Math 405 Exam #2 November 20, 2006

Answer any four of the five problems. Clearly indicate which four you want graded. To save writing, you may use $J_{k,c}$ to represent a $k \times k$ Jordan block with c on the diagonal. Give sufficient reasons for your answers.

1. (25) Let $A = \begin{bmatrix} 0 & 2 & 3 & 7 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 4-i \\ 0 & 0 & 0 & 3+i \end{bmatrix}$.

a) Find the characteristic polynomial of A.

Answer: $x^3(x-3-i)$

b) Find the minimal polynomial of A.

Answer: Since the minimal polynomial divides the characteristic polynomial and has the same factors, it is either $x(x-3-i)$ *or* $x^2(x-3-i)$ *or* $x^3(x-3-i)$. *Calculate* $A(A-(3+i)I)$ *and see it is nonzero, then calculate* $A^2(A-(3+i)I)$ *which is 0. So the minimal polynomial is* $x^2(x-3-i)$.

c) Find the Jordan form of A.

Answer: Since the minimal polynomial is $x^2(x-3-i)$ *one of the Jordan blocks is* $J_{2,0}$ *and another is* $J_{1,3+i}$. *There is only room for one more* 1×1 *block* $J_{1,0}$. *So the Jordan form of* A *has three Jordan blocks,* $J_{2,0}$, $J_{1,0}$, *and* $J_{1,3+i}$. *In other words, it is all zeros except for a 1 in row 2 column 1 and a* $3+i$ *in the fourth row fourth column. If you wanted to find a basis* $\mathcal{B} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ *so that* $[A]_{\mathcal{B}}$ *is in this Jordan form, you would take* α_1 *to be any vector in* $NS(A^2) - NS(A)$, *for example* $\alpha_1 = \epsilon_2$. *Then let* $\alpha_2 = A\alpha_1 = 2\epsilon_1$. *Then let* α_3 *be any vector in* $NS(A)$ *which is not a multiple of* α_2, *for example* $\alpha_3 = 3\epsilon_2 - 2\epsilon_3$. *Finally* α_4 *is any nonzero vector in* $NS(A-(3+i)I)$ *for example* $\alpha_4 = \epsilon_4 + \frac{4-i}{3+i}\epsilon_3 + \frac{4}{3+i}\epsilon_2 + \frac{41+4i}{(3+i)^2}\epsilon_1$. *Then if* P *is the matrix* $P = [\alpha_1\alpha_2\alpha_3\alpha_4]$ *we have* $P^{-1}AP$ *in Jordan form.*

2. (25) Let N be a nilpotent 3×3 real matrix.

a) Show that $(I+N)^{-1} = 1 - N + N^2$.

Answer: $(I+N)(I-N+N^2) = I - N + N^2 + N - N^2 + N^3 = I + N^3$. *But* $N^3 = 0$ *by the Cayley-Hamilton theorem since* N *is nilpotent. So* $(I+N)(I-N+N^2) = I$ *which implies* $(I+N)^{-1} = 1 - N + N^2$.

b) Suppose $N^3 + 3N^2 + N \neq 0$ and $N^7 - 5N^6 + 4N^3 - N^2 = 0$. What is the minimal polynomial of N?

Answer: Since $N^3 = 0$ *we see that* $3N^2 + N \neq 0$ *and* $-N^2 = 0$. *So* $N^2 = 0$ *but* $N \neq 0$ *which means the minimal polynomial of* N *is* x^2.

3. (25) Let $V \subset \mathbb{R}[x]$ be the polynomials of degree 2 or less. let $T\colon V \to V$ be the linear operator $T(p(x)) = (x+1)p'(x)$, so $T(x^2) = 2x(x+1)$, $T(x) = x+1$, and $T(1) = 0$.

a) Find the minimal polynomial of T.

Answer: The matrix of T *with respect to the standard basis* $\{1, x, x^2\}$ *is* $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$.

So the characteristic polynomial of T *is* $x(x-1)(x-2)$. *Since the minimal polynomial* *always has the same roots and not larger exponents, the minimal polynomial must also be* $x(x-1)(x-2)$.

b) Does T have a Jordan form? If so, find it. If not, explain why not.

Answer: Yes, since the minimal polynomial is a product of linear factors. In fact, since all exponents are 1 we know T is diagonalizable so the Jordan form is diagonal with $0, 1, 2$ *on the diagonal. A basis diagonalizing T is* $\{1, x+1, (x+1)^2\}$.

4. (25) Answer four of the following six short questions by either finding the requested matrices or subspaces, or showing they do not exist.

a) Find a 4×4 complex matrix which is not diagonalizable, and write down its minimal and characteristic polynomials.

Answer: For example $J_{4,0}$ with characteristic and minimal polynomials both x^4. What you need is that some linear factor in the minimal polynomial have exponent > 1.

b) Find a 4×4 real matrix which is not triangulable, and write down its minimal and characteristic polynomials.

Answer: The minimal and characteristic polynomials cannot be products of linear factors so they need to be divisible by a polynomial without a real root, for example x^2+1. So for example if $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ *is rotation by 90 degrees, then B^2 is rotation by 180 degrees, so $B^2 + I = 0$. So we could take the block matrix* $A = \begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix}$. *You can calculate the characteristic polynomial as $(x^2+1)x^2$. Since $(A^2+I)A = 0$ the minimal polynomial is* $(x^2+1)x$.

c) Find a 4×4 complex matrix which is not triangulable, and write down its minimal and characteristic polynomials.

Answer: Impossible, every complex matrix is triangulable since the minimal polynomial can be written as a product of linear factors.

d) Find a 3×3 matrix A and a subspace $W \subset \mathbb{R}^3$ and an $\alpha \in \mathbb{R}^3$ so that $S_A(\alpha; W)$ is not an ideal. (Recall $S_A(\alpha; W) = \{p \in \mathbb{R}[x] \mid p(A)\alpha = 0\}$.)

Answer: We know W cannot be invariant under A. So lets just try a random example and see if it works. Let W be the span of ϵ_1 and take any A so that $A\epsilon_1$ is not a multiple of ϵ_1, say $A\epsilon_1 = \epsilon_2$, $A\epsilon_2 = A\epsilon_3 = 0$. Note $A^2 = 0$ so computations will be easy. Then if $p(x) = d + ex + \cdots$ is any polynomial, $p(A)(a,b,c) = d(a,b,c) + e(0,a,0)$ so $p(A)(a,b,c) \in W$ if and only if $cd = 0$ and $bd + ae = 0$. So if we let $\alpha = (1,0,0)$ for example then $S_A(\alpha; W)$ is the set of all polynomials whose x coefficient is 0. This is not an ideal since for example $1 \in S_A(\alpha; W)$ but $x \cdot 1 \notin S_A(\alpha; W)$.

e) Find an upper triangular real matrix which is not similar to a real matrix in Jordan form.

Answer: Impossible, the characteristic polynomial is a product of linear factors, so the matrix is similar to a matrix in Jordan form.

f) Let $A = J_{3,0}$. Find all subspaces $W \subset \mathbb{R}^3$ invariant under A.

Answer: There are very few invariant subspaces. If $(a,b,c) \in W$ then $A(a,b,c) = (0,a,b) \in W$ and $A^2(a,b,c) = (0,0,a) \in W$. So if $a \neq 0$ then $W = \mathbb{R}^3$. So suppose $W \neq \mathbb{R}^3$. Then $a = 0$ so W must be contained in the span of ϵ_2, ϵ_3. If $b \neq 0$ then $(0,b,c) \in W$ and $(0,0,b) \in W$ so W is the span of ϵ_2, ϵ_3. So suppose W is neither $\mathbb{R}^3$ nor the span of ϵ_2, ϵ_3.

Then $a = b = 0$ and if $c \neq 0$ then W is the span of ϵ_1, otherwise $W = 0$. So there are only 4 subspaces invariant under A, namely $\mathbb{R}^3$, 0, the span of ϵ_2, ϵ_3 and the span of ϵ_3.

5. (25) Let $p(x) = x^2(x^2+4)(x-1)(x+3)^3$, $q(x) = (x+5)^3(x+1)^4(x+3)^4$, and $r(x) = x^2(x+1)(x+3)^2$. Suppose A is a 5×5 complex matrix and p and q are both in the annihilating ideal of A, so $p(A) = 0$ and $q(A) = 0$. Suppose also that $r(A) \neq 0$.

a) What are all possible characteristic polynomials of A?

Answer: The minimal polynomial of A must divide both p and q so it must be some $(x+3)^k$ with $k \leq 3$. Since the characteristic polynomial has the same roots and has degree 5 it must be $(x+3)^5$.

b) What are all possible minimal polynomials of A?

Answer: The minimal polynomial $(x+3)^k$ does not divide r so $k = 3$. So the minimal polynomial is $(x+3)^3$.

c) What are all possible Jordan forms of A?

Answer: There must be a 3×3 block $J_{3,-3}$ so the only possibilities are:

- *One $J_{3,-3}$ and one $J_{2,-3}$*
- *One $J_{3,-3}$ and two $J_{1,-3}$*