitian matrix. The characteristic polynomial is* $x^2 + 4$ *so the char values are* $\pm 2i$. *For* $2i$ *a char vector is in* $NS(A - 2iI) = span(i\ 1)$. *So a unit length char vector is* $(i/\sqrt{2},\ 1/\sqrt{2})$. *Similarly a char vector for* $-2i$ *is* $(-i/\sqrt{2},\ 1/\sqrt{2})$. *Make these the columns of P, so* $P = \begin{bmatrix} i/\sqrt{2} & -i/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$. *Other answers are possible but all are obtained by possibly switching the columns of P and multiplying each column by some some* $e^{i\theta}$, *for example by* $\pm i$ *or* -1.

d) Suppose $(1, i)$, $(0, 1)$ is an orthonormal basis for some inner product $(\ |\ )$ on $\mathbb{C}^2$. Find the orthogonal compliment of the subspace spanned by $(1, 0)$.

Answer: *Let $\alpha_1 = (1, i)$ and $\alpha_2 = (0, 1)$. Then $(1, 0) = \alpha_1 - i\alpha_2$ so the orth complement is the set of all $c\alpha_1 + d\alpha_2$ with $0 = c(\overline{1}) + d(\overline{-i}) = c + id$. So $d = ci$ and this is the span of $\alpha_1 + i\alpha_2 = (1, 2i)$.*

e) Suppose $(1, i)$, $(0, 1)$ is an orthonormal basis for some inner product $(\mid)$ on $\mathbb{C}^2$. Find the adjoint of the linear operator T which is represented in the standard ordered basis by the matrix $A = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$.

Answer: *Note $T\alpha_1 = (2i, -2) = 2i\alpha_1$ and $T\alpha_2 = (2, 0) = 2\alpha_1 - 2i\alpha_2$. So if $\mathcal{B} = \{\alpha_1, \alpha_2\}$ then $[T]_{\mathcal{B}} = \begin{bmatrix} 2i & 2 \\ 0 & -2i \end{bmatrix}$. So $[T^*]_{\mathcal{B}} = \begin{bmatrix} -2i & 0 \\ 2 & 2i \end{bmatrix}$. If you wanted the matrix of T^* in the standard basis it would be* $\begin{bmatrix} 1 & 0 \\ i & 1 \end{bmatrix} \begin{bmatrix} -2i & 0 \\ 2 & 2i \end{bmatrix} \begin{bmatrix} 1 & 0 \\ i & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ i & 1 \end{bmatrix} \begin{bmatrix} -2i & 0 \\ 2 & 2i \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -i & 1 \end{bmatrix}$

3. (50) Let $T: V \to V$ be a linear operator on a finite dimensional vector space V over $\mathbb{C}$. Suppose the characteristic polynomial of T is $(x-1)^2 x^3 (x+1)$ and the minimal polynomial of T is $(x-1)x^2(x+1)$. There is also a vector β in V so that $T^5\beta = \beta$.

a) What is the dimension of V?

Answer: 6, *the degree of the characteristic polynomial.*

b) Is T singular or nonsingular?

Answer: *T is singular since one of its characteristic values is 0.*

c) What are the possible Jordan forms of T?

Answer: *From the minimal polynomial, the largest Jordan blocks for the char values ± 1 is 1×1 and for 0 it is 2×2. The sum of the block sizes for a char value is the exponent in the char polynomial, so the only possible Jordan form has blocks $J_{1,1}, J_{1,1}, J_{2,0}, J_{1,0}, J_{1,-1}$.*

d) What is the dimension of the range of T?

Answer: *From the above Jordan form the null space has dimension 2, so the range has dimension $6 - 2$ or 4.*

e) What is the dimension of the range of $T + 2I$?

Answer: *Since -2 is not a characteristic value, $T + 2I$ is nonsingular so its range is all of V and thus has dimension 6.*

f) Find a generator of the ideal of polynomials p so that $p(T)\beta = 0$.

Answer: *We know the minimal polynomial $(x-1)x^2(x+1) = x^4 - x^2$ is in this ideal, and $T^5\beta = \beta$ tells us $x^5 - 1$ is in the ideal also. The generator must divide both these polynomials. Since 0 and -1 are not roots of $x^5 - 1$ this generator must be $x - 1$. You could also fool around a bit by noting that $x^5 - 1 - x(x^4 - x^2) = x^3 - 1$ is in the ideal. Hence $x^5 - 1 - x^2(x^3 - 1) = x^2 - 1$ is in the ideal. Hence $x^3 - 1 - x(x^2 - 1) = x - 1$ is in the ideal. Since I stated during the exam that $\beta \neq 0$, the generator has at least degree 1, so it must be $x - 1$.*

g) Show that β is a characteristic vector of T.

Answer: *Since $x - 1$ is in the ideal in part f, we know $(T - I)\beta = 0$ so $T\beta = \beta$ so β is a characteristic vector with characteristic value 1.*

4. (40) Suppose $S: V \to V$ and $T: V \to V$ are linear operators on a finite dimensional vector space V over $\mathbb{R}$ and $ST = TS$. Suppose the minimal polynomial of S is $x^2 - 1$ and the minimal polynomial of T is x^3.

a) What are the characteristic values of S?

Answer: ± 1, *the roots of the minimal polynomial.*

b) What are the characteristic values of T?

Answer: 0, the root of the minimal polynomial.

c) Show that $V = W_1 \oplus W_2$ where each subspace W_i is invariant under T and consists entirely of characteristic vectors for S.

Answer: *The primary decomposition theorem says $V = W_1 \oplus W_2$ where $W_1 = NS(S - I) =$ the char vectors for 1 and $W_2 = NS(S + I) =$ the char vectors for -1. So we must only show W_1 and W_2 are invariant under T. Suppose $\alpha \in W_1$, then $S\alpha = \alpha$. We have $(S - I)T\alpha = ST\alpha - T\alpha = TS\alpha - T\alpha = T\alpha - T\alpha = 0$ so $T\alpha \in W_1$ and W_1 is invariant under T. Likewise if $\beta \in W_2$, then $(S + I)T\beta = TS\beta + T\beta = T(-\beta) + T\beta = 0$ so W_2 is invariant under T. See section 6.5.*

d) Show that there is a basis $\mathcal{B}$ of V so that $[S]_{\mathcal{B}}$ is diagonal and $[T]_{\mathcal{B}}$ is upper triangular. For five points extra credit, you may instead show that there is a basis $\mathcal{B}$ of V so that $[S]_{\mathcal{B}}$ is diagonal and $[T]_{\mathcal{B}}$ is in Jordan form.

Answer: Let $T_1\colon W_1 \to W_1$ and $T_2\colon W_2 \to W_2$ be the restrictions of T. Since T_1 has minimal polynomial x^k it is triangulable so there is a basis $\mathcal{A}_1$ of W_1 so that $[T_1]_{\mathcal{A}_1}$ is an upper triangular matrix A_1. Likewise, there is a basis $\mathcal{A}_2$ of W_2 so that $[T_2]_{\mathcal{A}_2}$ is an upper triangular matrix A_2. Since $V = W_1 \oplus W_2$ we know the union $\mathcal{B} = \mathcal{A}_1 \cup \mathcal{A}_2$ is a basis of V. But $[T]_{\mathcal{B}} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$ is upper triangular. Jordan is done similarly (pun intended), since T_i is nilpotent we may instead choose our bases to make A_i be in Jordan form.

5. (20) (Extra credit) Let $T\colon V \to V$ be a normal operator on a finite dimensional inner product space V over $\mathbb{R}$. Suppose T has no (real) characteristic values. Show that for any integer $k > 0$ there is an operator S so that $S^k = T$.
Answer: We showed there is a basis $\mathcal{B}$ of V so that $[T]_{\mathcal{B}}$ is in block diagonal form with only 2×2 and 1×1 blocks on the diagonal. There are no 1×1 blocks since they correspond to real characteristic values. The 2×2 blocks are all of the form $B_{a,b} = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ where $a + bi$ is a char value of the complexification of T. Note that $B_{a,b}B_{c,d} = B_{e,f}$ where $e + fi = (a + bi)(c + di)$ so we have $B_{a,b} = B^k_{c,d}$ where $c + di = (a + bi)^{1/k}$. Alternatively, note that $B_{a,b}$ is the composition of a scalar multiplication by $\sqrt{a^2 + b^2}$ and rotation by some angle θ, and these operators commute. (In fact, $a + bi = re^{i\theta}$ where $r = \sqrt{a^2 + b^2}$.) Then $B_{a,b} = B^k_{c,d}$ where $c + di = (a + bi)^{1/k} = r^{1/k}e^{i\theta/k}$. That is, $B_{c,d}$ rotates by θ/k and scalar multiplies by $(a^2 + b^2)^{1/2k}$. So if $[T]_{\mathcal{B}}$ has blocks $B_{a_1,b_1}, B_{a_2,b_2}, \ldots, B_{a_n,b_n}$ we pick S so $[S]_{\mathcal{B}}$ has blocks $B_{c_1,d_1}, B_{c_2,d_2}, \ldots, B_{c_n,d_n}$ where $(c_j + id_j)^k = a + bi$ for all j. Then $S^k = T$.