MATH 405 EXAM # 2 May 2, 2005

INSTRUCTIONS: Work all problems in the answer book provided. Give sufficient reason for your answers. For example, a yes or no answer is insufficient. Instead say "Yes, because ... " and then show why.

1. [10] Let $\langle\ ,\ \rangle$ be a Hermitian product on $\mathbb{C}^2$ so that $\langle e_1, e_1\rangle = 4$, $\langle e_2, e_2\rangle = 1$, and $\langle e_1, e_2\rangle = 2\mathbf{i}$. Find $\langle(1, 2\mathbf{i}), (0, \mathbf{i})\rangle$ $(=\ \langle e_1 + 2\mathbf{i}e_2,\ \mathbf{i}e_2\rangle)$.

2. [25] Let $\langle\ ,\ \rangle$ be a scalar product on $\mathbb{R}^2$ so that $\langle e_1, e_1\rangle = 4$, $\langle e_2, e_2\rangle = 1$, and $\langle e_1, e_2\rangle = 2$.
   a) Find an orthogonal basis for $\mathbb{R}^2$ with this product.
   b) Is this product positive definite?
   c) Is this product nondegenerate?
   d) Sylvester's theorem says that the basis you found in part a) has something in common with all the other bases found correctly by other students taking this test. What do all these bases have in common?

3. [20] The matrix $A = \begin{pmatrix} -1 & 7 & 5 \\ -16 & 22 & 32 \\ -9 & 9 & -3 \end{pmatrix}$ has a characteristic polynomial which factors as $p_A(t) = (t+12)(t-24)(t-b)$ for some b.
   a) Which of the following are eigenvectors for A?
      i) ${}^t(2, 0, 1)$
      ii) ${}^t(1, 1, 0)$
      iii) ${}^t(-2, -2, 0)$
   b) What is b?
   c) Is there a real matrix S so that $S^{-1}AS$ is diagonal?
   d) Is there a real unitary matrix S so that $S^{-1}AS$ is diagonal?

4. [10] What can you say about the eigenvalues of a Hermitian matrix? That is, which numbers could possibly be eigenvalues? What can you say about the eigenvalues of a unitary matrix?

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5. [10] The characteristic polynomial of $A = \begin{pmatrix} 9 & 17 & 7 \\ 6 & 14 & 6 \\ -22 & -47 & -20 \end{pmatrix}$ is $p_A(t) = t^3 - 3t^2 + 4$.

Find $A^3 - 3A^2 + A$. (Hint: What does the Cayley-Hamilton theorem say about A?)

6. [25] Let V be a finite dimensional vector space over the real numbers with positive definite scalar product $\langle\ ,\ \rangle$. Let $S\colon V \to V$ and $T\colon V \to V$ be two symmetric linear transformations. Show that the following two conditions are equivalent:

a) $ST = TS$.

b) ST is symmetric.

In case $V = \mathbb{R}^n$, $S = L_A$ and $T = L_B$ for $n \times n$ matrices A and B, and $\langle\ ,\ \rangle$ is the usual dot product, show that the two conditions above are equivalent to:

c) There is a real unitary matrix S so that both $S^{-1}AS$ and $S^{-1}BS$ are diagonal matrices.