Math 405 Final Exam May 19, 2005

Work any five problems in the answer book provided. Be sure to clearly indicate which five you want counted, otherwise I will just count the first five I see. Give sufficient reason for your answers. For example, a yes or no answer is insufficient. Instead say "Yes, because ... " and then show why. In a multipart problem you may use results from earlier parts even though you may not have proven them.

1. (20) Let $\langle\ ,\ \rangle$ be a Hermitian product on $\mathbb{C}^2$ so that $\langle e_1, e_1\rangle = 3$, $\langle e_2, e_2\rangle = 1$, and $\langle e_1, e_2\rangle = 2\mathbf{i}$.
 a) Find an orthogonal basis for $\mathbb{C}^2$ with this product.
 b) Let W be the subspace generated by $\mathbf{i}e_1 + e_2$. Find $W^\perp$.

2. (20) Suppose $p(t)$ is a polynomial, $A\colon V \to V$ is a linear operator, and v is an eigenvector of A with eigenvalue λ. Show that v is an eigenvector of $p(A)$ and determine its eigenvalue.

3. (20) Recall that a linear operator $N\colon V \to V$ is nilpotent if $N^k = 0$ for some k. Suppose that $N\colon V \to V$ is nilpotent, $\dim V = 3$, and $N^2 \neq 0$.
 a) What is the characteristic polynomial of N?
 b) Show that $N^3 = 0$.
 c) Find all possible Jordan normal forms of N.
 d) Let $id\colon V \to V$ denote the identity. Show that $(id + N)^{-1} = id - N + N^2$.
 e) Suppose $v \in V$ and $N^2 v \neq 0$. Show that $\{v, Nv, N^2 v\}$ forms a basis of V. How does this relate to c)?

4. (20) Let A be a Hermitian or unitary matrix with entries in $\mathbb{C}$.
 a) Show that A has a square root, a matrix B so that $A = B^2$.
 b) Determine the condition on the eigenvalues of A which guarantees that the matrix B in part a) can be chosen to be Hermitian.

5. (20) Let V be the vector space generated by the functions e^t, te^t, and t^2e^t, and let $\mathcal{B} = \{e^t, te^t, t^2e^t\}$ and $\mathcal{C} = \{e^t, (t+1)e^t, (t^2+2t)e^t\}$ be two bases of V. Let $T\colon V \to V$ be the map $T(f) = df/dt$.
 a) Show that T is a linear transformation.
 b) Find the matrix $\mathcal{M}^{\mathcal{B}}_{\mathcal{C}}(T)$ of T relative to $\mathcal{B}$, $\mathcal{C}$.
 c) Find the matrix $\mathcal{M}^{\mathcal{B}}_{\mathcal{C}}(id)$ of the identity relative to $\mathcal{B}$, $\mathcal{C}$.

6. (20) Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{pmatrix}$. Find a matrix P so that $P^{-1}AP$ is in Jordan normal form. You may substitute this problem for any problem above, just tell me which one.

7. (20) Suppose a matrix A has characteristic polynomial $(t-1)^4(t-2)^3$ and minimal polynomial $(t-1)^2(t-2)$.
 a) What are the possible dimensions of $\mathrm{Ker}(A - I)$?
 b) What are the possible dimensions of $\mathrm{Ker}(A - 2I)$?
 c) What are the possible dimensions of $\mathrm{Ker}(A - 3I)$?
 d) What are the possible Jordan normal forms of A? For each such Jordan normal form, determine the dimension of $\mathrm{Ker}(A - 2I)$.