2.1:5 3a is not satisfied because $\alpha \oplus \beta = \alpha - \beta \neq \beta - \alpha = \beta \oplus \alpha$ if $\alpha \neq \beta$. 3b is not satisfied because $\alpha \oplus (\beta \oplus \gamma) = \alpha \oplus (\beta - \gamma) = \alpha - (\beta - \gamma) = \alpha - \beta + \gamma$ and $(\alpha \oplus \beta) \oplus \gamma = (\alpha - \beta) \oplus \gamma = \alpha - \beta - \gamma$ and so these are unequal if $\gamma \neq 0$. 3c is satisfied with 0 the usual 0 vector since $\alpha \oplus 0 = \alpha - 0 = \alpha$. 3d is satisfied with $-\alpha = \alpha$ since $\alpha \oplus \alpha = \alpha - \alpha = 0$. 4a is not satisfied since $1 \cdot \alpha = -1\alpha = -\alpha \neq \alpha$ if $\alpha \neq 0$. 4b is not satisfied since $(c_1 c_2) \cdot \alpha = -c_1 c_2 \alpha$ and $c_1 \cdot (c_2 \cdot \alpha) = c_1 \cdot (-c_2 \alpha) = -c_1(-c_2\alpha) = c_1 c_2 \alpha$ so these are unequal if $\alpha \neq 0$ and $c_i \neq 0$. 4c is satisfied since $c \cdot (\alpha \oplus \beta) = c \cdot (\alpha - \beta) = -c(\alpha - \beta) = -c\alpha + c\beta$ and $c \cdot \alpha \oplus c \cdot \beta = (-c\alpha) \oplus (-c\beta) = -c\alpha - (-c\beta) = -c\alpha + c\beta$. 4d is not satisfied because $(c_1 + c_2) \cdot \alpha = -(c_1 + c_2)\alpha = c_1\alpha - c_2\alpha$ and $c_1\alpha \oplus c_2\alpha = (-c_1\alpha) \oplus (-c_2\alpha) = -c_1\alpha - (-c_2\alpha) = -c_1\alpha + c_2\alpha$ so these are unequal if $\alpha \neq 0$ and $c_1 \neq c_2$.

2.1:6 I'll do this a slick way to avoid having to verify all those axioms. By exercise 3 we know the set of all complex valued function on the real line is a vector space over $\mathbb{C}$ with addition and scalar multiplication (2-5) and (2-6) the same as we want for this problem. Now I claim that if W is any vector space over $\mathbb{C}$ then by restricting the scalars to $\mathbb{R}$ we get a vector space over $\mathbb{R}$. In other words if we use the same vector addition and the same scalar multiplication (except that we only allow multiplication by real numbers) then all the vector space axioms are satisfied. This is a trivial observation since if the axioms all hold for complex scalars, in particular they must hold for real scalars. So by these two observations we see that the set U of complex valued functions on $\mathbb{R}$ is a vector space over $\mathbb{R}$. So we must only show that the set of those functions with $f(-t) = \overline{f(t)}$ for all t form a subspace of U. If f and g are in V and c is real then $(cf + g)(-t) = (cf)(-t) + g(-t) = cf(-t) + g(-t) = c\overline{f(t)} + \overline{g(t)} = \overline{c}\overline{f(t)} + \overline{g(t)} = \overline{cf(t) + g(t)} = \overline{(cf)(t) + g(t)} = \overline{(cf + g)(t)}$ so $cf + g \in V$. Since $0 \in V$ this means V is a subspace of U, hence a vector space. An example which is not real valued is $f(t) = it$. In fact any function whose real part is even and whose imaginary part is odd and nonzero will do.

2.1:7 This is not a vector space. For example $(0, 1) + (x, y) = (1 + x, 0) \neq (0, 1)$ for any (x, y) so there is no zero vector. Also $1(0, 1) = (0, 0) \neq (0, 1)$ so axiom 4a also fails. I think all the axioms besides 3c and 4a hold but I haven't written out details.

2.2:1

a) is not a subspace because for example $(-1)(1, 0, \dots, 0) = (-1, 0, \dots, 0) \notin$ the set.
b) This is a subspace by example 7 with $A = [1\ 3\ \ -1\ 0 \cdots 0]$.
c) is not a subspace. For example $(1, 1, 0, \dots)$ is in it but $2(1, 1, 0, \dots)$ is not.
d) is not a subspace because for example $(1, 0, 0, \dots)$ and $(0, 1, 0, \dots)$ are in it but $(1, 0, 0, \dots) + (0, 1, 0, \dots) = (1, 1, 0, \dots)$ is not.
e) The presumption is that the scalars are $\mathbb{R}$. Assuming the scalars are $\mathbb{R}$ this is not a subspace because for example $(0, 1, 0, \dots)$ is in it but $\sqrt{2}(0, 1, 0, \dots)$ is not. But if the scalars were the rationals $\mathbb{Q}$ then this would be a subspace.

2.2:2

a) This is not a subspace. For example, $f(x) = x$ is in it but $(2f)(x) = 2x$ is not.
b) This is a subspace. If f and g are in it and c is real then $(cf + g)(0) = cf(0) + g(0) = cf(1) + g(1) = (cf + g)(1)$ so $cf + g$ is in it. It contains 0 so it is nonempty.
c) This is not a subspace, for example 0 is not in it.
d) This is a subspace. If f and g are in it and $c \in \mathbb{R}$ then $(cf + g)(-1) = cf(-1) + g(-1) = c0 + 0 = 0$ so $cf + g$ is in it. It contains 0 so it is nonempty.
e) This is a subspace. A theorem of first semester calculus says that the sum of continuous functions is continuous and a constant times a continuous function is continuous. It contains 0 so it is nonempty.

2.2:5

a) not a subspace because for example 0 is not invertible.
b) It is not hard to find two noninvertible matrices whose sum is invertible so this is not a subspace. For example let $A_{ij} = 0$ if $i \neq j$ or $i \leq n/2$ and $A_{ii} = 1$ if $i > n/2$. Let $B_{ij} = 0$ if $i \neq j$ or $i > n/2$ and $B_{ii} = 1$ if $i \leq n/2$. Then A and B are noninvertible but $A + B = I$ is invertible.
c) This is a subspace. If A and A' are in it and $c \in F$ then $(cA + A')B = cAB + A'B = cBA + BA' = B(cA) + BA' = B(cA + A')$ so $cA + A'$ is in it. It contains 0 so it is nonempty.
d) this is not a subspace because for example the identity I is in it, but $2I$ is not.

2.2:7 If neither W_1 nor W_2 is contained in the other, we may find an α in W_1 so $\alpha \notin W_2$ and a $\beta \in W_2$ so $\beta \notin W_1$. Denote $\alpha + \beta = \gamma$. Since $W_1 \cup W_2$ is a subspace, we must have $\gamma \in W_1 \cup W_2$. If $\gamma \in W_1$ then $\gamma - \alpha \in W_1$ but $\gamma - \alpha = \beta \notin W_1$ so this is not possible. So we must have $\gamma \in W_2$. But then $\gamma - \beta \in W_2$ but $\gamma - \beta = \alpha \notin W_2$ which is not possible. So we have a contradiction, which means that either $W_1 \subset W_2$ or $W_2 \subset W_1$.

2.3:2 They are not linearly independent because when we try to solve $x_1\alpha_1 + x_2\alpha_2 + x_3\alpha_3 + x_4\alpha_4 = 0$ we get the Gaussian elimination problem:

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & -1 & -1 & 1 \\ 2 & -5 & -4 & 1 \\ 4 & 2 & 0 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -3 & -2 & -1 \\ 0 & -9 & -6 & -3 \\ 0 & -6 & -4 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -3 & -2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 2/3 & 1/3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1/3 & 4/3 \\ 0 & 1 & 2/3 & 1/3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which means that many nonzero solutions are possible, for example $x_4 = 3$, $x_3 = 0$, $x_2 = -1$, $x_1 = -4$. So $-4\alpha_1 - \alpha_2 + 3\alpha_4 = 0$.

2.3:3 One possible basis is $\{\alpha_1, \alpha_2\}$. By the calculation in problem 2 we see that $\alpha_4 = (4/3)\alpha_1 + (1/3)\alpha_2$ and $\alpha_3 = (-1/3)\alpha_1 + (2/3)\alpha_2$. So the subspace spanned by α_1 and α_2 contains α_3 and α_4 and hence equals the subspace spanned by $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$. On the other hand $\{\alpha_1, \alpha_2\}$ is linearly independent since by the calculation in problem 2, the only solution with $x_3 = x_4 = 0$ is $x_1 = x_2 = 0$. You also know from class that they are linearly independent because α_1 is not a multiple of α_2.

2.3:6 A basis is $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$. This spans $F^{2\times2}$ because for any $A \in F^{2\times2}$,

$$A = A_{11} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + A_{12} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + A_{21} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + A_{22} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

It is linearly independent because if $c_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + c_4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ then $\begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ so all $c_i = 0$.

2.3:7 W_1 is a subspace since it is the span of $\left\{ \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$. This set is linearly independent so $\dim W_1 = 3$. W_2 is a subspace since it is the span of $\left\{ \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$. This set is linearly independent so $\dim W_2 = 3$. $W_1 \cap W_2$ is all matrices of the form $\begin{bmatrix} x & -x \\ -x & z \end{bmatrix}$. So a basis for $W_1 \cap W_2$ is $\left\{ \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$. So $\dim(W_1 \cap W_2) = 2$. By theorem 6 we know $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2) = 3 + 3 - 2 = 4$. We could also see directly that $W_1 + W_2 = V$ and hence has dimension 4 by noting that for any $A \in F^2$, $A = \begin{bmatrix} A_{11} & -A_{11} \\ A_{21} & A_{22} \end{bmatrix} + \begin{bmatrix} 0 & A_{11} + A_{12} \\ 0 & 0 \end{bmatrix}$, the sum of vectors in W_1 and W_2.

2.3:11

a) Notice that by the argument given in problem 6 of 2.1 that $\mathbb{C}^{2\times2}$ can be thought of as a vector space over $\mathbb{R}$. So we only need show V is a subspace of this real vector space. If $A, B \in V$ and $c \in \mathbb{R}$ then $(cA+B)_{11} + (cA+B)_{22} = cA_{11} + B_{11} + cA_{22} + B_{22} = cA_{11} + cA_{22} + B_{11} + B_{22} = c(A_{11} + A_{22}) + B_{11} + B_{22} = c0 + 0 = 0$. So $cA + B \in V$. Since $0 \in V$ also then V is a subspace.

b) A possible basis is

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & i \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ i & 0 \end{bmatrix} \right\}$$

c) Note $0 \in W$. If $A, B \in W$ and $c \in \mathbb{R}$ then

$$(cA+B)_{21} = cA_{21} + B_{21} = -c\overline{A_{12}} + \overline{B_{12}} = \overline{-\overline{c}A_{12} + B_{12}} = \overline{-cA_{12} + B_{12}} = \overline{(cA+B)_{12}}$$

So $cA + B \in W$, so W is a subspace. A possible basis is

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \right\}$$

3.1:1

a) is not linear because $T(0) = (1,0) \neq 0$.

b) is linear, it is right multiplication by $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

c) is not linear, for example $T(2(1,0)) = (4,0) \neq (2,0) = 2T(1,0)$.

d) is not linear, for example $T(2(\pi/2,0)) = (0,0) \neq (2,0) = 2T(\pi/2,0)$.

e) is linear, it is right multiplication by $\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$.

3.1:2 The 0 transformation has range $\{0\}$, hence has rank 0. The null space is V so its nullity is the dimension of V. The identity on V has range V, hence its rank is $\dim V$. It has 0 null space, so its nullity is 0.

3.1:3 I shall interpret V as being the space of polynomials, since things get weird for some fields where some nonzero polynomials are identically 0. (For example $x^2 + x$ is identically 0 for the field with two elements. Yet its derivative is $2x + 1 = 0x + 1 = 1$ which is not the derivative of 0. So H and K do not really mean the polynomial functions, but just formal polynomials with coefficients in $\mathcal{F}$.) Since our guiding rule is that all fields in this course will be either $\mathbb{R}$ or $\mathbb{C}$, this makes no difference. Since $Df = 0$ implies f is a constant, the Null space of D is the constant polynomials $f(x) = c$. But the range is all V, since for any $\alpha \in V$ we may write $\alpha = \Sigma_{i=0}^n c_i x^i$ and then $\alpha = D(\Sigma_{i=0}^n c_i \frac{x^{i+1}}{i+1})$. For the integration T, note first that by the fundamental theorem of calculus for each continuous f we know $T(f)$ is differentiable and $T(f)' = f$. Then if $T(f) = 0$ we must have $0 = 0' = T(f)' = f$ so the null space of T is just $\{0\}$. Any g in the range must be differentiable as noted above, but also must satisfy $g(0) = 0$ since $T(f)(0) = \int_0^0 f(t)\,dt = 0$. I claim that the range of T is the set of differentiable functions $g\colon \mathbb{R} \to \mathbb{R}$ so that $g(0) = 0$ since for such a g, we have $T(g') = \int_0^x g'(t)\,dt = g(t)\Big]_0^x = g(x) - g(0) = g(x)$.

3.1:4 Yes, by Thm 1 since $\{(1,-1,1),(1,1,1)\}$ is a linearly independent set. You could also explicitly find a linear transformation, for example right multiplication by $\begin{bmatrix} .5 & .5 \\ -.5 & .5 \\ 0 & 0 \end{bmatrix}$.

3.1:8 For example, right multiplication by $\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix}$.

3.1:9 If A and A' are in $\mathcal{F}^{n \times n}$ and $c \in \mathcal{F}$ then

$$T(cA + A') = (cA + A')B - B(cA + A') = cAB + A'B - B(cA) - BA' = cAB + A'B - cBA - BA'$$

$$= c(AB - BA) + A'B - BA' = cT(A) + T(A')$$

So T is linear.

3.1:12 Since $R_T = NS(T)$ we have $\dim V = \dim(R_T) + \dim(NS(T)) = 2\dim(R_T)$ so $\dim V$ is even.

3.2:1

a) T reflects about the line $x_1 = x_2$ and U projects straight down to the subspace spanned by $\{e_1\}$ (usually known as the x axis).

b) $(U+T)(x_1, x_2) = (x_1 + x_2, x_1)$ (I don't see a good geometric description though). $UT(x_1, x_2) = (x_2, 0)$ it projects to the y axis and rotates 90° clockwise. $TU(x_1, x_2) = (0, x_1)$ it projects to the x axis and rotates 90° counterclockwise. $T^2(x_1, x_2) = (x_1, x_2)$ is the identity. $U^2(x_1, x_2) = (x_1, 0)$ it is the same as U. Note, as I read it the geometric descriptions were not asked for in the problem but I included them anyway.

3.2:6 We know the nullity of T is $3 - \dim(R_T) \geq 3 - \dim(\mathbb{R}^2) = 1$ by theorem 2 and the fact that $R_T \subset \mathbb{R}^2$ so $\dim(R_T) \leq 2$. Since T has nonzero nullity we know that there is an $\alpha \in \mathbb{R}^3$ so that $T\alpha = 0$ and $\alpha \neq 0$. But then $UT\alpha = U0 = 0$ so UT is not one to one so it is not invertible. To generalize, suppose $T\colon V \to W$, $U\colon W \to Z$ are linear transformations, V is finite dimensional and $\dim W < \dim V$. Then UT is not invertible.

3.3:3 Let us give this map a name, T. So $T(x,y,z,t) = \begin{bmatrix} t+x & y+iz \\ y-iz & t-x \end{bmatrix}$. We must show T is linear, one to one, and onto. (If we knew that the 2×2 Hermitian matrices had dimension 4 over the reals we could skip either one to one or onto.) To show linearity,

$$T(c(x,y,z,t) + (u,v,w,s)) = T(cx+u, cy+v, cz+w, ct+s) =$$

$$= \begin{bmatrix} ct+s+cx+u & cy+v+i(cz+w) \\ cy+v-i(cz+w) & ct+s-(cx+u) \end{bmatrix} = \begin{bmatrix} ct+cx & cy+i(cz) \\ cy-i(cz) & ct-(cx) \end{bmatrix} + \begin{bmatrix} s+u & v+i(w) \\ v-i(w) & s-(u) \end{bmatrix}$$

$$= c\begin{bmatrix} t+x & y+iz \\ y-iz & t-x \end{bmatrix} + \begin{bmatrix} s+u & v+i(w) \\ v-i(w) & s-(u) \end{bmatrix} = cT(x.y.z.t) + T(u,v,w,s)$$

To show one to one, suppose $T(x,y,z,t) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Then $t+x=0$, $y+iz=0$, $y-iz=0$, and $t-x=0$. Solving we get $t = .5(t+x) + .5(t-x) = 0+0 = 0$, $x = (t+x) - t = 0 - 0 = 0$, $y = .5(y+iz) + .5(y-iz) = 0+0 = 0$ and $z = -i(y+iz) + iy = 0+0 = 0$. So $T\alpha = 0$ implies $\alpha = (0,0,0,0)$ which means T is one to one. Finally we must show T is onto. Let $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ be any 2×2 Hermitian matrix. Since $\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^* = \begin{bmatrix} \overline{A_{11}} & \overline{A_{21}} \\ \overline{A_{12}} & \overline{A_{22}} \end{bmatrix}$ we know $A_{11} = \overline{A_{11}}$, $A_{12} = \overline{A_{21}}$, $A_{21} = \overline{A_{12}}$, and $A_{22} = \overline{A_{22}}$. I will finish this problem two ways. The first way is to solve for (x,y,z,t) in terms of the A_{ij}. Since $A_{ii} = \overline{A_{ii}}$ we know the diagonal entries A_{ii} are real. So we may set $t = (A_{11} + A_{22})/2$ and $x = (A_{11} - A_{22})/2$. Then $t + x = A_{11}$ and $t - x = A_{22}$. Set $y =$ the real part of A_{12} and $z =$ the imaginary part of A_{12} and we then have $T(x,y,z,t) = A$ and thus T is onto. The second way is to note that

$$A = A_{11}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + A_{22}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + real(A_{12})\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + imaginary(A_{12})\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$$

and thus W is the span of four vectors and hence must have dimension ≤ 4 by Cor 2b, p 45. But the range R_T has dimension 4 by Thm 2 p 71 and is a subspace of W. Consequently W has dimension 4 and hence T is an isomorphism by Thm 9 p 81.

3.3:6 Suppose V and W are isomorphic. Let $T\colon V \to W$ be and isomorphism. By Thm 2 p 71, $\dim V = rank(T) + nullity(T)$. But $nullity(T) = 0$ since T is one to one and $rank(T) = \dim(R_T) = \dim W$ since T is onto (so $R_T = W$). So $\dim V = \dim W + 0 = \dim W$. Conversely, suppose $\dim V = \dim W$. Pick bases $\{\alpha_1, \ldots, \alpha_n\}$ for V and $\{\beta_1, \ldots, \beta_n\}$ for W. By Thm 1 p 69, there is a linear transformation $T\colon V \to W$ so that $T(\alpha_i) = \beta_i$ for all i. Note T is onto since for every $\beta \in W$ we may write $\beta = \sum_{i=1}^n c_i\beta_i$ for some scalars c_i, but then $T(\sum_{i=1}^n c_i\alpha_i) = \beta$ by linearity. Then by Thm 9 p 81 we know T is an isomorphism. So V and W are isomorphic.

3.3:7 Let's give this map a name, ϕ, so $\phi\colon L(V,V) \to L(W,W)$ is given by $\phi(T) = UTU^{-1}$. Since U is an isomorphism we know that V and W have the same dimension. Hence $L(V,V)$ and $L(W,W)$ have the same dimension, namely $\dim(V)^2$. Note $\phi(cT + T') = U(cT + T')U^{-1} = UcTU^{-1} + UT'U^{-1} = cUTU^{-1} + UT'U^{-1} = c\phi(T) + \phi(T')$ so ϕ is linear. Also if $\phi(T) = 0$ then $0 = UTU^{-1}$ so $0 = U^{-1}0U = U^{-1}UTU^{-1}U = T$, so ϕ is one to one. So ϕ is an isomorphism by Thm 9, p 81.

3.4:1 The matrix P taking $\mathcal{B}'$ coordinates to $\mathcal{B}$ coordinates is $\begin{bmatrix} 1 & -i \\ i & 2 \end{bmatrix}$ so the matrix taking $\mathcal{B}$ coordinates to $\mathcal{B}'$ coordinates is its inverse $\begin{bmatrix} 2 & i \\ -i & 1 \end{bmatrix}$.

a) $T(\varepsilon_1) = (1,0)$ which has $\mathcal{B}'$ coordinates $\begin{bmatrix} 2 & i \\ -i & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -i \end{bmatrix}$. $T(\varepsilon_2) = (0,0)$ which has coordinates $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$. So the matrix of T relative to $\mathcal{B}, \mathcal{B}'$ is $\begin{bmatrix} 2 & 0 \\ -i & 0 \end{bmatrix}$.

b) $T\alpha_1 = (1,0)$ with $\mathcal{B}$ coordinates $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $T\alpha_2 = (-i,0)$ with $\mathcal{B}$ coordinates $\begin{bmatrix} -i \\ 0 \end{bmatrix}$. So the matrix of T relative to $\mathcal{B}', \mathcal{B}$ is $\begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix}$.

c) $T\alpha_1 = (1,0)$ with $\mathcal{B}'$ coordinates $\begin{bmatrix} 2 \\ -i \end{bmatrix}$ and $T\alpha_2 = (-i,0)$ with $\mathcal{B}'$ coordinates $-i\begin{bmatrix} 2 \\ -i \end{bmatrix} = \begin{bmatrix} -2i \\ -1 \end{bmatrix}$. So the matrix of T relative to $\mathcal{B}', \mathcal{B}$ is $\begin{bmatrix} 2 & -2i \\ -i & -1 \end{bmatrix}$.

d) If you see it, the easy way is to take the matrix of part c) and switch the two rows and switch the two columns, since all you are doing is switching α_1 and α_2. Or, just say $T(\alpha_2) = (-i,0) = -\alpha_2 - 2i\alpha_1$ so the first column is $\begin{bmatrix} -1 \\ -2i \end{bmatrix}$ and $T\alpha_1 = (1,0) = -i\alpha_2 + 2\alpha_1$ so the matrix is $\begin{bmatrix} -1 & -i \\ -2i & 2 \end{bmatrix}$.

3.4:8 The hint says to solve $T\alpha_1 = e^{i\theta}\alpha_1$, so

$$\alpha_1 \in NS(T - e^{i\theta}I) = NS\begin{bmatrix} \cos\theta - e^{i\theta} & -\sin\theta \\ \sin\theta & \cos\theta - e^{i\theta} \end{bmatrix} = NS\begin{bmatrix} -i\sin\theta & -\sin\theta \\ \sin\theta & -i\sin\theta \end{bmatrix}$$

so $\alpha_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}$ will work. Similarly, $\alpha_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$ will work. $\{\alpha_1, \alpha_2\}$ is linearly independent because $\det\begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} = 2i \neq 0$. Then

$$\begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} = \frac{1}{2i}\begin{bmatrix} 1 & i \\ -1 & i \end{bmatrix}\begin{bmatrix} i\cos\theta - \sin\theta & -i\cos\theta - \sin\theta \\ i\sin\theta + \cos\theta & -i\sin\theta + \cos\theta \end{bmatrix}$$

$$= \frac{1}{2i}\begin{bmatrix} 2i\cos\theta - 2\sin\theta & 0 \\ 0 & 2\sin\theta + 2i\cos\theta \end{bmatrix} = \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}$$

3.4:10 If $\alpha \in R_S$ then $S\alpha = \alpha$ because $\alpha = S\beta$ for some β and then $S\alpha = S^2\beta = S\beta = \alpha$. So if $\dim R_S = 2$ then $R_S = \mathbb{R}^2$ so then $S = I$. If $\dim R_S = 0$ then $R_S = 0$ so $S = 0$. So we may as well assume from now on that $\dim R_S = 1$. If $\mathcal{B} = \{\alpha_1, \alpha_2\}$ and $[S]_\mathcal{B} = A$ then $S\alpha_1 = 1\alpha_1 + 0\alpha_2 = \alpha_1$ and $S\alpha_2 = 0\alpha_1 + 0\alpha_2 = 0$. So we could pick α_1 any nonzero vector in R_S. We have $\dim NS(S) = 2 - \dim R_S = 2 - 1 = 1$ so we may also pick a nonzero vector α_2 in $NS(S)$. Suppose $c_1\alpha_1 + c_2\alpha_2 = 0$. Then $0 = S0 = c_1S(\alpha_1) + c_2S(\alpha_2) = c_1\alpha_1$ so $c_1 = 0$. But then $0 = c_1\alpha_1 + c_2\alpha_2 = c_2\alpha_2$ so $c_2 = 0$. So $\mathcal{B} = \{\alpha_1, \alpha_2\}$ is linearly independent and hence a basis, and $[S]_\mathcal{B} = A$.

3.5:1 The matrix transforming from the standard coordinates to the $\{\alpha_1, \alpha_2, \alpha_3\}$ coordinates is the inverse of $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & -2 & 0 \end{bmatrix}$ which is $\begin{bmatrix} 2 & -2 & -1 \\ 1 & -1 & -1 \\ 1 & -2 & -1 \end{bmatrix}$.

a) $f(a,b,c) = f((2a-2b-c)\alpha_1+(a-b-c)\alpha_2+(a-2b-c)\alpha_3) = (2a-2b-c)1+(a-b-c)(-1)+(a-2b-c)3 = 4a-7b-3c$

b) Similar to a) but with different values, we could take $f(a,b,c) = a-2b-c$.

c) $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & -2 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix}$ so $(2,3,1) = \alpha_1 + 2\alpha_2 - 4\alpha_3$ so $f(2,3,1) = -4f(\alpha_3) \neq 0$.

3.5:2 Let $\alpha_1^*(a,b,c) = xa + yb + zb$ and solve for x, y, z. We have $x - z = 1$, $x + y + z = 0$ and $2x + 2y = 0$ which has solution $x = 1$, $y = -1$, $z = 0$. So $\alpha_1^*(a,b,c) = a - b$. Similarly $\alpha_2^*(a,b,c) = a - b + c$ and $\alpha_3^*(a,b,c) = -a/2 + b - c/2$. You can also get the coefficients from the columns of the inverse of the matrix whose rows are the α_i.

3.5:7 The annihilator of W is the same as the annihilator of $\{\alpha_1, \alpha_2\}$ so it is all such linear functionals where $c_1 - c_3 + 2c_4 = 0$ and $2c_1 + 3c_2 + c_3 + c_4 = 0$. If you want you can solve more completely. After row reduction, we see this is the same as where $c_1 - c_3 + 2c_4 = 0$ and $c_2 + c_3 - c_4 = 0$ so they are of the form $(c_3 - 2c_4)x_1 + (c_4 - c_3)x_2 + c_3x_3 + c_4x_4$. In other words the span of $x_1 - x_2 + x_3$ and $-2x_1 + x_2 + x_4$.

4.2:1 $A^2 = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ -5 & 8 \end{bmatrix}$. $A^3 = \begin{bmatrix} 3 & 5 \\ -5 & 8 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 18 \\ -18 & 19 \end{bmatrix}$. So:

a) $A^2 - A + 2I = \begin{bmatrix} 3 & 4 \\ -4 & 7 \end{bmatrix}$.

b) $A^3 - I = \begin{bmatrix} 0 & 18 \\ -18 & 18 \end{bmatrix}$.

c) $A^2 - 5A + 7I = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

4.4:1

a) Not an ideal, x^2 is in it, but $x \cdot x^2 = x^3$ is not.

b) Not an ideal, $x^5 + 1$ and $-x^5$ are in it, but $x^5 + 1 - x^5 = 1$ is not. Also 0 is not in it.

c) Is an ideal. If $f(0) = 0$ and $g(0) = 0$ then $(cf + g)(0) = cf(0) + g(0) = c0 + 0 = 0$ so it is a subspace. But $(xf)(0) = 0f(0) = 0$ so it is closed under multiplication by x, hence an ideal. We know $f(x) = x$ is in it because $f(0) = 0$. But no nonzero polynomial of smaller degree than x is in it because it would have to be degree 0, hence a nonzero constant. So the monic generator is x.

d) By a similar argument to c) except evaluating at 2 and 4, we see this is an ideal. Note $f(x) = (x-2)(x-4)$ is in it. Suppose a smaller degree polynomial is in it, it must have the form $ax + b$ with $2a + b = 0$ and $4a + b = 0$. The only solution is $a = b = 0$. So the smallest degree monic polynomial in it is $(x-2)(x-4) = x^2 - 6x + 8$ so this is the generator.

e) This is the same as c) so the generator is x. Note that anything in the range of T has no constant term so it evaluates to 0 at 0. But if $f(0) = 0$, and $f(x) = \Sigma_{i=1}^n d_i x^i$ then $f(x) = T(\Sigma_{i=1}^n i d_i x^{i-1})$. So it is the ame as c).

4.4:4 $A^2 = \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -8 \\ 0 & 9 \end{bmatrix}$. But note $A^2 = \begin{bmatrix} 1 & -8 \\ 0 & 9 \end{bmatrix} = 4A - 3I$ so $A^2 - 4A + 3I = 0$. No degree one polynomial is in the annihilating ideal since A is not a multiple of the identity. So $x^2 - 4x + 3$ generates the annihilating ideal.

6.2:1 For $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\det(xI - A) = x(x-1)$ so the characteristic values of T and U are both 0 and 1. For $c = 0$ the characteristic vectors are $NS(T) = Span(\epsilon_2)$ and the same for U. So a basis is $\{\epsilon_2\}$ in each case. For $c = 1$ the characteristic vectors are $NS(T - I) = Span(\epsilon_1)$ and the same for U. So a basis is $\{\epsilon_1\}$ in each case.

For $A = \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}$, $\det(xI - A) = x^2 - 3x + 5$. Since this has no real roots there are no characteristic values for T. The characteristic values for U are $c = \frac{3 \pm \sqrt{-11}}{2}$. For $c = \frac{3+\sqrt{-11}}{2}$ then $NS(cI - A) =$

$NS(\begin{bmatrix} \frac{-1+\sqrt{-11}}{2} & -3 \\ 1 & \frac{1+\sqrt{-11}}{2} \end{bmatrix})$ which is spanned by $\begin{bmatrix} \frac{1+\sqrt{-11}}{2} \\ -1 \end{bmatrix}$. Similarly, for $c = \frac{3-\sqrt{-11}}{2}$ then $NS(cI - A) =$ $NS(\begin{bmatrix} \frac{-1-\sqrt{-11}}{2} & -3 \\ 1 & \frac{1-\sqrt{-11}}{2} \end{bmatrix})$ which is spanned by $\begin{bmatrix} \frac{1-\sqrt{-11}}{2} \\ -1 \end{bmatrix}$.

For $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $\det(xI - A) = x^2 - 2x = x(x-2)$. So the characteristic values of T and U are both 0 and 2. For $c = 0$ the characteristic vectors are $NS(T) = Span(\begin{bmatrix} 1 \\ -1 \end{bmatrix})$ and the same for U. So a basis is $\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\}$ in each case. For $c = 2$ the characteristic vectors are $NS(2T - I) = Span(\begin{bmatrix} 1 \\ 1 \end{bmatrix})$ and the same for U. So a basis is $\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\}$ in each case.

6.2:5 $\det(xI - A) = \det \begin{bmatrix} x-6 & 3 & 2 \\ -4 & x+1 & 2 \\ -10 & 5 & x+3 \end{bmatrix} = (x-6)((x+1)(x+3)-10) + 4(3(x+3)-10) - 10(6 - 2(x+1)) = x^3 - 2x^2 + x - 2$. But by trying small integers we see that 2 is a root of this polynomial so $x^3 - 2x^2 + x - 2 = (x-2)(x^2+1)$ and x^2+1 has roots $\pm\sqrt{-1}$ so we cannot factor the characteristic polynomial over $\mathbb{R}$ into linear factors, so A is not similar over $\mathbb{R}$ to a diagonal matrix. But A is similar over $\mathbb{C}$ to a diagonal matrix by Theorem 2 since there are three different characteristic values $c_1 = 2, c_2 = i, c_3 = -i$ and if $W_j = NS(A - c_jI)$ then $\dim W_j \geq 1$ so $3 \geq \dim(W_1 + W_2 + W_3) = \dim W_1 + \dim W_2 + \dim W_3 \geq 3 = \dim \mathbb{C}^3$ (so in fact $\dim W_j = 1$). You can use Matlab to do such calculations using the following commands:

A=[6 -3 -2;4 -1 -2;10 -5 -3]
poly(A)
[V D] = eig(A)

6.2:11 Note $0 = \det(0) = \det(N^2) = \det(N)^2$ so $\det(N) = 0$ so N has rank 0 or 1. If N has rank 0 then $N = 0$. If N has rank 1 then the null space $NS(N)$ has dimension 1 so we may pick some vector $\beta_1 \in \mathbb{C}^2$ which is not in $NS(N)$. Let $\beta_2 = N(\beta_1)$. Note that $\beta_2 \neq 0$ since β_1 is not in the null space of N. But β_2 is in the null space since $N(\beta_2) = N^2(\beta_1) = 0$. Since β_1 is not a multiple of β_2 we know $\{\beta_1, \beta_2\}$ is a linearly independent set. You can also show this by setting $0 = c_1\beta_1 + c_2\beta_2$, then $0 = N(0) = c_1N(\beta_1) + c_2N(\beta_2) = c_1\beta_2$ so $c_1 = 0$. But then $0 = c_2\beta_2$ so $c_2 = 0$. Since $N\beta_1 = 0\beta_1 + 1\beta_2$ and $N\beta_2 = 0\beta_1 + 0\beta_2$ we know the matrix of N in the basis $\{\beta_1, \beta_2\}$ is $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.

6.2:15 So we have $T(B) = AB$. If c is a characteristic value of T then we have a nonzero matrix B so that $T(B) = AB = cB$. If β_i is the i-th column of B then $[A\beta_1 \cdots A\beta_n] = A[\beta_1 \cdots \beta_n] = c[\beta_1 \cdots \beta_n] = [c\beta_1 \cdots c\beta_n]$ so we have $A\beta_i = c\beta_i$ for all i. Since $B \neq 0$ we know some $\beta_i \neq 0$ which means that c is a characteristic value of A. On the other hand, if c is a characteristic value of A, take a nonzero characteristic vector β so $A\beta = c\beta$. Let $B = [\beta\beta \cdots \beta]$ then $T(B) = AB = cB$ so c is also a characteristic vector of T. Consequently, A and T have the same characteristic values.

6.3:2

$$\det(xI - A) = \det \begin{bmatrix} x & 0 & -c \\ -1 & x & -b \\ 0 & -1 & x-a \end{bmatrix} = x \det \begin{bmatrix} x & -b \\ -1 & x-a \end{bmatrix} - c \det \begin{bmatrix} -1 & x \\ 0 & -1 \end{bmatrix}$$

$$= x(x(x-a) - b) - c = x^3 - ax^2 - bx - c$$

So the characteristic polynomial is $x^3 - ax^2 - bx - c$. $A^2 = \begin{bmatrix} 0 & 0 & c \\ 1 & 0 & b \\ 0 & 1 & a \end{bmatrix} \begin{bmatrix} 0 & 0 & c \\ 1 & 0 & b \\ 0 & 1 & a \end{bmatrix} = \begin{bmatrix} 0 & c & ac \\ 0 & b & c+ab \\ 1 & a & b+a^2 \end{bmatrix}$. So the first column of $c_2A^2 + c_1A + c_0I$ is $\begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix}$. Consequently if $c_2A^2 + c_1A + c_0I = 0$ then this first column must be 0 so all $c_i = 0$. So the minimal polynomial of A must have degree > 2. But it must also divide the characteristic polynomial which has degree 3, so it must equal the characteristic polynomial.

6.3:4 Similar matrices have the same minimal polynomial and the same characteristic polynomial. The minimal polynomial of a diagonal matrix is a product of distinct linear factors, which A does not have, but we don't know this officially until later so I will ignore that general fact. If A is similar to a diagonal matrix, that diagonal matrix must have the same characteristic polynomial as A, namely $x^2(x-1)^2$, so the diagonal entries must be $0, 0, 1, 1$ or some permutation. So A would be similar to $B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ which has minimal polynomial $x^2 - x$ since $B^2 - B = 0$. So A and B would have different minimal polynomials which contradicts their being similar. So A is not similar to a diagonal matrix.

6.3:6 We need a 3×3 matrix A so that $A^2 = 0$ but $A \neq 0$. So there is an $\alpha_1 \neq 0$ so that $A\alpha_1 \neq 0$ but $A^2\alpha_1 = 0$. To make computations easiest, we may as well take $\alpha_1 = \epsilon_1$ and $A\alpha_1 = \epsilon_2$ and $A\epsilon_2 = 0$. We have some freedom in choosing $A\epsilon_3$, (it could be any multiple of ϵ_2) but we'll let it be 0. So the matrix is $A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. In fact as we will see when we study the Jordan canonical form, any 3×3 matrix with minimal polynomial x^2 will be similar to $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

6.4:1 a) The only subspaces of $\mathbb{R}^2$ which are not $\mathbb{R}^2$ or 0 must have dimension 1. So if A has an invariant subspace W with $W \neq \mathbb{R}^2$ and $W \neq 0$, we must have $W = Span\{\alpha\}$ for some $\alpha \neq 0$. Invariance of W implies $T\alpha = c\alpha$ for some c, i.e., c is a characteristic value of T and hence of A. But the characteristic polynomial of A is $x^2 - 3x + 4$ whose roots are $x = \frac{3 \pm \sqrt{-7}}{2}$ which are not real so T has no characteristic values, and hence no nontrivial invariant subspaces. Warning– in higher dimensions it is quite possible that an operator have no characteristic values and yet still have nontrivial invariant subspaces.

b) The calculation above shows that U has two characteristic values. Thus it has two invariant subspaces, namely the two spaces of charateristic vectors $NS(A - \frac{3+\sqrt{-7}}{2}I)$ and $NS(A - \frac{3-\sqrt{-7}}{2}I)$. The first is the span of $\begin{bmatrix} 1 \\ \frac{-1-\sqrt{-7}}{2} \end{bmatrix}$ and the second is the span of $\begin{bmatrix} 1 \\ \frac{-1+\sqrt{-7}}{2} \end{bmatrix}$.

6.4:3 T_W is multiplication by the scalar c, also could be denoted cI.

6.4:5 If $A^2 = A$ then $x^2 - x = x(x-1)$ is in the annihilating ideal of A, so the minimal polynomial of A divides $x(x-1)$, so the minimal polynomial of A is either x, $x - 1$, or $x(x-1)$. In each case the minimal polynomial is a product of distinct linear factors, so by theorem 6 we know A is similar to a diagonal matrix.

6.4:9 If $f(x)$ is a polynomial function $f(x) = \Sigma_{i=0}^{n} c_n x^n$ then $(Tf)(x) = \Sigma_{i=0}^{n} \frac{c_n}{n+1} x^{n+1}$ is also a polynomial so the subspace of polynomial functions is invariant under T. By the fundamental theorem of calculus, the indefinite integral of a continuous function is differentiable, so the differentiable functions are invariant under T. The space of functions with $f(1/2) = 0$ is not invariant however. For example, $2x - 1$ is in this subspace, but $T(2x-1) = x^2 - x$ is not in the subspace because $(1/2)^2 - 1/2 = -1/4 \neq 0$.

6.5:1a Following the proof of Thm 8, we let W_1 and W_2 be the characteristic vectors for the two characteristic values of one of the operators, say A. Then each W_i is also invariant under B. In this case, since each W_i must be one dimensional that means that vectors in W_i are also characteristic vectors for B (but with probably different characteristic values). To be more specific, the characteristic values of A are 1 and 2, so $W_1 = NS(A - I) = Span(\epsilon_1)$ and $W_2 = NS(A - 2I) = Span(\begin{bmatrix} 2 \\ 1 \end{bmatrix})$. Then ϵ_1 and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ will be the columns of P. Checking, we get

$$P^{-1}AP = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$P^{-1}BP = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -8 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$$

6.5:2 This was harder than I intended. Details later.

6.6:1 Choose a basis $\{\beta_1, \ldots, \beta_k\}$ for W_1. Extend this basis to a basis $\{\beta_1, \ldots, \beta_n\}$ for V. Let W_2 be the subspace spanned by $\{b_{k+1}, \ldots, \beta_n\}$. Then $V = W_1 + W_2$ and $W_1 \cap W_2 = \{0\}$ so $V = W_1 \oplus W_2$.

6.6:4 False. Take for example $E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ which projects to the x axis, and $E_2 = E_1$. Then $(E_1 + E_2)^2 = 4E_1 \neq E_2 + E_2$ so $E_1 + E_2$ is not a projection. In order to be a projection, you need $E_1 + E_2 = (E_1 + E_2)^2 = E_1^2 + E_1E_2 + E_2E_1 + E_2^2 = E_1 + E_1E_2 + E_2E_1 + E_2$ so you need $E_1E_2 + E_2E_1 = 0$.

6.6:5 Let $f(x) = a_0 + a_1x + \cdots + a_kx^k$. Then $f(E) = a_0I + a_1E + \cdots + a_kE^k$. But $E^2 = E$, $E^3 = E^2E = EE = E$, and in general $E^k = E$ for all $k \geq 1$. So $f(E) = a_0I + (a_1 + a_2 + \cdots + a_k)E$. So a is the constant coefficient and b is the sum of the remaining coefficients.

6.7:2

a) If $\alpha \in W_1$ then $\alpha = c\varepsilon_1$ for some c, so $T\alpha = T(c\varepsilon_1) = 2c\varepsilon_1 \in W_1$. So W_1 is invariant under T.
b) Suppose W_2 is complimentary and invariant. Since $\dim W_1 + \dim W_2 = \dim R^2 = 2$ we must have $\dim W_2 = 1$ so W_2 is the span of some nonzero vector β. Since W_2 is invariant under T we must have $T\beta \in W_2$, so $T\beta = c\beta$ for some scalar c. So c is a characteristic value of T, hence a root of the characteristic polynomial which is $(x-2)^2$. So $c = 2$. So $\beta \in NS(T - 2I) = W_1$ which means $W_1 \cap W_2 \neq 0$ which contradicts the independence of W_1 and W_2. So no such invariant complementary W_2 exists.

6.7:6 The c_i are the characteristic values of A so we may take $c_1 = 0$, $c_2 = 2$, $c_3 = -2$. The E_i are projections to the subspaces of characteristic vectors of A. The null space of A is spanned by $\alpha_1 = [1\ 0\ -1\ 0]^T, \alpha_2 = [0\ 1\ 0\ -1]^T$. The null space of $A - 2I$ is spanned by $\alpha_3 = [1\ 1\ 1\ 1]^T$. The null space of $A + 2I$ is spanned by $\alpha_4 = [1\ -1\ 1\ -1]^T$, Then $E_1 = P \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} P^{-1}$, $E_2 = P \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} P^{-1}$, $E_3 = P \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} P^{-1}$, where $P = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ -1 & 0 & 1 & 1 \\ 0 & -1 & 1 & -1 \end{bmatrix}$. This works because P^{-1} finds the $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ coordinates of a vector, the middle matrix projects to the appropriate subspace, and P changes back to standard coordinates.

6.8:1 $T^2 = \begin{bmatrix} 6 & -3 & -2 \\ 4 & -1 & -2 \\ 10 & -5 & -3 \end{bmatrix} \begin{bmatrix} 6 & -3 & -2 \\ 4 & -1 & -2 \\ 10 & -5 & -3 \end{bmatrix} = \begin{bmatrix} 4 & -5 & 0 \\ 0 & -1 & 0 \\ 10 & -10 & -1 \end{bmatrix}$. Using Matlab I found the characteristic polynomial of T is $x^3 - 2x^2 + x - 2 = (x-2)(x^2+1)$. This must also be the minimal polynomial of T since the minimal polynomial divides the characteristic polynomial and has the same complex roots 2, $\pm i$. Another reason is that T^2, T, I are linearly independent since their first rows are linearly independent. So we may let $p_1(x) = x - 2$ and $p_2(x) = x^2 + 1$. We calculate $NS(T - 2I) = Span[1\ 0\ 2]^T$. $T^2 + I = \begin{bmatrix} 5 & -5 & 0 \\ 0 & 0 & 0 \\ 10 & -10 & 0 \end{bmatrix}$ so by inspection its null space has basis $\{[1\ 1\ 0]^T, [0\ 0\ 1]^T\}$. The matrix of T_1 is $[2]$ since $T[1\ 0\ 2]^T = 2[1\ 0\ 2]^T$. The matrix of T_2 is $\begin{bmatrix} 3 & -2 \\ 5 & -3 \end{bmatrix}$ since $T[1\ 1\ 0]^T = 3[1\ 1\ 0]^T + 5[0\ 0\ 1]^T$ and $T[0\ 0\ 1]^T = -2[1\ 1\ 0]^T - 3[0\ 0\ 1]^T$. Different matrices for T_2 are possible of course if a different basis was chosen.

7.1:1 Suppose α is a nonzero vector in $\mathcal{F}^2$ which is not a characteristic vector for T. Then $T\alpha$ is not a multiple of α which means that $\{\alpha, T\alpha\}$ is a linearly independent set, and hence a basis of $\mathcal{F}^2$. So $\mathcal{F}^2$ is the span of $\{\alpha, T\alpha\}$. So $\mathcal{F}^2 = Span\{\alpha, T\alpha\} \subset Span\{\alpha, T\alpha, T^2\alpha, \ldots\} = Z(\alpha; T) \subset \mathcal{F}^2$ so we must have

$\mathcal{F}^2 = Z(\alpha; T)$. Alternatively, you could note that if the characteristic polynomial of T is $x^2 - ax - b$ then $T^2 = aT + bI$ so $T^2\alpha = aT\alpha + b\alpha$ which implies $Z(\alpha; T) = Span\{\alpha, T\alpha\}$.

Given the above, if T does not have a cyclic vector, then every nonzero vector is a characteristic vector. If α and β are nonzero characteristic vectors with different characteristic values $a \neq b$ then $\alpha + \beta$ is not characteristic because $T(\alpha + \beta) = a\alpha + b\beta \neq c(\alpha + \beta)$ for any c. So if T does not have a cyclic vector, then every nonzero vector is a characteristic vector and there is just one characteristic value c. So $T\alpha = c\alpha$ for all α and $T = cI$.

7.1:5 In my posted notes on the Cyclic Decomposition theorem, I proved that $\{\alpha, N\alpha, \ldots, N^{n-1}\alpha\}$ is a linearly independent set. Its Span is n dimensional and hence must be all of V. So α is a cyclic vector for N. The matrix of N in this basis is just the $n \times n$ Jordan block with 0 everywhere except for 1s just below the diagonal.

7.2:1 In mathematics, two sets are disjoint if their intersection is empty. So actually, no two subspaces of a vector space are disjoint since they have zero in common. But he must mean something else here. The index says "Disjoint subspaces (see Independent: subspaces)" which seems to imply the book wants disjoint subspaces to be the same as independent subspaces. I apologize for not catching this beforehand. So let us prove that $Z(\alpha_2; T)$ and $Z(\alpha_1; T)$ are never independent. We will in fact show that $Z(\alpha_2; T)$ always contains $Z(\alpha_1; T)$ if $\alpha_2 \neq 0$.

$T\alpha_1 = 0$ so $Z(\alpha_1; T)$ is the span of α_1 which is not $\mathcal{F}^2$. Pick any $\alpha_2 = (a, b) \neq (0, 0)$. Then $T\alpha_2 = (0, a) = a\alpha_1$. So if $a \neq 0$ $Z(\alpha_2; T) \supset SpanT\alpha_2 = Span\alpha_1 = Z(\alpha_1; T)$. But if $a = 0$ then $\alpha_2 = b\alpha_1$ with $b \neq 0$ so $Z(\alpha_2; T) = Z(\alpha_1; T)$.

7.3:1 Similar matrices have the same characteristic polynomial so one direction is immediate. So suppose that N_1 and N_2 have the same minimal polynomial. The characteristic polynomial of each N_i is x^3 so the minimal polynomial divides this and thus must be x, x^2, or x^3. If the minimal polynomials of N_i are both x then $N_i = 0$ so N_1 and N_2 are similar, and in fact equal. If the minimal polynomials are both x^2 then the Jordan form of each N_i must have a 2×2 Jordan block, so the Jordan forms of N_1 and N_2 are the same, one 2×2 block and one 1×1 block. Since N_i are similar to the same Jordan form matrix they are themselves similar. If the minimal polynomials are both x^3 then the Jordan form of each N_i must have a 3×3 Jordan block, so the Jordan forms of N_1 and N_2 are the same, one 3×3 block. Since N_i are similar to the same Jordan form matrix they are themselves similar.

Beware that the corresponding result for larger matrices does not hold. The nilpotent Jordan form matrix with two 2×2 blocks is not similar to the matrix with one 2×2 block and two 1×1 bocks, even though each has minimal polynomial x^2.

7.3:3 We know from the $(x-2)^3$ in the characteristic polynomial that there are three dimensions worth of Jordan blocks with diagonal entry 2, and we know from the $(x-2)^2$ in the minimal polynomial that the largest of these blocks is 2×2 so there must be one 2×2 block and one 1×1 block with diagonal entry 2. Likewise, there will be two 1×1 blocks with diagonal entry -7. So the Jordan form of A is
$$\begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & -7 & 0 \\ 0 & 0 & 0 & 0 & -7 \end{bmatrix}.$$

7.3:4 As I have done in my notes I will let $J_{k,c}$ represent a $k \times k$ Jordan block with c on the diagonal. There are 5 possibilities for the upper left 4×4 part with -2 on the diagonal:

1) $J_{4,-2}$.
2) $J_{3,-2}$ and $J_{1,-2}$.
3) $J_{2,-2}$ and $J_{2,-2}$.
4) $J_{2,-2}$, $J_{1,-2}$ and $J_{1,-2}$.
5) $J_{1,-2}$, $J_{1,-2}$, $J_{1,-2}$ and $J_{1,-2}$.

There are 2 possibilities for the lower right 2×2 part with 1 on the diagonal:

a) $J_{2,1}$.
b) $J_{1,1}$ and $J_{1,1}$.

So all in all there are $5 \cdot 2 = 10$ possible Jordan forms.

7.3:5 If D is differentiation then x^3 is a cyclic vector for D since the vector space is spanned by $\{x^3, Dx^3, D^2x^3, D^3x^3\} = \{x^3, 3x^2, 6x, 6\}$. So the Jordan form is one 4×4 block with 0 on the diagonal, $J_{4,0}$.

7.3:8 We did this in class. We may as well suppose that A is in Jordan form since if B is similar to A then $B^3 = I$ also (since similar operators have the same annihilating polynomials). The Jordan form of a 3×3 matrix would be either $A_1 = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ or $A_2 = \begin{bmatrix} a & 0 & 0 \\ 1 & a & 0 \\ 0 & 0 & c \end{bmatrix}$ or $A_3 = \begin{bmatrix} a & 0 & 0 \\ 1 & a & 0 \\ 0 & 1 & a \end{bmatrix}$. But $A_3^3 = \begin{bmatrix} a^3 & 0 & 0 \\ 3a^2 & a^3 & 0 \\ 3a & 3a^2 & a^3 \end{bmatrix} \neq I$ and $A_2^3 = \begin{bmatrix} a^3 & 0 & 0 \\ 3a^2 & a^3 & 0 \\ 0 & 0 & c^3 \end{bmatrix} \neq I$. So the only possibility is the diagonalizable A_1 with $a^3 = b^3 = c^3 = 1$. Let $\omega = e^{2\pi i/3} = \frac{\sqrt{3}+i}{2}$. Then a, b, c must be ω^k for some $k = 0, 1, 2$. There are 10 distinct nonsimilar possibilities:
1) $a, b, c = 1, 1, 1$.
2) $a, b, c = 1, 1, \omega$.
3) $a, b, c = 1, 1, \omega^2$.
4) $a, b, c = 1, \omega, \omega$.
5) $a, b, c = 1, \omega, \omega^2$.
6) $a, b, c = 1, \omega^2, \omega^2$.
7) $a, b, c = \omega, \omega, \omega$.
8) $a, b, c = \omega, \omega, \omega^2$.
9) $a, b, c = \omega, \omega^2, \omega^2$.
10) $a, b, c = \omega^2, \omega^2, \omega^2$.

7.3:10 This was also done in class. You could use the Jordan form as we did at first in class, but it seems easier without. I posted notes which determine exactly when a complex matrix has a square root if you are interested. If $N = A^2$ then $0 \neq N^{n-1} = A^{2n-2}$. In particular, A is also nilpotent so we know its characteristic polynomial is x^n, so $A^n = 0$. But since $A^{2n-2} \neq 0$ we must have $2n - 2 < n$, so $n < 2$, but we were told $n \geq 2$, so no such A exists.

7.3:14 What is wrong is the assertion that $A^t = -A$ implies that $J^t = -J$. There is no reason to conclude this. Indeed it is false. Consider for example $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Then A has characteristic values $\pm i$ and thus its Jordan form is $J = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$. But $J^t = J \neq -J$.

8.1:1
a) $(0|\beta) = (0\beta|\beta) = 0(\beta|\beta) = 0$.
b) Let $\beta = \alpha$. If $\alpha \neq 0$ then $0 = (\alpha|\beta) = (\alpha|\alpha) > 0$, a contradiction. So $\alpha = 0$.

8.1:2 Let $(\ |\ )$ and $<\ |\ >$ be two inner products on V. Define a third inner product by $[\alpha|\beta] = (\alpha|\beta) + <\alpha|\beta>$. Then $[\alpha + \beta|\gamma] = (\alpha + \beta|\gamma) + <\alpha + \beta|\gamma> = (\alpha|\gamma) + (\beta|\gamma) + <\alpha|\gamma> + <\beta|\gamma> = [\alpha|\gamma] + [\beta|\gamma]$ so property (a) on page 271 is true. $[c\alpha|\beta] = (c\alpha|\beta) + <c\alpha|\beta> = c(\alpha|\beta) + c<\alpha|\beta> = c[\alpha|\beta]$ so (b) is true. $[\beta|\alpha] = (\beta|\alpha) + <\beta|\alpha> = \overline{(\alpha|\beta)} + \overline{<\alpha|\beta>} = \overline{(\alpha|\beta) + <\alpha|\beta>} = \overline{[\alpha|\beta]}$ so (c) is true. Finally, if $\alpha \neq 0$ then $[\alpha|\alpha] = (\alpha|\alpha) + <\alpha|\alpha> > 0$ so (d) holds. So $[\ |\ ]$ is an inner product. If we took the difference instead of the sum, then (a), (b), and (c) would still be true, but (d) might not be. For example the difference of an inner product with itself is 0 which would not satisfy (d). Now let's show a positive multiple of an inner product is an inner product. Take any real $d > 0$ and redefine $[\alpha|\beta] = d(\alpha|\beta)$. Then $[\alpha + \beta|\gamma] = d(\alpha + \beta|\gamma) = d(\alpha|\gamma) + d(\beta|\gamma) = [\alpha|\gamma] + [\beta|\gamma]$ so property (a) is true. $[c\alpha|\beta] = d(c\alpha|\beta) = dc(\alpha|\beta) = c[\alpha|\beta]$ so (b) is true. $[\beta|\alpha] = d(\beta|\alpha) = d\overline{(\alpha|\beta)} = \overline{d(\alpha|\beta)} = \overline{[\alpha|\beta]}$ so (c) is true. Finally, if $\alpha \neq 0$ then $[\alpha|\alpha] = d(\alpha|\alpha) > 0$ so (d) holds. So $[\ |\ ]$ is an inner product.

8.1:6 Note $T\varepsilon_1 = \varepsilon_2$ so $[\varepsilon_1|\varepsilon_2] = 0$. Also $T(\varepsilon_1 + \varepsilon_2) = -\varepsilon_1 + \varepsilon_2$ so $0 = [\varepsilon_1 + \varepsilon_2| - \varepsilon_1 + \varepsilon_2] = -[\varepsilon_1|\varepsilon_1] + [\varepsilon_2|\varepsilon_2]$ so $[\varepsilon_1|\varepsilon_1] = [\varepsilon_2|\varepsilon_2]$. Let $d = [\varepsilon_1|\varepsilon_1] = [\varepsilon_2|\varepsilon_2]$. Note $d > 0$ by property (d). Then $[(a_1, a_2)|(b_1, b_2)] =$

$a_1\overline{b_1}[\varepsilon_1|\varepsilon_1] + a_2\overline{b_2}[\varepsilon_2|\varepsilon_2] + a_1\overline{b_2}[\varepsilon_1|\varepsilon_2] + a_2\overline{b_1}[\varepsilon_2|\varepsilon_1] = d(a_1\overline{b_1} + a_2\overline{b_2})$. So [|] is just a positive multiple of the standard inner product.

8.1:9 $||\alpha+\beta||^2 + ||\alpha-\beta||^2 = (\alpha+\beta|\alpha+\beta) + (\alpha-\beta|\alpha-\beta) = (\alpha|\alpha) + (\beta|\alpha) + (\alpha|\beta) + (\beta|\beta) + (\alpha|\alpha) - (\beta|\alpha) - (\alpha|\beta) + (\beta|\beta) = 2(\alpha|\alpha) + 2(\beta|\beta) = 2||\alpha||^2 + 2||\beta||^2$.

8.1:12 This would be easier to do if we had the Gram-Schmidt process at this point, but that does not come until the next section, so I'll not use it. We need to find d_i, $i = 1, \ldots, n$ so that if $\alpha = \sum_{i=1}^n d_i\alpha_i$ then $(\alpha|\alpha_j) = c_j$ for all j. Moreover we must show that these d_i are unique. This is just a system linear equations which can be written in matrix form as $Ad = c$ where d and c are column vectors with i-th entry d_i and c_i and the row j column i entry of A is $(\alpha_i|\alpha_j)$. We need to show A is nonsingular, and then the solution d will exist and be unique. If $Ax = 0$ then $x^*Ax = 0$, but $0 = x^*Ax = (\sum_{i=1}^n x_i\alpha_i|\sum_{i=1}^n x_i\alpha_i)$ which means $\sum_{i=1}^n x_i\alpha_i = 0$ by property (d). So $x = 0$ by linear independence of the α_i and thus A is nonsingular. So the coefficients d_i exist and are unique. Hence $\alpha = \sum_{i=1}^n d_i\alpha_i$ exists and is unique.

8.2:3 Use the Gram-Schmidt process. First I'll find an orthogonal basis $\{\alpha_1, \alpha_2\}$. Let $\alpha_1 = (1, 0, i)$ and $\alpha_2 = (2, 1, 1+i) - \big((2, 1, 1+i) \mid (1, 0, i)\big)(1, 0, i)/||(1, 0, i)||^2 = (2, 1, 1+i) - (3-i)(1, 0, i)/2 = (1+i, 2, 1-i)/2$. We have $||\alpha_1|| = \sqrt{2}$ and $||\alpha_2|| = \sqrt{1+1+4+1+1}/2 = \sqrt{2}$ so an orthonormal basis is $(\frac{1}{\sqrt{2}}, 0, \frac{i}{\sqrt{2}})$, $(\frac{1+i}{2\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1-i}{2\sqrt{2}})$.

8.2:6

a) By theorem 4(iii), $E(x_1, x_2) = \frac{3x_1+4x_2}{25}(3, 4)$.

b) The columns are $E(1, 0)$ and $E(0, 1)$ so the matrix is $\begin{bmatrix} .36 & .48 \\ .48 & .64 \end{bmatrix}$.

c) $W^\perp = NS(E) =$ the span of $(4, -3)$.

d) If the orthonormal basis is $\{\beta_1, \beta_2\}$ then $E\beta_1 = \beta_1$ and $E\beta_2 = 0$, so $\beta_1 \in W$ and $\beta_2 \in W^\perp$. So we may take the basis $\{(.6, .8),\ (.8, -.6)\}$.

8.2:7 We have by page 274 (8-3), $\big((a, b)|(c, d)\big) = ||(a+c, b+d)||^2/4 - ||(a-c, b-d)||^2/4 = ((a+c-b-d)^2 + 3(b+d)^2 - (a-c-b+d)^2 - 3(b-d)^2)/4 = (a-b)(c-d) + 3bd$.

a) $||(3, 4)||^2 = 1 + 3\cdot 16 = 49$ and $((x_1, x_2)|(3, 4)) = x_2 - x_1 + 12x_2 = 13x_2 - x_1$ so $E(x_1, x_2) = \frac{13x_2-x_1}{49}(3, 4)$.

b) $\begin{bmatrix} -3/49 & 39/49 \\ -4/49 & 52/49 \end{bmatrix}$

c) The span of (13,1).

d) We have $||(13, 1)|| = \sqrt{12^2 + 3\cdot 1^2} = \sqrt{147}$ so an orthonormal basis is $\{3/7, 4/7), (13/\sqrt{147}, 1/\sqrt{147})\}$.

8.2:10 By the formula on page 272 the orthogonal complement of the diagonal matrices is the subspace of matrices whose diagonal entries are all 0. To fill in the details, let D_i be the matrix which is all 0 except for a 1 in the i-th row, i-th column. Then $(A|D_i)$ is the i-th diagonal entry of A. So if $0 = (A|D_i)$ for all i, then all diagonal entries of A are 0. Conversely, if all diagonal entries of A are 0 then $(A|D) = 0$ for any diagonal matrix.

8.2:17 Recall that an even function f satisfies $f(x) = f(-x)$ for all x. First note that $\int_{-1}^0 f(t)\,dt = \int_1^0 -f(-u)\,du$ substituting $u = -t$. But if f is odd, then $\int_1^0 -f(-u)\,du = \int_1^0 f(u)\,du = -\int_0^1 f(u)\,du$. So $\int_{-1}^1 f(t)\,dt = \int_{-1}^0 f(t)\,dt + \int_0^1 f(t)\,dt = -\int_0^1 f(u)\,du + \int_0^1 f(t)\,dt = 0$. Since the product of an odd and an even function is odd, this means that $(f|g) = 0$ if f is even and g is odd. So the orthogonal complement of the odd functions contains the even functions. Now take any g in the orthogonal complement of the odd functions. Let $g_o(t) = (g(t) - g(-t))/2$ and $g_e(t) = (g(t) + g(-t))/2$. Note that $g = g_o + g_e$ and g_o is odd and g_e is even. We have $0 = (g|g_o) = (g_e + g_o|g_o) = (g_e|g_o) + (g_o|g_o) = 0 + (g_o|g_o)$. So $(g_o|g_o) = 0$ which means $g_o = 0$. Consequently $g = g_e$ and g is even. So the orthogonal complement of the odd functions is the even functions.

8.3:1 By the corollary on page 294 the matrix of T^* is the conjugate transpose of the matrix of T, so $T^*(x_1, x_2) = x_1(1, -i) + x_2(-2, -1)$. Just to check, we should have $(T(a, b)|(c, d)) = ((a, b)|T^*(c, d))$. Then

$$((a, b)|T^*(c, d)) = ((a, b)|c(1, -i) + d(-2, -1)) = \bar{c}(a + bi) + \bar{d}(-2a - b)$$

$$(T(a,b)|(c,d)) = (a(1,-2)+b(i,-1)|(c,d)) = a(\bar{c}-2\bar{d})+b(i\bar{c}-\bar{d}) = \bar{c}(a+bi)+\bar{d}(-2a-b)$$

8.3:4 We discussed several strategies in class. Here's another. If $\alpha \in R_{T^*}$ and $\gamma \in NS(T)$ then $\alpha = T^*\beta$ for some β and $(\alpha|\gamma) = (T^*\beta|\gamma) = (\beta|T\gamma) = (\beta|0) = 0$, so $R_{T^*} \subset NS(T)^\perp$. Now suppose $\alpha' \in R_{T^*}^\perp$. Then $(\alpha'|T^*\beta) = 0$ for all $\beta \in V$. But then $0 = (\alpha'|T^*\beta) = (T\alpha'|\beta)$ for all $\beta \in V$ (for example $\beta = T\alpha'$) which means $T\alpha' = 0$. So $R_{T^*}^\perp \subset NS(T)$. But $A \subset B$ implies $B^\perp \subset A^\perp$ which means $NS(T)^\perp \subset R_{T^*}^{\perp\perp}$. Since we are in finite dimensions, $R_{T^*}^{\perp\perp} = R_{T^*}$, so we have $R_{T^*} \subset NS(T)^\perp \subset R_{T^*}^{\perp\perp} = R_{T^*}$. So $R_{T^*} = NS(T)^\perp$. There are couple claims above you must verify though.

8.3:6 $T(c\alpha_1+\alpha_2) = (c\alpha_1+\alpha_2|\beta)\gamma = (c(\alpha_1|\beta)+(\alpha_2|\beta))\gamma = cT\alpha_1+T\alpha_2$ so T is linear. We need $(\alpha|T^*\delta) = (T\alpha|\delta) = (\alpha|\beta)(\gamma|\delta)$ for all α and δ, so we may let $T^*\delta = \overline{(\gamma|\delta)}\beta = (\delta|\gamma)\beta$, in other words just switch β and γ. The j,k-th entry is $(T\varepsilon_k|\varepsilon_j) = ((\varepsilon_k|\beta)\gamma|\varepsilon_j) = \bar{y}_k x_j$. The range of T is the span of γ and the rank is the dimension of the range. So if $\gamma \neq 0$ then the rank is 1,and if $\gamma = 0$ the rank is 0.

8.3:9 Let A be the matrix of D^* with respect to the usual basis $\{1,x,x^2,x^3\}$. Note $(x^i|x^j) = \frac{1}{i+j+1}$. Suppose the first column of A has entries a,b,c,d. this means $D^*1 = a+bx+cx^2+dx^3$ so $\frac{a}{i}+\frac{b}{i+1}+\frac{c}{i+2}+\frac{d}{i+3} = (x^i|a+bx+cx^2+dx^3) = (x^i|D^*1) = (Dx^i|1) = 1$ if $i>0$ and $=0$ if $i=0$. Let X be the matrix $X = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{bmatrix}$. Then we need to solve the equation $X\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ so $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = X^{-1}\begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. Likewise the coefficients of D^*x^j are the $j+1$-th column of A and are given by $X^{-1}\begin{bmatrix} (D1|x^j) \\ (Dx|x^j) \\ (Dx^2|x^j) \\ (Dx^3|x^j) \end{bmatrix} = X^{-1}\begin{bmatrix} 0 \\ \frac{1}{j+1} \\ \frac{2}{j+2} \\ \frac{3}{j+3} \end{bmatrix}$.

So $A = X^{-1}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ 1 & \frac{2}{3} & \frac{2}{4} & \frac{2}{5} \\ 1 & \frac{3}{4} & \frac{3}{5} & \frac{3}{6} \end{bmatrix}$. Using Matlab to calculate this, I get $D^*(1) = -120+180x-420x^2+280x^3$, $D^*(x) = -5+60x-180x^2+140x^3$, $D^*(x^2) = -4+58x-180x^2+140x^3$, and $D^*(x^3) = -4+60x-183x^2+140x^3$.

Another way to do this is probably the way H&K were thinking of, because it uses problem 8. By the calculations in example 21 we know $(f|D^*g+Dg) = f(1)g(1)-f(0)g(0)$ for all f and g. So if we had h_0 and h_1 so that $(f|h_0) = f(0)$ and $(f|h_1) = f(1)$ for all f then $f(1)g(1)-f(0)g(0) = (f|g(1)h_1-g(0)h_0)$ so we must have $D^*g+Dg = g(1)h_1-g(0)h_0$. Thus $D^*g = -Dg+g(1)h_1-g(0)h_0$. To calculate $h_0(x) = a+bx+cx^2+dx^3$ we must have $(x^i|h_0) = 0$ for $i>0$ and $(1|h_0) = 1$. So $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = X^{-1}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. Likewise the coefficients of h_1 are $X^{-1}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. I presume H&K had some trick in mind for calculating this which I don't see at the moment.

8.4:1 Note an $n \times n$ matrix is orthogonal if and only if the sum of the squares of the entries in each column is 0 and the dot product (without conjugating) of any two different columns is 0. So $[i]$ is unitary but not orthogonal for example since $[i]^t[i] = [-1]$ but $[i]^*[i] = [1]$. $\begin{bmatrix} \sqrt{2} & i \\ i & -\sqrt{2} \end{bmatrix}$ is orthogonal but not unitary since $\begin{bmatrix} \sqrt{2} & i \\ i & -\sqrt{2} \end{bmatrix}^t \begin{bmatrix} \sqrt{2} & i \\ i & -\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ but $\begin{bmatrix} \sqrt{2} & i \\ i & -\sqrt{2} \end{bmatrix}^* \begin{bmatrix} \sqrt{2} & i \\ i & -\sqrt{2} \end{bmatrix} = \begin{bmatrix} 3 & 2\sqrt{2}i \\ -2\sqrt{2}i & 3 \end{bmatrix}$.

8.4:2 Recall that $tr(AB) = tr(BA)$ by exercise 3, page 105. Suppose first that M is unitary. Then for any matrices A and B in V, $(T_MA|T_MB) = (MA|MB) = tr(MA(MB)^*) = tr(MAB^*M*) = tr(M^*MAB^*) = tr(AB^*) = (A|B)$. Thus T_M is unitary and one direction is proven.

Now suppose that T_M is unitary. We have $(T_MA|T_MB) = (A|B)$ for all A and B, so by the above formulae we have $tr(M^*MAB^*) = tr(AB^*)$ for all A and B. Letting $C = AB^*$ and $N = M^*M$ we then have $tr(NC) = tr(C)$ for all C. Let C_{ij} be the matrix which has all 0 entries except for a 1 in the ij place. Let N_{ij} denote the ij-th entry of N. The j-th column of NC_{ij} is the i-th column of N and all other columns are 0. So $tr(NC_{ij}) = N_{ji}$. Also $tr(C_ij) = \delta_{ij} = 0$ for $i \neq j$ and $= 1$ for $i = j$. So $N_{ji} = \delta_{ij}$ and thus N is the identity. But then $I = N = M^*M$ so M is unitary.

8.4:6 a) Let α_1 and α_2 be two vectors in V and suppose $\alpha_i = \beta_i + \gamma_i$ with $\beta_i \in W$ and $\gamma_i \in W^\perp$. Note that $(\beta_i|\gamma_j) = 0$. Then $(U\alpha_1|\alpha_2) = (\beta_1 - \gamma_1|\beta_2 + \gamma_2) = (\beta_1|\beta_2) - (\gamma_1|\gamma_2) = (\beta_1 + \gamma_1|\beta_2 - \gamma_2) = (\alpha_1|U\alpha_2)$ so U is self adjoint. Then $U^*U(\alpha_1) = U^*(\beta_1 - \gamma_1) = U(\beta_1 - \gamma_1) = \beta_1 + \gamma_1 = \alpha_1$ so U^*U is the identity and thus U is unitary.

b) We have $(1,0,0) = .5(1,0,1) + .5(1,0,-1)$ so $U(1,0,0) = .5(1,0,1) - .5(1,0,-1) = (0,0,1)$. Note $U^2 = U^*U = I$, so $U(0,0,1) = U^2(1,0,0) = (1,0,0)$. Also $(0,1,0) \in W^\perp$ so $U(0,1,0) = -(0,1,0)$. So the matrix of U in the standard basis is $\begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

8.4:10 a) $(T_{\alpha,\beta}\gamma|\delta) = ((\gamma|\beta)\alpha|\delta) = (\gamma|\beta)(\alpha|\delta) = (\gamma|\overline{(\alpha|\delta)}\beta) = (\gamma|(\delta|\alpha)\beta) = (\gamma|T_{\beta,\alpha}\delta)$ for all γ and δ so $T^*_{\alpha,\beta} = T_{\beta,\alpha}$.

b) To find the trace of an operator, we take some basis $\mathcal{A}$ of V and take the trace of the matrix of the operator with respect to that basis. We may as well take an orthonormal basis $\mathcal{A} = \{\alpha_1, \ldots, \alpha_n\}$ of V. The i-th coordinate of $T_{\alpha,\beta}\alpha_j$ is then $(T_{\alpha,\beta}\alpha_j|\alpha_i) = (\alpha_j|\beta)(\alpha|\alpha_i) = (\alpha|(\beta|\alpha_j)\alpha_i)$. This is the ij-th entry of $[T_{\alpha,\beta}]_\mathcal{A}$. So the trace is $\sum_{i=1}^n (\alpha|(\beta|\alpha_i)\alpha_i) = (\alpha|\sum_{i=1}^n(\beta|\alpha_i)\alpha_i) = (\alpha|\beta)$.

c) $T_{\alpha,\beta}T_{\gamma,\delta}\phi = T_{\alpha,\beta}((\phi|\delta)\gamma) = (\phi|\delta)T_{\alpha,\beta}\gamma = (\phi|\delta)(\gamma|\beta)\alpha = (\phi|\overline{(\gamma|\beta)}\delta)\alpha = (\phi|(\beta|\gamma)\delta)\alpha = T_{\alpha,(\beta|\gamma)\delta}\phi$ for all ϕ so $T_{\alpha,\beta}T_{\gamma,\delta} = T_{\alpha,(\beta|\gamma)\delta}$.

d) If $T_{\alpha,\beta}$ is self adjoint then $T_{\alpha,\beta} = T^*_{\alpha,\beta} = T_{\beta,\alpha}$. So for all γ we have $(\gamma|\beta)\alpha = (\gamma|\alpha)\beta$. So α and β must be linearly dependent. If $\alpha = 0$ then $T_{0,\beta} = 0$ is self adjoint. If $\alpha \neq 0$ then $\beta = c\alpha$ for some c and $(\gamma|\beta)\alpha = \bar{c}(\gamma|\alpha)\alpha$ and $(\gamma|\alpha)\beta = c(\gamma|\alpha)\alpha$. So $T_{\alpha,\beta}$ is self adjoint if and only if either $\alpha = 0$ or β is a real multiple of α.

8.5:1 We just need to find characteristic vectors. For $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ the characteristic values are 0 and 2 with char vectors $[1\ -1]^t$ and $[1\ 1]^t$. Do Grahm-Schmidt on these vectors (i.e., normalize them) and they are the columns of $P = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

For $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ the characteristic values are -1 and 3 with char vectors $[1\ -1]^t$ and $[1\ 1]^t$. Do Grahm-Schmidt on these vectors (i.e., normalize them) and they are the columns of $P = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

For $\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$ the characteristic values are 1 and -1 with char vectors $[\sin\theta\ 1-\cos\theta]^t$ and $[-\sin\theta\ 1+\cos\theta]^t$. Do Grahm-Schmidt on these vectors (i.e., normalize them) and they are the columns of $P = \begin{bmatrix} \frac{\sin\theta}{\sqrt{2-2\cos\theta}} & \frac{-\sin\theta}{\sqrt{2+2\cos\theta}} \\ \sqrt{\frac{1-\cos\theta}{2}} & \sqrt{\frac{1+\cos\theta}{2}} \end{bmatrix} = \begin{bmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix}$, (the second equality technically requires $0 \le \theta \le \pi$ but P will still work for any θ).

8.5:4 $A^*A = \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ and $AA^* = \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}\begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ so $T^*T = TT^*$ and thus T is normal. The characteristic polynomial of T is $x^2 - 2x + 2$ with roots $1 \pm i$. For $1+i$ the characteristic vectors are $NS(A-(1+i)I) = NS\begin{bmatrix} -i & i \\ i & -i \end{bmatrix} = span\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. For $1-i$ the characteristic vectors are $NS(A-(1-i)I) = NS\begin{bmatrix} i & i \\ i & i \end{bmatrix} = span\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. So an orthonormal basis of characteristic vectors is $\{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}\}$.

8.5:8 This problem should also specify that T is an operator on a vector space V which is defined over $\mathbb{C}$. First of all, if such a T has an adjoint, then I claim there are unique self adjoint operators T_1 and T_2 on V so that $T = T_1 + iT_2$. If such T_j exist then by Theorem 9, page 297 we have $T^* = T_1^* - iT_2^* = T_1 - iT_2$. So we may solve for T_j and get $T_1 = \frac{T+T^*}{2}$ and $T_2 = \frac{T-T^*}{2i}$. But by Theorem 9, $T_1^* = (\frac{T+T^*}{2})^* = \frac{T^*+T}{2} = T_1$ and $T_2^* = (\frac{T-T^*}{2i})^* = \frac{T^*-T}{-2i} = \frac{T-T^*}{2i} = T_2$.

So for any T with an adjoint, $T_1 = \frac{T+T^*}{2}$ and $T_2 = \frac{T-T^*}{2i}$ are the unique self adjoint operators on V so that $T = T_1 + iT_2$.

Now

$$\begin{aligned} T^*T - TT^* &= (T_1 - iT_2)(T_1 + iT_2) - (T_1 + iT_2)(T_1 - iT_2) \\ &= T_1^2 - iT_2T_1 + iT_1T_2 + T_2^2 - T_1^2 - iT_2T_1 + iT_1T_2 - T_2^2 \\ &= 2i(T_1T_2 - T_2T_1) \end{aligned}$$

So $T^*T - TT^* = 0$ if and only if $T_1T_2 - T_2T_1 = 0$, i.e., T is normal if and only if T_1 and T_2 commute.

8.5:9 By the corollary on page 314, there is an invertible matrix P so that $P^{-1}AP$ is diagonal. Let D be the diagonal matrix whose diagonal entries are the cube roots of the entries of $P^{-1}AP$. Then $D^3 = P^{-1}AP$. let $B = PDP^{-1}$. Then

$$B^3 = PDP^{-1}PDP^{-1}PDP^{-1} = PD^3P^{-1} = PP^{-1}APP^{-1} = A$$