Math 461 Test # 2

1. (25) For each set H below determine whether or not it is a subspace. If it is not a subspace, show why it is not. If it is a subspace and is finite dimensional, write down a basis for H and determine the dimension of H.

a) H is the set of upper triangular matrices in $\mathbb{M}_{3\times 3}$.

Answer: *The sum of two upper triangular matrices is upper triangular. If you multiply an upper triangular matrix by a scalar, it is still upper triangular. The 0 matrix is upper triangular. So H is a subspace. An example of a basis is $\mathcal{B} = \{E_{11}, E_{12}, E_{13}, E_{22}, E_{23}, E_{33}\}$ where E_{ij} is the matrix which is zero in all but the entry in the i-th row and j-th column, and this entry is 1. If A is an upper triangular 3×3 matrix with ij-th entry a_{ij} then $A = a_{11}E_{11} + a_{12}E_{12} + a_{13}E_{13} + a_{22}E_{22} + a_{23}E_{23} + a_{33}E_{33}$ so $\mathcal{B}$ spans H. But $\mathcal{B}$ is also linearly independent since if $c_{11}E_{11}+c_{12}E_{12}+c_{13}E_{13}+c_{22}E_{22}+c_{23}E_{23}+c_{33}E_{33} = 0$ then all $c_{ij} = 0$. So $\mathcal{B}$ is a basis and thus H has dimension 6.*

b) $H = Span\{v_1, v_2, v_3\}$ where v_i are nonzero vectors in a vector space V, $v_1 = 2v_2 - 5v_3$, and v_2 is not a scalar multiple of v_3.

Answer: *We know H is a subspace because the span of a bunch of vectors is always a subspace. Since v_1 is a linear combination of v_2 and v_3 we know that $Span\{v_2, v_3\} = Span\{v_1, v_2, v_3\} = H$. But since v_2 is not a scalar multiple of v_3 we also know that $\{v_2, v_3\}$ is linearly independent. Since $\mathcal{B} = \{v_2, v_3\}$ is linearly independent and spans H, we know it is a basis of H. Then H has dimension 2. Actually, $\{v_1, v_2\}$ and $\{v_1, v_3\}$ and $\{v_1 + v_2, v_2 - v_3\}$ are other examples of bases of H, but the reasoning to show it is more involved.*

c) H is the set of polynomials p in $\mathbb{P}_3$ so that $p(0) = 1$.

Answer: *This is not a subspace for several reasons:*

1) The zero polynomial is not in H,

2) If $p(0) = 1$ and $q(0) = 1$ then $(p + q)(0) = 1 + 1 = 2$ so $p + q$ is not in H.

3) If $p(0) = 1$ then $cp(0) = c$ so cp is not in H if $c \neq 1$.

d) $H = \left\{ \begin{bmatrix} 2s+3t \\ s-2t \\ 5t \end{bmatrix} : s, t \text{ in } \mathbb{R} \right\}$.

Answer: *Note that $H = Span\{(2\ 1\ 0)^T, (3\ \ -2\ 5)^T\}$ so H is a subspace. Since $\{(2\ 1\ 0)^T, (3\ -2\ 5)^T\}$ is linearly independent it forms a basis of H and H has dimension 2.*

2. (15) Let $\mathcal{B} = \{[1\ 2]^T, [0\ 1]^T\}$ and $\mathcal{C} = \{[0\ 1]^T, [1\ 0]^T\}$ be two bases of $\mathbb{R}^2$. Find the coordinate change matrix $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$ from $\mathcal{C}$ to $\mathcal{B}$ coordinates.

Answer: *We know $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = \left(\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\right)^{-1}$. Since $c_1[1\ 2]^T + c_2[0\ 1]^T = (2c_1+c_2)[0\ 1]^T + c_1[1\ 0]^T$ we know that $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$. So $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$. You could also solve* *this by noting that* $[0\ 1]^T = 0 \cdot [1\ 2]^T + 1 \cdot [0\ 1]^T$ *so the first column is* $[0\ 1]^T$, *and* $[1\ 0]^T = 1 \cdot [1\ 2]^T - 2 \cdot [0\ 1]^T$ *so the second column is* $[1\ -2]^T$.

3. (15) Solve the equation $\begin{bmatrix} I_5 & X \\ 0 & I_5 \end{bmatrix} \begin{bmatrix} A \\ Y \end{bmatrix} = \begin{bmatrix} B \\ C \end{bmatrix}$ for X and Y in terms of A, B, and C. Assume that A, B, and C are invertible 5×5 matrices.

Answer: *This is a chapter 2 problem.*

4. (10) Determine whether or not $\{2t^2,\ (t-2)^2,\ t-1\}$ is a basis for $\mathbb{P}_2$. If it is a basis, find the coordinate vector of $p(t) = t+1$ relative to this basis.

Answer: *Solve* $a2t^2 + b(t-2)^2 + c(t-1) = 0$ *and we get* $(2a+b)t^2 + (-4b+c)t + (4b-c) = 0$. *The solution is* $c = 4b$, $a = -b/2$. *In particular we may take* $a = -1$, $b = 2$, $c = 8$ *and get a nontrivial linear combination equals 0. So the vectors are linearly dependent and thus could not be a basis.*

5. (30) Indicate whether each statement is true or false.

a) $\text{rank}(A) = \text{rank}(A^T)$.

Answer: *True*

b) If A is invertible, then $\text{rank}(A) = \text{rank}(A^{-1})$.

Answer: *True, since the rank in each case must be* n *if* A *is* $n \times n$.

c) Any four vectors which span a four dimensional vector space V form a basis for V.

Answer: *True*

d) If v_1, v_2, v_3 are linearly independent vectors in a four dimensional vector space V, then there is a vector v_4 so that v_1, v_2, v_3, v_4 is a basis for V.

Answer: *True*

e) Every vector space has a basis.

Answer: *False, it must be finite dimensional. (Actually this is true with a more general definition of basis which you are not likely to see in an undergraduate course. But as far as our definition of basis is concerned it is false.)*

f) The vector spaces $\mathbb{M}_{2\times 3}$ and $\mathbb{P}_5$ are isomorphic.

Answer: *True, since they both have dimension 6.*

g) $\det(2A) = 2\det(A)$.

Answer: *False.* $\det(2A) = 2^n \det(A)$ *if* A *is* $n \times n$.

h) $\det(A) = \det(A^T)$.

Answer: *True*

i) If $T\colon V \to W$ is a linear transformation, then the kernel of T is a subspace of V.

Answer: *True*

j) If $T\colon V \to W$ is a linear transformation, then the kernel of T is a subspace of W.

Answer: *False, (though the Range of* T *is a subspace of* W*).*

6. (10) Short answer.

a) If A is an $m \times n$ matrix, then $\text{rank}(A) + \dim Nul(A) =$

Answer: n

b) If B is obtained from A by switching two rows of A, then $\det(B) =$

Answer: $-\det(A)$

7. (20) Let $T: V \to W$ be a linear transformation. Given a subspace U of V, let $T(U)$ denote the set of all images of the form $T(x)$, where x is in U. Show that $T(U)$ is a subspace of W.

Answer: *If w_1 and w_2 are in $T(U)$ then there are x_1 and x_2 in U so that $T(x_1) = w_1$ and $T(x_2) = w_2$. Then $w_1 + w_2 = T(x_1) + T(x_2) = T(x_1 + x_2)$. But $x_1 + x_2$ is in U since U is a subspace. So $w_1 + w_2$ is in $T(U)$. Likewise, for any scalar c we have $cw_1 = cT(x_1) = T(cx_1)$ which is in $T(U)$ since cx_1 is in U. Finally $0 = T(0)$ so 0 is in $T(U)$. So $T(U)$ is a subspace of W since it satisfies the three subspace conditions.*