Math 405 Final Exam May 16, 2002

Work all problems in the answer book provided. Give sufficient reason for your answers. For example, a yes or no answer is insufficient. Instead say "Yes, because ... " and then show why.

1. (26) We'll call a linear mapping $P\colon V \to V$ a *projection* if $P^2 = P$. Suppose P is a projection and let $Q = \mathrm{id} - P$. For parts f and g, suppose V is equipped with a positive definite Hermitian product.
 a) Show that Q is a projection.
 b) Show that $V = \mathrm{Im}(P) \oplus \mathrm{Im}(Q)$.
 c) Show whether or not the set of projections is a subspace of $\mathcal{L}(V, V)$, the vector space of linear mappings from V to itself.
 d) If V is finite dimensional, show that $\dim \mathrm{Ker}(P) + \dim \mathrm{Ker}(Q) = \dim V$.
 e) If V is finite dimensional, show that any projection which not either the identity or 0 must have exactly two eigenvalues. What are these eigenvalues?
 f) If P is Hermitian, show that Q is Hermitian also.
 g) If P is Hermitian, show that $\mathrm{Ker}(P)$ and $\mathrm{Ker}(Q)$ are orthogonal subspaces of V.

2. (15) Let $L : V \to V$ be a a linear operator where V has a positive definite Hermitian product $<,>$. Suppose that $L^* = \sqrt{-1}L$.
 a) Show that $(1 + \sqrt{-1}) < Lv, v >$ is real for any $v \in V$.
 b) Show that if λ is an eigenvalue of L, then $(1 + \sqrt{-1})\lambda$ is real.
 c) Give an example of such an L when $V = \mathbb{C}^2$ with standard Hermitian product.

3. (10) Suppose that $L : V \to V$ is a linear operator and 1 is not an eigenvalue of L. Suppose further that $v \in V$ and $L^4 v = v$. Show that $L^3 v + L^2 v + Lv + v = 0$. Hint: $t^4 - 1 = (t - 1)(t^3 + t^2 + t + 1)$.

4. (16) Suppose A is a square matrix and $A^4 = 8A^2 - 16I$ but $A^2 \neq 4I$. Suppose A has two different eigenvalues.
 a) What are the eigenvalues of A?
 b) Recall that the minimal polynomial of a matrix A is the smallest degree monic polynomial p so that $p(A) = 0$. What are the possible minimal polynomials of A?
 c) Give, if possible, an example of such an A. If this is not possible, say why not.
 d) Give, if possible, an example of such an A which is diagonalizable. If this is not possible, say why not.

5. (18) Let A be a 7×7 matrix with characteristic polynomial $p_A(t) = (t - 1)^3(t^2 + 4)^2$.
 a) What is the smallest possible dimension of $\mathrm{Ker}(A - I)^3$?
 b) Give an example of such a matrix with $\dim \mathrm{Ker}(A - I) = 2$ and $\dim \mathrm{Ker}(A - 2\sqrt{-1}I) = 1$. Extra credit for an example with all real entries.
 c) Write down the Jordan normal form for A if its minimal polynomial is $(t - 1)^2(t^2 + 4)$.

6. (15) True-False, short answer
 a) When proving the existence of Jordan normal form we used the fact that if p and q are polynomials without a common root, then there are polynomials p' and q' so that _______.
 b) If $L : V \to V$ has eigenvalue λ and $p(t)$ is a polynomial, then $p(L)$ has eigenvalue _______.
 c) If $P\colon V \to W$ then $\dim \mathrm{Ker}(P) + \dim \mathrm{Im}(P) = ?$.
 d) True or false: If V and W are isomorphic finite dimensional vector spaces, then $\dim V$ must equal $\dim W$.
 e) Suppose V has dimension 3 and W has dimension 4. Suppose $\{T_1, T_2, \ldots, T_k\}$ are linearly independent linear mappings in $\mathcal{L}(V, W)$. Then $k \leq$ _______.

7. (() 10) (Optional) If you wish you may substitute this problem for ten points of any problem above. Just tell me which problem you want it to apply to. Let $A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$. Find e^A.