**MATH 405** **EXAM # 1** **March 9, 2005**

**INSTRUCTIONS:** Work all problems in the answer book provided. Give sufficient reason for your answers. If there is insufficient information given to determine an answer, say why. In all problems, $U$, $V$, and $W$ are vector spaces are over a subfield $K$ of the complex numbers. However the $U$, $V$, and $W$ in one numbered problem have no relation to the $U$, $V$, and $W$ in another numbered problem.

1. [20] Suppose $U$ and $W$ are subspaces of $V$ and every vector in $V$ can be written as a sum $u + w$ with $u \in U$ and $w \in W$. Suppose the dimensions of $U$, $W$, and $V$ are 3, 4, and 5 respectively.
   a) Find the dimension of $U \cap W$.
   b) Find the dimension of $\mathcal{L}(U, W)$.
   c) Find the dimension of $U + W$.
   d) The dual space $V^*$ of $V$ is defined to be $\mathcal{L}(V, K)$, the space of linear transformations from $V$ to $K^1$. Find the dimension of $V^*$.

2. [30] Let $V = \mathrm{Mat}_{2\times 2}(K)$ be the space of $2\times 2$ matrices with entries in $K$. Recall that if $A$ is a square matrix, then its trace $\mathrm{tr}(A)$ is the sum of its diagonal entries. Define $T\colon V \to V$ by $T(A) = 2A - \mathrm{tr}(A)I$ where $I$ is the identity matrix. So for example $T\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} -3 & 4 \\ 6 & 3 \end{pmatrix}$.
   a) Show that $T$ is a linear transformation.
   b) Find $\mathrm{Ker}(T)$.
   c) Find a basis for $\mathrm{Ker}(T)$.
   d) Find the dimension of $\mathrm{Im}(T)$.

3. [50] Let $T\colon V \to W$ be a one-to-one, onto, linear transformation. Suppose that $\mathcal{B} = \{v_1, \ldots, v_n\}$ is a basis of $V$.
   a) Let $w_i = T(v_i)$. Prove that $\mathcal{B}' = \{w_1, \ldots, w_n\}$ is a basis of $W$.
   b) What is the dimension of $V$?
   c) What is the dimension of $W$?
   d) Find the matrix $\mathcal{M}^{\mathcal{B}}_{\mathcal{B}'}(T)$ associated to $T$ relative to $\mathcal{B}$, $\mathcal{B}'$.
   e) If possible, give an example of such a $T$ where $T^{-1}$ is not linear.

4. [20] (optional) If you wish you may do this problem and reduce the number of points in problem 2 to 25 and the points in problem 3 to 35.
   a) For which $x$ are the vectors $\{(x, 1, 2, 0), (4, x, 4, 0), (0, 1, 2, 3)\}$ linearly independent?
   b) For which $x$ are the three vectors above a basis of $K^4$?
   c) Let $V$ be the space of polynomials of degree $\leq 4$ and let $T\colon V \to V$ be differentiation. Show that $T$ is a linear transformation and find its kernel.