Math 405 Exam #1 October 6, 2006

1. (25) Let $W_1 \subset \mathbb{R}^{2\times 2}$ be the set of matrices with trace 0, that is, matrices of the form $\begin{bmatrix} x & y \\ z & -x \end{bmatrix}$. Let $W_2 \subset \mathbb{R}^{2\times 2}$ be the set of symmetric matrices, that is, matrices of the form $\begin{bmatrix} x & y \\ y & z \end{bmatrix}$.

a) Show that W_1 and W_2 are both subspaces.

$c\begin{bmatrix} x & y \\ z & -x \end{bmatrix}+\begin{bmatrix} x' & y' \\ z' & -x' \end{bmatrix} = \begin{bmatrix} cx+x' & cy+y' \\ cz+z' & -cx-x' \end{bmatrix}$. *Since* $-(cx+x') = -cx-x'$, *this linear combination of elements of* W_1 *is still in* W_1. *Since* $0 \in W_1$ *we then know* W_1 *is a subspace.* $c\begin{bmatrix} x & y \\ y & z \end{bmatrix} + \begin{bmatrix} x' & y' \\ y' & z' \end{bmatrix} = \begin{bmatrix} cx+x' & cy+y' \\ cy+y' & cz+z' \end{bmatrix}$. *Since the off diagonal entries are equal, this linear combination of elements of* W_2 *is still in* W_2, *and since* W_2 *is nonempty we know* W_2 *is a subspace.*

b) Find the dimensions of W_1, W_2, $W_1 \cap W_2$ and $W_1 + W_2$ (and give sufficient reasons).

A basis of W_1 *is* $\{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\}$ *since they are linearly independent and span* W_1 *so* $\dim W_1 = 3$. *A basis of* W_2 *is* $\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\}$ *since they are linearly independent and span* W_2 *so* $\dim W_2 = 3$. $W_1 \cap W_2$ *is all matrices of the form* $\begin{bmatrix} x & y \\ y & -x \end{bmatrix}$ *so it has basis* $\{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\}$ *and hence has dimension 2.* $\dim(W_1+W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2) = 3+3-2 = 4$. *Alternatively, since* W_1 *is not contained in* W_2 *we know* $\dim(W_1+W_2) > \dim W_2 = 3$. *But* $\dim(W_1+W_2) \leq \dim \mathbb{R}^{2\times 2} = 4$ *so we must have* $\dim(W_1+W_2) = 4$. *Then* $\dim(W_1 \cap W_2) = \dim W_1 + \dim W_2 - \dim(W_1+W_2) = 3+3-4 = 2$.

c) For 10 points extra credit, you may instead do this problem where W_1 is the trace 0 matrices in $\mathbb{C}^{2\times 2}$, W_2 is the Hermitian matrices in $\mathbb{C}^{2\times 2}$ and the field is $\mathbb{R}$.

$\dim W_1 = 6$ *with basis* $\{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & i \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ i & 0 \end{bmatrix}\}$,

$\dim W_2 = 4$ *with basis* $\{\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & i\\-i & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\}$, $\dim(W_1 \cap W_2) = 3$ *with basis* $\{\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}, \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & i\\-i & 0\end{bmatrix}\}$, $\dim(W_1 + W_2) = 6 + 4 - 3 = 7$. $W_1 + W_2$ *is the set of matrices whose trace is real.*

2. (25) Let T be the linear operator on $\mathbb{C}^2$ defined by

$$T(x_1, x_2) = (-x_1 + x_2, x_1 - x_2).$$

a) Show that $\mathcal{B} = \{(1,1), (i, -i)\}$ is a basis for $\mathbb{C}^2$.

$a(1,1) + b(i,-i) = (0,0)$ *implies* $a + bi = 0$ *and* $a - bi = 0$ *so* $2a = a + bi + a - bi = 0$ *so* $a = 0$ *so* $0 + bi = 0$ *so* $b = 0$. *So* $\{(1,1), (i,-i)\}$ *is linearly independent and hence forms a basis of the 2 dimensional space* $\mathbb{C}^2$.

b) Find the matrix $[T]_\mathcal{B}$ of T in the ordered basis $\{(1,1), (i,-i)\}$.

$T(1,1) = (0,0) = 0(1,1) + 0(i,-i)$ *so* $[T(1,1)]_\mathcal{B} = \begin{bmatrix}0\\0\end{bmatrix}$. $T(i,-i) = (-2i, 2i) = 0(1,1) - 2(i,-i)$ *so* $[T(i,-i)]_\mathcal{B} = \begin{bmatrix}0\\-2\end{bmatrix}$. *So* $[T]_\mathcal{B} = \begin{bmatrix}0 & 0\\0 & -2\end{bmatrix}$.

3. (25) Let V be a finite dimensional vector space and let $W \subset V$ be a subspace. Show that there is a linear transformation $T: V \to V$ so that the range R_T of T is W and so $T^2 = T$. What is the dimension of the null space of T?

Choose a basis $\{\beta_1, \dots, \beta_k\}$ *of* W. *Extend this to a basis* $\{\beta_1, \dots, \beta_n\}$ *of* V. *There is a unique linear transformation* $T: V \to V$ *so that* $T(\beta_i) = \beta_i$ *and* $i \leq k$ *and* $T(\beta_i) = 0$ *for* $i > k$. *Note that* $T^2(\beta_i) = T(\beta_i)$ *for all* i *so by uniqueness we know* $T^2 = T$. *The range of* T *is the subspace spanned by all* β_i *for* $i \leq k$, *which is* W. $\dim NS(W) = \dim V - \dim R_T = \dim V - \dim W$.

4. (25) Let $\mathcal{B} = \{\beta_1, \beta_2, \dots, \beta_n\}$ be an ordered basis for V and let $\{\beta_1^*, \beta_2^*, \dots, \beta_n^*\}$ be its dual basis. Show that the $\mathcal{B}$ coordinates of any

$\alpha \in V$ are given by $[\alpha]_{\mathcal{B}} = \begin{bmatrix} \beta_1^*(\alpha) \\ \beta_2^*(\alpha) \\ \cdots \\ \beta_n^*(\alpha) \end{bmatrix}$.

If $\alpha = \sum_{i=1}^{n} c_i \beta_i$ *then* $\beta_j^*(\sum_{i=1}^{n} c_i \beta_i) = \sum_{i=1}^{n} c_i \beta_j^*(\beta_i) = c_j$ *since* $\beta_j^*(\beta_i) = 0$ *for* $j \neq i$ *and* $\beta_j^*(\beta_j) = 1$. *So* $\begin{bmatrix} \beta_1^*(\alpha) \\ \beta_2^*(\alpha) \\ \cdots \\ \beta_n^*(\alpha) \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \cdots \\ c_n \end{bmatrix} = [\alpha]_{\mathcal{B}}$.